Conciseness of words and equationally Noetherian groups

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Conciseness and strong conciseness

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The *verbal subgroup* of w in G: $w(G) = \langle G_w \rangle$.

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A word w is *concise* in a class $\mathcal{C}$ of groups if for every $G \in \mathcal{C}$,

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Theorem (P. Hall, '50s)

Every non-commutator word is concise in the class of all groups

A non-commutator word w is a word such that $w \notin F_k'$.

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Theorem (Turner-Smith, 1966)

Every word is concise in the class of all group all of whose quotients are residually finite.

In particular, every word is concise in the class of finitely generated virtually abelian-by-polycyclic groups.

Abelian-by-polycyclic groups and linear groups

Corollary

Every word is concise in the class of:

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Theorem (Merzlyakov, 1967)

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Theorem (Wilson, 1974)

Every outer commutator word is concise in the class of all groups.

A new conjecture

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Theorem (Ivanov, 1989)

The word $w = [[x^{pn}, y^{pn}]^n, y^{pn}]^n$ *is not concise in a certain (non-residually finite) group for, big enough* $n, p \in \mathbb{N}$ *with* n *odd and* p *a prime.*

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1. *(Guralnick, Shumyatsky, 2018) Rational words. For example,* $[\ldots[x_1^{n_1}, x_2]^{n_2}, \ldots, x_k]^{n_k}$, *with* $n_1, \ldots, n_k \in \mathbb{Z}$.

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2. *(Fernández-Alcober, Pintonello, 2023)* $\gamma_n(w_1, \dots, w_n)$ *and* $\delta_n(w_1, \dots, w_{2^n})$, *with* $n \in \mathbb{N}$ *and* w_i *non-commutator words.*

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1. *(Acciarri, Shumyatsky, 2024)* w^q *and* $[w^q, y, \overset{n}{\ldots}, y]$, *with* $n, q \in \mathbb{N}$ *and* w *an outer commutator word.*

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   Here, coprime commutator "more or less" means that we consider commutators of the form $[x, y]$ *with* $(o(x), o(y)) = 1$.

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Conjecture

Every word is concise in the class of all profinite groups.

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Theorem

Let $\mathcal{C}$ be the class of all profinite groups. The following words are strongly concise:

1. *(Detomi, Klopsch, Shumyatsky, 2020) Outer commutator words, x^6, $[x^n, z_1, \ldots, z_r]$ and $[x^n, y, y, z_1, \ldots, z_r]$, with $n \in \{2, 3\}$ and $r \geq 0$.*

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3. *(Azevedo, Shumyatsky, 2021) The words* $w = [v, y, \overset{n}{\ldots}, y]$ *and* $w = [y, v, \overset{n}{\ldots}, v]$, *with* $n \in \mathbb{N}$ *and* $v = \gamma_n(x_1, \ldots, x_n)$, *if* $w(G)$ *is finitely generated.*

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- *Virtually abelian-by-polycyclic profinite groups? (What is this?)*

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Definitions and analogies

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Let $S \subseteq G * F_k$. The *set of solutions* of S in G^k is

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Analogy: If $Y \subseteq K^k$, the ideal of polynomials vanishing at Y

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Definition

A group G is called *equationally Noetherian* if for every $k \in \mathbb{N}$ and every subset $S \subseteq G * F_k$ there exists a finite subset $S_0 \subseteq S$ such that

$$V_G(S) = V_G(S_0).$$

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Analogy: Irreducible component G_0 of an algebraic group.

Equationally Noetherian groups

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6. *(Valiunas, 2024) Free-by-cyclic groups.*

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   Main reason: (Hall, 1954) If H is virtually polycyclic, then $\mathbb{Z}[H]$ is Noetherian.

Strong conciseness
and
equationally Noetherian groups

Strong conciseness and equationaly Noetherianity

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Theorem (IH, Zozaya)

Let G be a profinite group and let w be a word. Suppose that G has an equationally Noetherian subgroup that is dense in G with respect to the profinite topology. Then, $|G_w| < 2^{\aleph_0}$ implies $|G_w| < \infty$.

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Every word is strongly concise in the class of groups consisting of profinite completions of residually finite linear groups.

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We have:

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Question

Is every word strongly concise in the class of virtually abelian-by-nilpotent groups?

Our methods do not seem to apply to this case.

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What is the profinite version of virtually abelian-by-polycylic groups?

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Actually,

Proposition (IH, Zozaya)

A group is polyprocyclic if and only if it is the inverse limit of polycyclic groups of a given "cyclic length".

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Finitely generated virtually abelian-by-polycyclic groups are equationally Noetherian, so:

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Corollary (IH, Zozaya)

Every word is strongly concise in the class of groups consisting of profinite completions of finitely generated virtually abelian-by-polycyclic groups.

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If G is a procyclic group, is $\mathbb{Z}[G]$ profinite-Noetherian (meaning that every closed ideal is topologically finitely generated).

Eskerrik asko!!

Grazie mille!!