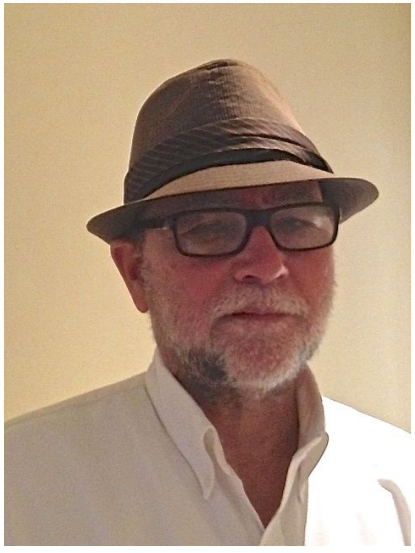
Local-global generation property of commutators in finite groups

Cristina Acciarri

University of Modena and Reggio Emilia

AGTA Workshop - Reinhold Baer Prize 2026
Napoli, October 27-28 2026

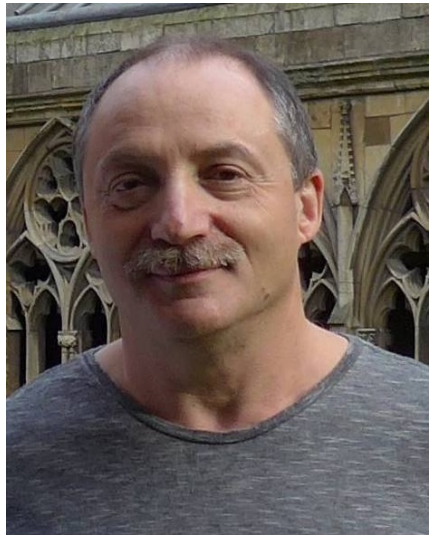
based on joint works with



Robert M. Guralnick (USC)



Evgeny Khukhro (Univ. Lincoln)



Pavel Shumyatsky (UnB)

Background on Rank

Def. (also known as the Prüfer rank)

The *rank* of a group G is the least positive integer r such that every finitely generated subgroup of G can be generated by r elements.

Background on Rank

Def. (also known as the Prüfer rank)

The *rank* of a group G is the least positive integer r such that every finitely generated subgroup of G can be generated by r elements.

Bounds for the rank of (finite) groups impose strong restrictions on their structure.

Rank (cont.)

Having a bound for the rank is crucial in many situations:

Rank (cont.)

Having a bound for the rank is crucial in many situations:

- a finite p-group of rank r has a powerful subgroup of (p, r)- bounded index;

Rank (cont.)

Having a bound for the rank is crucial in many situations:

- a finite p-group of rank r has a powerful subgroup of (p, r)- bounded index;
- a residually finite group of finite rank is virtually locally soluble;

Rank (cont.)

Having a bound for the rank is crucial in many situations:

- a finite p-group of rank r has a powerful subgroup of (p, r)- bounded index;
- a residually finite group of finite rank is virtually locally soluble;
- in representation theory of finite groups, the ranks are related to the dimensions of linear spaces arising as elementary abelian sections of the group;

Rank (cont.)

Having a bound for the rank is crucial in many situations:

- a finite p-group of rank r has a powerful subgroup of (p, r)- bounded index;
- a residually finite group of finite rank is virtually locally soluble;
- in representation theory of finite groups, the ranks are related to the dimensions of linear spaces arising as elementary abelian sections of the group;
- a finite soluble group of rank r has r-bounded Fitting height;

Rank (cont.)

Having a bound for the rank is crucial in many situations:

- a finite p-group of rank r has a powerful subgroup of (p, r)- bounded index;
- a residually finite group of finite rank is virtually locally soluble;
- in representation theory of finite groups, the ranks are related to the dimensions of linear spaces arising as elementary abelian sections of the group;
- a finite soluble group of rank r has r-bounded Fitting height;
- Lubotzky's criterion of linearity states that a finitely generated group G is linear (of finite degree over a field of characteristic zero) if and only if there exists a prime p and an integer r such that G has a filtration by normal subgroups $\{N_i\}_{i\in\mathbb{N}}$ satisfying G/N_1 is finite, N_1/N_i is a finite p-group with rank $\leq r$ for any $i \geq 1$ and $\bigcap_{i=1}^{\infty} N_i = 1$.

Commutator set

Notation

For a subgroup S of a group G, let $I_G(S)$ denote the set of commutators

$$I_G(S) = \{[g, s] = g^{-1}g^s \mid g \in G, s \in S\}.$$

Commutator set

Notation

For a subgroup S of a group G, let $I_G(S)$ denote the set of commutators

$$I_G(S) = \{[g, s] = g^{-1}g^s \mid g \in G, s \in S\}.$$

Then the subgroup $[G, S]$ is generated by $I_G(S)$.

In most cases, $[G, S] \neq I_G(S)$.

Commutator set

Notation

For a subgroup S of a group G, let $I_G(S)$ denote the set of commutators

$$I_G(S) = \{[g,s] = g^{-1}g^s \mid g \in G, s \in S\}.$$

Then the subgroup $[G,S]$ is generated by $I_G(S)$.

In most cases, $[G,S] \neq I_G(S)$.

For a group A acting by automorphisms on a group G, the subset $I_G(A) = \{[g,a] = g^{-1}g^a \mid g \in G, a \in A\}$ and the subgroup $[G,A]$ have the same meaning as above in the natural semidirect product GA.

Local-global generating properties: order

Local-global generating properties: order

We write "$(a, b, c \dots)$-bounded" to abbreviate "bounded above by some function depending only on the parameters $a, b, c \dots$"

Local-global generating properties: order

We write "$(a, b, c \dots)$-bounded" to abbreviate "bounded above by some function depending only on the parameters $a, b, c \dots$"
In general, a small number of generators does not mean that the group is small. But the following holds for the set of commutators

Example

If there are only finitely many, say m, commutators in a group G, then $[G, G]$ is finite of m-bounded order

Proof Let $I = \{[g_1, g_2], \dots, [g_{2m-1}, g_{2m}]\}$ be all the commutators in G. Note that $[G : C_G(g_i)] \leq m$ for any i. Since I generates $[G, G]$ we can assume that $G = \langle g_1, \dots, g_{2m} \rangle$.
Then $\bigcap_{i=1}^{2m} C_G(g_i)$ is a central subgroup of m-bounded index in G.
Hence the derived subgroup $[G, G]$ is finite of m-bounded order by Schur's theorem. $\square$

Local-global generating properties

In general a small number of generators does not mean that the group has small rank.

Local-global generating properties

In general a small number of generators does not mean that the group has small rank.

Main objective: to study the ranks of subgroups generated by commutators of elements of finite groups.

Local-global generating properties

In general a small number of generators does not mean that the group has small rank.

Main objective: to study the ranks of subgroups generated by commutators of elements of finite groups.

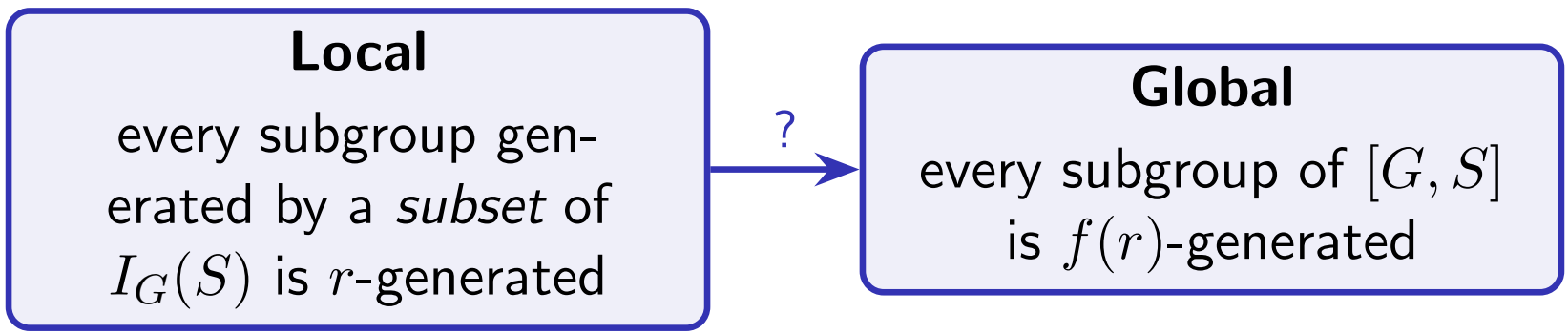
Local-global generating properties

In general a small number of generators does not mean that the group has small rank.

Main objective: to study the ranks of subgroups generated by commutators of elements of finite groups.

Local-global property

for a 'distinguished' subgroup S of G, the r-generation by subsets of $I_G(S)$ implies $f(r)$-generation of all subgroups of $[G, S]$.



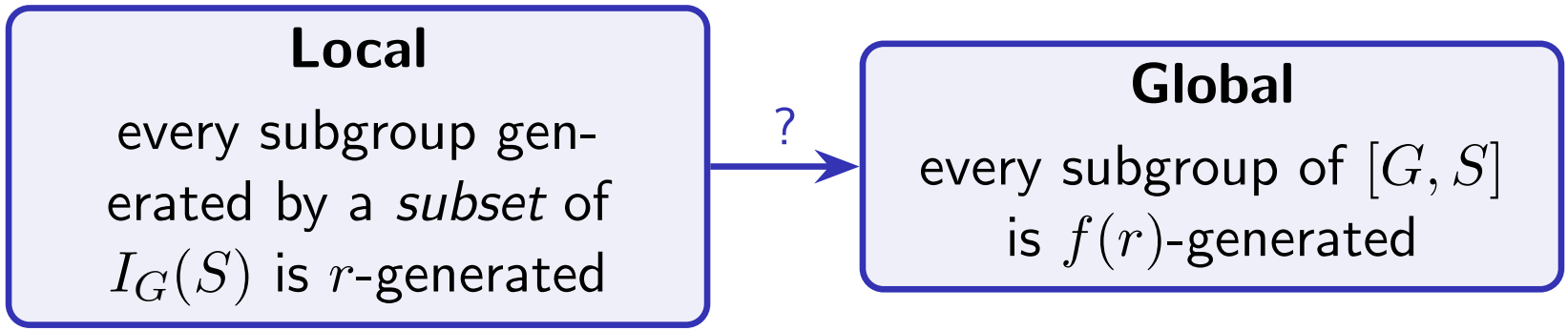
Local-global generating properties

In general a small number of generators does not mean that the group has small rank.

Main objective: to study the ranks of subgroups generated by commutators of elements of finite groups.

Local-global property

for a 'distinguished' subgroup S of G, the r-generation by subsets of $I_G(S)$ implies $f(r)$-generation of all subgroups of $[G, S]$.



For which "distinguished" subgroups $S \leq G$?

$S = P$, **a Sylow p-subgroup** $S = C$, **a Carter subgroup**

Case $S = P$ a Sylow p-subgroup in a p-soluble group

For a prime p, $O_p(G)$ denotes the largest normal p-subgroup of a group G, and $O_{p'}(G)$ the largest normal p'-subgroup.

Case $S = P$ a Sylow p-subgroup in a p-soluble group

For a prime p, $O_p(G)$ denotes the largest normal p-subgroup of a group G, and $O_{p'}(G)$ the largest normal p'-subgroup.
$O_{p',p}(G)$ denotes the inverse image of $O_p(G/O_{p'}(G))$
$O_{p',p,p'}(G)$ is the inverse image of $O_{p'}(G/O_{p',p}(G))$, and so on.

Case $S = P$ a Sylow p-subgroup in a p-soluble group

For a prime p, $O_p(G)$ denotes the largest normal p-subgroup of a group G, and $O_{p'}(G)$ the largest normal p'-subgroup.
$O_{p',p}(G)$ denotes the inverse image of $O_p(G/O_{p'}(G))$
$O_{p',p,p'}(G)$ is the inverse image of $O_{p'}(G/O_{p',p}(G))$, and so on.
These subgroups form the upper p-soluble series of G and a group G is said to be p-soluble, if this series terminates with $O_{p',p,p'\ldots}(G) = G$.

Case $S = P$ a Sylow p-subgroup in a p-soluble group

For a prime p, $O_p(G)$ denotes the largest normal p-subgroup of a group G, and $O_{p'}(G)$ the largest normal p'-subgroup.
$O_{p',p}(G)$ denotes the inverse image of $O_p(G/O_{p'}(G))$
$O_{p',p,p'}(G)$ is the inverse image of $O_{p'}(G/O_{p',p}(G))$, and so on.
These subgroups form the upper p-soluble series of G and a group G is said to be p-soluble, if this series terminates with $O_{p',p,p'\ldots}(G) = G$.

Theorem 1 - (AGKS1,2024)

Let p be a prime, r a positive integer, G a p-soluble finite group, and P a Sylow p-subgroup of G.
Suppose that any subgroup generated by a subset of $I_G(P) = \{[g,h] = g^{-1}g^h \mid g \in G, h \in P\}$ can be generated by r elements.
Then $[G,P]$ has r-bounded rank.

Case $S = P$ a Sylow p-subgroup in a p-soluble group

For a prime p, $O_p(G)$ denotes the largest normal p-subgroup of a group G, and $O_{p'}(G)$ the largest normal p'-subgroup.
$O_{p',p}(G)$ denotes the inverse image of $O_p(G/O_{p'}(G))$
$O_{p',p,p'}(G)$ is the inverse image of $O_{p'}(G/O_{p',p}(G))$, and so on.
These subgroups form the upper p-soluble series of G and a group G is said to be p-soluble, if this series terminates with $O_{p',p,p'\ldots}(G) = G$.

Theorem 1 - (AGKS1,2024)

Let p be a prime, r a positive integer, G a p-soluble finite group, and P a Sylow p-subgroup of G.
Suppose that any subgroup generated by a subset of $I_G(P) = \{[g,h] = g^{-1}g^h \mid g \in G, h \in P\}$ can be generated by r elements.
Then $[G,P]$ has r-bounded rank.

The set of commutators $I_G(P)$ has this local-global property:
r-generation for subgroups generated by subsets of this generating set implies $f(r)$-generation of all subgroups of $[G,P]$.

Counterexamples for $S = P$

Examples show that Theorem 1 does not hold without the assumption of p-solubility.

Counterexamples for $S = P$

Examples show that Theorem 1 does not hold without the assumption of p-solubility.

- **Construction (for $p = 2$)** $G = \mathrm{PSL}_2(q^e)$ where $q \equiv 3 \pmod 8$ and e is odd.

Counterexamples for $S = P$

Examples show that Theorem 1 does not hold without the assumption of p-solubility.

- **Construction (for $p = 2$)** $G = \mathrm{PSL}_2(q^e)$ where $q \equiv 3 \pmod 8$ and e is odd.

- **Properties:**
  - Sylow 2-subgroup P is elementary abelian of order 4.
  - Every subgroup generated by a subset of $I_G(P)$ is generated by at most 3 elements
  - However, $\mathrm{rank}([G, P]) = \mathrm{rank}(G) \geq e$ is unbounded as $e \to \infty$.

We produce similar counterexamples when p is odd.

Case $S = P$ a Sylow p-subgroup for arbitrary finite groups

The p-solubility condition can be dropped at the expense of a *stronger* local generation condition.

Case $S = P$ a Sylow p-subgroup for arbitrary finite groups

The p-solubility condition can be dropped at the expense of a *stronger* local generation condition.

Theorem 2 - (AGKS1,2024)

Let p be a prime, r a positive integer, G a finite group, and P a Sylow p-subgroup of G. Suppose that

(a) *any subgroup generated by a subset of $I_G(P)$ can be generated by r elements; and*

Case $S = P$ a Sylow p-subgroup for arbitrary finite groups

The p-solubility condition can be dropped at the expense of a *stronger* local generation condition.

Theorem 2 - (AGKS1,2024)

Let p be a prime, r a positive integer, G a finite group, and P a Sylow p-subgroup of G. Suppose that

(a) *any subgroup generated by a subset of $I_G(P)$ can be generated by r elements; and*

(b) *for* **any** *$x \in I_G(P)$, any subgroup generated by a subset of $I_G(x)$ can be generated by r elements.*

Then $[G, P]$ has r-bounded rank.

Case $S = P$ a Sylow p-subgroup for arbitrary finite groups

The p-solubility condition can be dropped at the expense of a *stronger* local generation condition.

Theorem 2 - (AGKS1,2024)

Let p be a prime, r a positive integer, G a finite group, and P a Sylow p-subgroup of G. Suppose that

(a) *any subgroup generated by a subset of $I_G(P)$ can be generated by r elements; and*

(b) *for* **any** *$x \in I_G(P)$, any subgroup generated by a subset of $I_G(x)$ can be generated by r elements.*

Then $[G, P]$ has r-bounded rank.

Remark: the additional condition (b) is needed here to deal with simple groups of order divisible by p, and it cannot be dropped in view of the aforementioned examples.

Local-global result for the rank of the derived subgroup

As a Corollary we also get this result that holds for arbitrary finite groups.

Theorem 3 - (AGKS1,2024)

Let G be a finite group. Suppose that for every prime p dividing $|G|$ for any Sylow p-subgroup P, any subgroup generated by a subset of $I_G(P)$ can be generated by r elements.
Then the derived subgroup G' has r-bounded rank.

Key tool: coprime automorphisms

Theorem 4 - (AGKS1, 2024)

Suppose that G is a finite group admitting a group of coprime automorphisms A such that, for a positive integer r, any subgroup generated by a subset of $I_G(A)$ can be generated by r elements. Then $[G, A]$ has r-bounded rank.

The overall strategy

- nilpotent case: powerful p-groups (Lubotzky–Mann);

Key tool: coprime automorphisms

Theorem 4 - (AGKS1, 2024)

Suppose that G is a finite group admitting a group of coprime automorphisms A such that, for a positive integer r, any subgroup generated by a subset of $I_G(A)$ can be generated by r elements. Then $[G, A]$ has r-bounded rank.

The overall strategy

- nilpotent case: powerful p-groups (Lubotzky–Mann);
- semisimple case: CFSG and an involution trick;

Key tool: coprime automorphisms

Theorem 4 - (AGKS1, 2024)

Suppose that G is a finite group admitting a group of coprime automorphisms A such that, for a positive integer r, any subgroup generated by a subset of $I_G(A)$ can be generated by r elements. Then $[G, A]$ has r-bounded rank.

The overall strategy

- nilpotent case: powerful p-groups (Lubotzky–Mann);
- semisimple case: CFSG and an involution trick;
- a Hall–Higman-type bound: the exponent of the automorphisms induced by A on $G/F^*(G)$ is r-bounded, where $F^*(G)$ is the generalized Fitting subgroup;

Key tool: coprime automorphisms

Theorem 4 - (AGKS1, 2024)

Suppose that G is a finite group admitting a group of coprime automorphisms A such that, for a positive integer r, any subgroup generated by a subset of $I_G(A)$ can be generated by r elements. Then $[G, A]$ has r-bounded rank.

The overall strategy

- nilpotent case: powerful p-groups (Lubotzky–Mann);
- semisimple case: CFSG and an involution trick;
- a Hall–Higman-type bound: the exponent of the automorphisms induced by A on $G/F^*(G)$ is r-bounded, where $F^*(G)$ is the generalized Fitting subgroup;
- a bound in terms of r for the Fitting height of soluble A-invariant sections with $H = [H, A]$;

Key tool: coprime automorphisms

Theorem 4 - (AGKS1, 2024)

Suppose that G is a finite group admitting a group of coprime automorphisms A such that, for a positive integer r, any subgroup generated by a subset of $I_G(A)$ can be generated by r elements. Then $[G, A]$ has r-bounded rank.

The overall strategy

- nilpotent case: powerful p-groups (Lubotzky–Mann);
- semisimple case: CFSG and an involution trick;
- a Hall–Higman-type bound: the exponent of the automorphisms induced by A on $G/F^*(G)$ is r-bounded, where $F^*(G)$ is the generalized Fitting subgroup;
- a bound in terms of r for the Fitting height of soluble A-invariant sections with $H = [H, A]$;
- G can be assumed to be generated by r-boundedly many orbits of elements of $I_G(A)$ under the action of A (long and technical);

Key tool: coprime automorphisms

Theorem 4 - (AGKS1, 2024)

Suppose that G is a finite group admitting a group of coprime automorphisms A such that, for a positive integer r, any subgroup generated by a subset of $I_G(A)$ can be generated by r elements. Then $[G, A]$ has r-bounded rank.

The overall strategy

- nilpotent case: powerful p-groups (Lubotzky–Mann);
- semisimple case: CFSG and an involution trick;
- a Hall–Higman-type bound: the exponent of the automorphisms induced by A on $G/F^*(G)$ is r-bounded, where $F^*(G)$ is the generalized Fitting subgroup;
- a bound in terms of r for the Fitting height of soluble A-invariant sections with $H = [H, A]$;
- G can be assumed to be generated by r-boundedly many orbits of elements of $I_G(A)$ under the action of A (long and technical);
- reduction to the case where A is p-group of r-bounded derived length and induction on this derived length.

Key tool: coprime automorphisms (cont.)

Theorem 4 - (AGKS1, 2024)

Suppose that G is a finite group admitting a group of coprime automorphisms A such that, for a positive integer r, any subgroup generated by a subset of $I_G(A)$ can be generated by r elements. Then $[G, A]$ has r-bounded rank.

Why we need it. Allows control over the coprime action of P (Sylow) or C(Carter) on some specific sections.

Key tool: coprime automorphisms (cont.)

Theorem 4 - (AGKS1, 2024)

Suppose that G is a finite group admitting a group of coprime automorphisms A such that, for a positive integer r, any subgroup generated by a subset of $I_G(A)$ can be generated by r elements. Then $[G, A]$ has r-bounded rank.

Why we need it. Allows control over the coprime action of P (Sylow) or C(Carter) on some specific sections.

For example, for p-soluble G (Theorem 1) one argues by induction on the length of a $(p \times p')$-series.

If N is its second term, with Sylow p-subgroup P_0 and Hall p'-subgroup H_0, then P_0 acts coprimely on $O_{p'}(N)$ and H_0 acts coprimely on $O_p(N)$, and both $I_{O_{p'}(N)}(P_0)$ and $I_{O_p(N)}(H_0)$ lie in $I_G(P)$.

The key tool bounds the ranks of $[O_{p'}(N), P_0]$ and $[O_p(N), H_0]$, and factoring out their product shortens the series.

Carter subgroups

Def. Carter subgroup

A Carter subgroup C of a finite group G is a nilpotent subgroup that coincides with its normalizer, that is $C = N_G(C)$.

Carter subgroups

Def. Carter subgroup

A Carter subgroup C of a finite group G is a nilpotent subgroup that coincides with its normalizer, that is $C = N_G(C)$.

Theorem (RW Carter - EP Vdovin)

- Every soluble finite group has a Carter subgroup.
- If a finite group G has a Carter subgroup C, then all Carter subgroups of G are conjugate and the image of C in any quotient group G/N is a Carter subgroup of G/N

Vdovin also obtained a complete description of almost simple groups that have Carter subgroups (a finite group G is said to be almost simple if $S \leq G \leq \mathrm{Aut}(S)$ for a non-abelian simple group S).

Carter subgroups

Def. Carter subgroup

A Carter subgroup C of a finite group G is a nilpotent subgroup that coincides with its normalizer, that is $C = N_G(C)$.

Theorem (RW Carter - EP Vdovin)

- Every soluble finite group has a Carter subgroup.
- If a finite group G has a Carter subgroup C, then all Carter subgroups of G are conjugate and the image of C in any quotient group G/N is a Carter subgroup of G/N

Vdovin also obtained a complete description of almost simple groups that have Carter subgroups (a finite group G is said to be almost simple if $S \leq G \leq \mathrm{Aut}(S)$ for a non-abelian simple group S).

Property: In a finite group G we have $G' = [G, C]$, when C is a Carter subgroup.

The case $S = C$ a Carter subgroup in soluble groups

Theorem A - (AGKS2,2026)

Let r be a positive integer, G a soluble finite group, and C a Carter subgroup of G. Suppose that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements. Then the derived subgroup G' has r-bounded rank.

The case $S = C$ a Carter subgroup in soluble groups

Theorem A - (AGKS2,2026)

Let r be a positive integer, G a soluble finite group, and C a Carter subgroup of G. Suppose that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements. Then the derived subgroup G' has r-bounded rank.

The set of commutators $I_G(C)$ has this local-global property: r-generation for subgroups generated by subsets of this generating set implies $f(r)$-generation of all subgroups of $[G, C] = G'$.

Theorem A, step 1: a small minimal normal subgroup

Claim. If G is soluble and C satisfies the hypothesis, then G has a minimal normal subgroup of rank $\leq r$.

Let $M \cong C_p^k$ be minimal normal.

- $[M, C]$ is abelian and generated by a subset of $I_G(C)$, so $\mathrm{rk}[M, C] \leq r$.

Theorem A, step 1: a small minimal normal subgroup

Claim. If G is soluble and C satisfies the hypothesis, then G has a minimal normal subgroup of rank $\leq r$.

Let $M \cong C_p^k$ be minimal normal.

- $[M, C]$ is abelian and generated by a subset of $I_G(C)$, so $\mathrm{rk}[M, C] \leq r$.
- If $C_M(C) = 1$, then $M = [M, C]$.

Theorem A, step 1: a small minimal normal subgroup

Claim. If G is soluble and C satisfies the hypothesis, then G has a minimal normal subgroup of rank $\leq r$.

Let $M \cong C_p^k$ be minimal normal.

- $[M, C]$ is abelian and generated by a subset of $I_G(C)$, so $\mathrm{rk}[M, C] \leq r$.
- If $C_M(C) = 1$, then $M = [M, C]$.
- If $A := C_M(C) \neq 1$, then $A \leq N_G(C) = C$, so $I_G(A) \subseteq I_G(C)$ and $\mathrm{rk}[G, A] \leq r$: either $[G, A] = M$, or $A \leq Z(G)$ and any subgroup of order p of A is minimal normal.

Theorem A, step 1: a small minimal normal subgroup

Claim. If G is soluble and C satisfies the hypothesis, then G has a minimal normal subgroup of rank $\leq r$.

Let $M \cong C_p^k$ be minimal normal.

- $[M, C]$ is abelian and generated by a subset of $I_G(C)$, so $\mathrm{rk}[M, C] \leq r$.
- If $C_M(C) = 1$, then $M = [M, C]$.
- If $A := C_M(C) \neq 1$, then $A \leq N_G(C) = C$, so $I_G(A) \subseteq I_G(C)$ and $\mathrm{rk}[G, A] \leq r$: either $[G, A] = M$, or $A \leq Z(G)$ and any subgroup of order p of A is minimal normal.

Consequence: the Fitting height of G is r-bounded.

Theorem A, step 1: a small minimal normal subgroup

Claim. If G is soluble and C satisfies the hypothesis, then G has a minimal normal subgroup of rank $\leq r$.

Let $M \cong C_p^k$ be minimal normal.

- $[M, C]$ is abelian and generated by a subset of $I_G(C)$, so $\mathrm{rk}[M, C] \leq r$.
- If $C_M(C) = 1$, then $M = [M, C]$.
- If $A := C_M(C) \neq 1$, then $A \leq N_G(C) = C$, so $I_G(A) \subseteq I_G(C)$ and $\mathrm{rk}[G, A] \leq r$: either $[G, A] = M$, or $A \leq Z(G)$ and any subgroup of order p of A is minimal normal.

Consequence: the Fitting height of G is r-bounded. Indeed, $G/C_G(M)$ acts faithfully as a linear group on M (as a vector space of dimension at most r over F_q for some prime q) and by Zassenhaus's theorem the derived length d of $G/C_G(M)$ is r-bounded. Repeating inductively the argument one gets $h(G) \leq d + 1$.

Theorem A: roadmap

- **Nilpotent pieces.** M nilpotent, $[G, M] \leq F(G) \Rightarrow \mathrm{rk}[G, M]$ r-bounded (powerful p-groups; the p'-part of M is handled by the key tool).

Theorem A: roadmap

- **Nilpotent pieces.** M nilpotent, $[G, M] \leq F(G) \Rightarrow \mathrm{rk}[G, M]$ r-bounded (powerful p-groups; the p'-part of M is handled by the key tool).
- **Fitting height** is r-bounded (previous slide).

Theorem A: roadmap

- **Nilpotent pieces.** M nilpotent, $[G, M] \leq F(G) \Rightarrow \mathrm{rk}[G, M]$ r-bounded (powerful p-groups; the p'-part of M is handled by the key tool).
- **Fitting height** is r-bounded (previous slide).
- **Induction on $h(G)$.** Factor out $[G, C \cap F]$; then $C_{\bar{F}}(\bar{C}) = 1$, so every Sylow Q of $\bar{F}$ is $[Q, K]$ with $K \leq \bar{C}$ a q'-group: key tool.

Theorem A: roadmap

- **Nilpotent pieces.** M nilpotent, $[G, M] \leq F(G) \Rightarrow \mathrm{rk}[G, M]$ r-bounded (powerful p-groups; the p'-part of M is handled by the key tool).
- **Fitting height** is r-bounded (previous slide).
- **Induction on $h(G)$.** Factor out $[G, C \cap F]$; then $C_{\bar{F}}(\bar{C}) = 1$, so every Sylow Q of $\bar{F}$ is $[Q, K]$ with $K \leq \bar{C}$ a q'-group: key tool.
- **End.** Lubotzky–Mann for G'', then the metabelian case.

Counterexamples for $S = C$

Theorem A does not hold without the assumption of solubility, as shown by counterexamples.

Counterexamples for $S = C$

Theorem A does not hold without the assumption of solubility, as shown by counterexamples.

- **Construction:** $G = \mathrm{PSL}_2(p^e)$ with $p \equiv 7 \pmod{16}$ and e odd.

Counterexamples for $S = C$

Theorem A does not hold without the assumption of solubility, as shown by counterexamples.

- **Construction:** $G = \mathrm{PSL}_2(p^e)$ with $p \equiv 7 \pmod{16}$ and e odd.
- **Properties:**
  - If $x \in G$ is a 2-element, then every subgroup generated by a subset of $I_G(x)$ can be generated by 2 elements.
  - Dihedral group $K \cong D_8$ of order 8 (a self-normalizing Sylow 2-subgroup) is a Carter subgroup of G.
  - Subgroups generated by subsets of $I_G(K)$ are at most $(2 \cdot 7 =)$14-generated.
  - Yet $r(G') = r(G) \geq e$ grows without bound.

Counterexamples for $S = C$

Theorem A does not hold without the assumption of solubility, as shown by counterexamples.

- **Construction:** $G = \mathrm{PSL}_2(p^e)$ with $p \equiv 7 \pmod{16}$ and e odd.
- **Properties:**
  - If $x \in G$ is a 2-element, then every subgroup generated by a subset of $I_G(x)$ can be generated by 2 elements.
  - Dihedral group $K \cong D_8$ of order 8 (a self-normalizing Sylow 2-subgroup) is a Carter subgroup of G.
  - Subgroups generated by subsets of $I_G(K)$ are at most $(2 \cdot 7 =)$14-generated.
  - Yet $r(G') = r(G) \geq e$ grows without bound.

However, such examples exist only for a small class of almost simple groups, and it is possible to obtain a bound for the rank of G' in terms of the ranks of these obstructions.

Case $S = C$ for a finite group: parametrizing obstructions

Theorem B - (AGKS2,2026)

Let r and l be positive integers. Suppose that a finite group G contains a Carter subgroup C such that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements.

Case $S = C$ for a finite group: parametrizing obstructions

Theorem B - (AGKS2,2026)

Let r and l be positive integers. Suppose that a finite group G contains a Carter subgroup C such that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements.
Let l be the maximum rank of composition factors of G isomorphic to $PSL_2(q)$ for $q \equiv 7 \pmod 8$.

Case $S = C$ for a finite group: parametrizing obstructions

Theorem B - (AGKS2,2026)

Let r and l be positive integers. Suppose that a finite group G contains a Carter subgroup C such that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements.
Let l be the maximum rank of composition factors of G isomorphic to $PSL_2(q)$ for $q \equiv 7 \pmod 8$.
Then the derived subgroup G' has (r, l)-bounded rank.

Note that the rank of $PSL_2(q)$, where $q = p^k$ for a prime p, is equal to k, unless $k = 1$ and then the rank is 2.

Case $S = C$ for a finite group: parametrizing obstructions

Theorem B - (AGKS2,2026)

Let r and l be positive integers. Suppose that a finite group G contains a Carter subgroup C such that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements.
Let l be the maximum rank of composition factors of G isomorphic to $PSL_2(q)$ for $q \equiv 7 \pmod 8$.
Then the derived subgroup G' has (r, l)-bounded rank.

Note that the rank of $PSL_2(q)$, where $q = p^k$ for a prime p, is equal to k, unless $k = 1$ and then the rank is 2.

The 'local' property of r-generation of any subgroup generated by a subset of $I_G(C)$ implies the 'global' property of (r, l)-bounded generation of every subgroup of G'.

The role of simple groups in parametrizing the obstructions

The proof of Theorem B (and its Corollaries, see later) depends on the CFSG.

We use the classification for proving two results on finite simple groups containing elements x such that every subgroup generated by a subset of $I_G(x)$ is r-generated.

These two results are proved in greater generality than needed in the proof of Theorem B.

Simple groups with elements x and r-generated $I_G(x)$

Theorem E - (x is an involution), (AGKS2,2026)

Let r be a positive integer, G be a finite simple group, and $x \in G$ an involution. Suppose that every subgroup generated by a subset of $I_G(x)$ can be generated by at most r elements. Then one of the following holds:

1. the rank of G is r-bounded; or
2. $G = PSL_2(q)$ with $q \equiv 3 \pmod 4$.

Simple groups with elements x and r-generated $I_G(x)$

Theorem E - (x is an involution), (AGKS2,2026)

Let r be a positive integer, G be a finite simple group, and $x \in G$ an involution. Suppose that every subgroup generated by a subset of $I_G(x)$ can be generated by at most r elements. Then one of the following holds:

1. the rank of G is r-bounded; or
2. $G = PSL_2(q)$ with $q \equiv 3 \pmod 4$.

Theorem F - (x has odd prime order p), (AGKS2,2026)

Let p be an odd prime, and r a positive integer. Let G be a finite simple group of order divisible by p, and $x \in G$ an element of order p. Suppose that every subgroup generated by a subset of $I_G(x)$ can be generated by at most r elements. Then one of the following holds:

1. the rank of G is r-bounded; or
2. $G = PSL_2(q)$ with $q \equiv -1 \pmod p$.

Simple groups: where the obstructions live

Let G be finite simple and $x \in G$ of prime order p such that subsets of $I_G(x)$ generate r-generated subgroups. Then $\operatorname{rk} G$ is r-bounded, unless

- **Theorem E** ($p = 2$): $G = \mathrm{PSL}_2(q)$, $q \equiv 3 \pmod 4$;
- **Theorem F** (p odd): $G = \mathrm{PSL}_2(q)$, $q \equiv -1 \pmod p$ (bound independent of p).

Simple groups: where the obstructions live

Let G be finite simple and $x \in G$ of prime order p such that subsets of $I_G(x)$ generate r-generated subgroups. Then $\mathrm{rk}\, G$ is r-bounded, unless

- **Theorem E** ($p = 2$): $G = \mathrm{PSL}_2(q)$, $q \equiv 3 \pmod 4$;
- **Theorem F** (p odd): $G = \mathrm{PSL}_2(q)$, $q \equiv -1 \pmod p$ (bound independent of p).

The Sylow example ($p = 2$) corresponds exactly to $q \equiv 3 \pmod 8$.

Simple groups: where the obstructions live

Let G be finite simple and $x \in G$ of prime order p such that subsets of $I_G(x)$ generate r-generated subgroups. Then $\operatorname{rk} G$ is r-bounded, unless

- **Theorem E** ($p = 2$): $G = \mathrm{PSL}_2(q)$, $q \equiv 3 \pmod 4$;
- **Theorem F** (p odd): $G = \mathrm{PSL}_2(q)$, $q \equiv -1 \pmod p$ (bound independent of p).

The Sylow example ($p = 2$) corresponds exactly to $q \equiv 3 \pmod 8$.

For Carter subgroups our counterexample $G = \mathrm{PSL}_2(p^e)$ with $p \equiv 7 \pmod{16}$ and e odd lies in the exceptional family $q \equiv 3 \pmod 4$

Theorem B: roadmap

Theorem B - (AGKS2,2026)

Let r and l be positive integers. Suppose that a finite group G contains a Carter subgroup C such that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements.
Let l be the maximum rank of composition factors of G isomorphic to $\mathrm{PSL}_2(q)$ for $q \equiv 7 \pmod 8$.
Then the derived subgroup G' has (r, l)-bounded rank.

- If $R(G) = 1$, then $F^*(G) = T_1 \times \cdots \times T_n$, $n \leq 3r + 1$; each T_i has (r, l)-bounded rank (Theorems E, F and Vdovin); $|G : F^*(G)|$ is bounded.

Theorem B: roadmap

Theorem B - (AGKS2,2026)

Let r and l be positive integers. Suppose that a finite group G contains a Carter subgroup C such that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements.
Let l be the maximum rank of composition factors of G isomorphic to $\mathrm{PSL}_2(q)$ for $q \equiv 7 \pmod 8$.
Then the derived subgroup G' has (r,l)-bounded rank.

- If $R(G) = 1$, then $F^*(G) = T_1 \times \cdots \times T_n$, $n \leq 3r+1$; each T_i has (r,l)-bounded rank (Theorems E, F and Vdovin); $|G : F^*(G)|$ is bounded.
  G is generated by (r,l)-boundedly many conjugates of C (Hall–Liebeck–Seitz, Guralnick–Saxl, Belyaev–Hartley).

Theorem B: roadmap

Theorem B - (AGKS2,2026)

Let r and l be positive integers. Suppose that a finite group G contains a Carter subgroup C such that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements.
Let l be the maximum rank of composition factors of G isomorphic to $\mathrm{PSL}_2(q)$ for $q \equiv 7 \pmod 8$.
Then the derived subgroup G' has (r,l)-bounded rank.

- If $R(G) = 1$, then $F^*(G) = T_1 \times \cdots \times T_n$, $n \leq 3r+1$; each T_i has (r,l)-bounded rank (Theorems E, F and Vdovin); $|G : F^*(G)|$ is bounded.
  G is generated by (r,l)-boundedly many conjugates of C (Hall–Liebeck–Seitz, Guralnick–Saxl, Belyaev–Hartley).
- G general case: $G = R(G)\langle C_1, \ldots, C_m\rangle$ and m is (r,l)-bounded; Theorem A applies to $R(G)C_i$ with Carter subgroup C_i so $\mathrm{rk}[R(G)C_i, C_i]$ has r-bounded rank and in particular $\mathrm{rk}[R(G), C_i]$ is r-bounded; modulo $\prod[R(G), C_i]$ the radical $R(G)$ is central. Also $G/R(G)$ has (r,l)-bounded rank; Lubotzky–Mann.

Corollaries of Theorem B

We state two corollaries of Theorem B, which do not require restrictions on composition factors under additional hypotheses.

Corollary C - (AGKS2,2026)

Let r be a positive integer. Suppose that a finite group G has a Carter subgroup C of order not divisible by 8 such that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements. Then the derived subgroup G' has r-bounded rank.

Corollaries of Theorem B

We state two corollaries of Theorem B, which do not require restrictions on composition factors under additional hypotheses.

Corollary C - (AGKS2,2026)

Let r be a positive integer. Suppose that a finite group G has a Carter subgroup C of order not divisible by 8 such that any subgroup generated by a subset of $I_G(C)$ can be generated by r elements. Then the derived subgroup G' has r-bounded rank.

If $8 \nmid |C|$, then no such $\mathrm{PSL}_2(q)$ composition factors exist, and $r(G')$ is purely r-bounded.

Corollaries of Theorem B (cont.)

Another consequence of the proof of Theorem B gives a global rank-type conclusion under a stronger local condition.

Corollary D - (AGKS2,2026)

Let r be a positive integer. Suppose that a finite group G has a Carter subgroup C such that

(a) any subgroup generated by a subset of $I_G(C)$ can be generated by r elements; and

(b) for **any** $x \in I_G(C)$, any subgroup generated by a subset of $I_G(x)$ can be generated by r elements.

Then the derived subgroup G' has r-bounded rank.

Observation

Corollary D: on a factor $\mathrm{PSL}_2(q)$, $q \equiv 7 \pmod 8$, the Carter subgroup contains a Sylow 2-subgroup T, and condition (b) for C gives condition (b) for T: the Sylow Theorem 2 with (b) bounds the rank.

Remark for (p-)soluble case: a more general phenomenon

	$S = P$ **Sylow**	$S = C$ **Carter**
soluble case	G p-soluble $\Rightarrow \mathrm{rk}[G, P] \leq f(r)$	G soluble $\Rightarrow \mathrm{rk}\, G' \leq f(r)$
obstruction	$\mathrm{PSL}_2(q)$, $q \equiv 3 \pmod 8$	$\mathrm{PSL}_2(q)$, $q \equiv 7 \pmod 8$

Remark for (p-)soluble case: a more general phenomenon

	$S = P$ **Sylow**	$S = C$ **Carter**
soluble case	G p-soluble $\Rightarrow \mathrm{rk}[G, P] \leq f(r)$	G soluble $\Rightarrow \mathrm{rk}\, G' \leq f(r)$
obstruction	$\mathrm{PSL}_2(q)$, $q \equiv 3 \pmod 8$	$\mathrm{PSL}_2(q)$, $q \equiv 7 \pmod 8$

Recall that, for a set of primes π, a finite group is said to be π-soluble if it has a normal series whose factors are either π'-groups or soluble π-groups.

Remark for (p-)soluble case: a more general phenomenon

	$S = P$ **Sylow**	$S = C$ **Carter**
soluble case	G p-soluble $\Rightarrow \mathrm{rk}[G,P] \leq f(r)$	G soluble $\Rightarrow \mathrm{rk}\, G' \leq f(r)$
obstruction	$\mathrm{PSL}_2(q)$, $q \equiv 3 \pmod 8$	$\mathrm{PSL}_2(q)$, $q \equiv 7 \pmod 8$

Recall that, for a set of primes π, a finite group is said to be π-soluble if it has a normal series whose factors are either π'-groups or soluble π-groups.

Corollary (AGKS3,2026 Corollary C)

Suppose that H is a π-subgroup of a finite π-soluble group G such that any subset of $I_G(H)$ generates a subgroup that can be generated by r elements. Then the rank of $[G,H]$ is bounded in terms of r.

Remark for (p-)soluble case: a more general phenomenon

	$S = P$ **Sylow**	$S = C$ **Carter**
soluble case	G p-soluble $\Rightarrow \mathrm{rk}[G,P] \leq f(r)$	G soluble $\Rightarrow \mathrm{rk}\, G' \leq f(r)$
obstruction	$\mathrm{PSL}_2(q)$, $q \equiv 3 \pmod 8$	$\mathrm{PSL}_2(q)$, $q \equiv 7 \pmod 8$

Recall that, for a set of primes π, a finite group is said to be π-soluble if it has a normal series whose factors are either π'-groups or soluble π-groups.

Corollary (AGKS3,2026 Corollary C)

Suppose that H is a π-subgroup of a finite π-soluble group G such that any subset of $I_G(H)$ generates a subgroup that can be generated by r elements. Then the rank of $[G,H]$ is bounded in terms of r.

- $\pi = \{p\}$, $H = P$: the Sylow theorem for p-soluble groups: Thm 1;
- $\pi =$ all primes, $H = C$: Thm A.

Examples in AGKS1 show that the π-solubility condition on G cannot be dropped in the result above.

References

- **AGKS1** C. Acciarri, R. M. Guralnick, E. I. Khukhro, and P. Shumyatsky, Rank type conditions on commutators in finite groups, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2026, https://doi.org/10.2422/2036-2145.202501_002.
- **AGKS2** C. Acciarri, R. M. Guralnick, E. I. Khukhro, and P. Shumyatsky, Rank properties of commutators in finite groups, arXiv:2609.28309, 2026 https://doi.org/10.48550/arXiv.2609.28309
- **AGKS3**. Acciarri, R. M. Guralnick, E. I. Khukhro, and P. Shumyatsky, Local–global generation properties of commutators in finite π-soluble groups, J. Algebra 692 (2026), 519–549. https://doi.org/10.1016/j.jalgebra.2025.12.020

Thank you for your attention!