The Model Theory of Groups

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Model Theory

Model theory studies mathematical structures from the point of view of their *definable sets*, i.e. the sets of realizations of some first-order formula. These allow negation (complement), *finite* conjunction (intersection) and disjunction (union), existential quantification (projections) and universal quantification.

Examples

- $C_G(g) = \{x \in G : xg = gx\}$, the centraliser of $g \in G$.
- $Z(G) = \{x \in G : \forall y\, xy = yx\}$, the centre of G.
- $G[n] = \{x \in G : x \cdots_n x = 1\}$, the elements of order $\mid n$.

Non-examples

- $\langle g \rangle$, the subgroup generated by g.
- $\mathrm{tor}(G)$, the torsion part of G.
- $\mathbb{Z}G$, the group ring over $\mathbb{Z}$.

Theories and Morphisms

Rather than analysing a single structure, model theory usually studies classes of structures given by complete or incomplete *theories*, i.e. sets of first order sentences. A particular case are group varieties which are given by positive quantifier-free sentences.

Similarly, substructures and morphisms are *elementary*, i.e. they preserve not only the positive quantifier-free structure, but all first-order sentences with parameters.

Examples

- $(\mathbb{Q}, +) \prec (\mathbb{R}, +)$.
- $(\mathbb{Z}, +) \prec (\mathbb{Z} \oplus \mathbb{Q}, +)$.
- $(2\mathbb{Z}, +) \not\prec (\mathbb{Z}, +)$.

Language

The available fist-order formulas depend, of course, on the *language* $\mathcal{L}$ used. If $\mathcal{L} = \{1, \times, =\}$ is reduced to the group language, we are dealing with *pure* groups. However, one often considers groups in expanded languages, for instance the ring language $\mathcal{L} = \{0, 1, +, -, \times, =\}$, or the R-module language $\mathcal{L} = \{0, +, =, \lambda_r : r \in R\}$, where R is a ring and λ_r is scalar multiplication by $r \in R$.

One might also add a predicate for certain, otherwise undefinable, subgroups, or for specific subsets.

Thus, the model theory of groups encompasses the model theory of rings, fields, modules, algebras, etc.

Given a structure $\mathfrak{M}$, its *theory* $\mathrm{Th}(\mathfrak{M})$ is the collection of all $\mathcal{L}$-sentences true in $\mathfrak{M}$.

Conversely, a structure $\mathfrak{M}$ satisfying all sentences of a theory T is a *model* of T.

Ultraproducts

Recall that if I is an infinite index set, a collection $\mathcal{F}$ of subsets of I is a *filter* if it is closed under intersection and superset and does not contain $\emptyset$. It is an *ultrafilter* if it is maximal, i.e. contains either X or $I \setminus X$ for any $X \subseteq I$.

The axiom of choice implies that any filter can be completed to an ultrafilter.

Examples

- A *principal* ultrafilter $\{X \subseteq I : i_0 \in X\}$ for some $i_0 \in I$.
- The *Fréchet* filter $\{X \subseteq I : I \setminus X \text{ finite}\}$.
- More generally, $\{X \subseteq I : |I \setminus X| < |I|\}$.

Łos' Theorem

Let $\mathcal{U}$ be an ultrafilter on I, and suppose for each $i \in I$ we are given an $\mathcal{L}$-structure $\mathfrak{M}_i$. On the product $\prod_I \mathfrak{M}_i$ define an equivalence relation

$$(m_i)_I \sim (n_i)_I \Leftrightarrow \{i \in I : m_i = n_i\} \in \mathcal{U}.$$

Then the *ultraproduct* $\prod_{\mathcal{U}} \mathfrak{M}_i = \prod_I \mathfrak{M}_i/\sim$ canonically carries an $\mathcal{L}$-structure, and for any formula $\varphi(\bar{x})$ and parameters $\bar{m} = (\bar{m}_i)_I$

$$\prod_{\mathcal{U}} \mathfrak{M}_i \models \varphi(\bar{m}) \Leftrightarrow \{i \in I : \mathfrak{M}_i \models \varphi(\bar{m}_i)\} \in \mathcal{U}.$$

If $\mathfrak{M}_i = \mathfrak{M}$ for all $i \in I$, the canonical diagonal embedding $\mathfrak{M} \hookrightarrow \prod_{\mathcal{U}} \mathfrak{M}$ is elementary.

If $\mathrm{Th}(\mathfrak{M}_i) = T$ for all $i \in I$, then $\mathrm{Th}(\prod_{\mathcal{U}} \mathfrak{M}_i) = T$.

Ultraproducts are a good tool to study asymptotic behaviour.

Compactness

The *Compactness Theorem* is the most fundamental tool in model theory.

Theorem

Suppose Φ is a collection of sentences such that any finite part has a model. Then Φ has a model.

Proof.

Let I be the set of finite parts of Φ, and for $i \in I$ let $\mathfrak{M}_i$ be a model of i. For $i \in I$ put

$$I_i = \{j \in I : i \subseteq j\} \quad \text{and} \quad \mathcal{F} = \{X \subseteq I : \exists i \in I\ I_i \subseteq X\}.$$

Then $\mathcal{F}$ is a filter; let $\mathcal{U}$ be a completion to an ultrafilter. Then by Łos' Theorem $\prod_{\mathcal{U}} \mathfrak{M}_i$ is a model of Φ. □

Consequences

Corollary (Automatic uniformity)

- *Any object proven to be finite for all models of a theory is* uniformly *finite.*
- *Any object proven to be definable for all models of a theory is* uniformly *definable.*

In particular:

- There is no axiomatisation of all finite groups.
- There is no axiomatisation of all torsion groups.
- There is no axiomatisation of all nilpotent or all soluble groups.

Quantifier Elimination

Definition

Let $\mathcal{C}$ be a class of formulas (quantifier-free, existential, $\forall\exists$,...). A theory T has *$\mathcal{C}$-elimination of quantifiers* if every formula is equivalent modulo T to a formula in $\mathcal{C}$.

If we understand well sets defined by a formula in $\mathcal{C}$, then $\mathcal{C}$-elimination of quantifiers yields a good comprehension of all definable sets.

Examples

- R-modules eliminate down to boolean combinations of positive-primitive formulas (particular existential formulas).
- Algebraically closed fields eliminate quantifiers.
- Real closed fields have existential elimination of quantifiers.
- Non-abelian free groups eliminate down to boolean combinations of $\forall\exists$-formulas.

Tameness

Of course, it is impossible to prove anything meaningful about the class of *all* groups.

One can study particular theories:

- Abelian groups, or more generally R-modules.
- Various theories of fields with operators (algebraically closed, real closed, separably closed, differentially closed, generic difference fields, Henselian valued fields), and groups definable in those structures.
- Non-abelian free groups.

The aim usually is to prove some quantifier-elimination result in a suitable language, and to deduce the properties of the class of definable sets.

Alternatively, we can add some *tameness* condition on the class of definable sets.

Tameness conditions

- **Minimality.** An infinite structure is *minimal* if every definable subset in one variable is finite or co-finite.
- **Categoricity.** A theory is κ-*categorical* if all its models of cardinality κ are isomorphic.
- **Pseudofiniteness.** A structure is *pseudofinite* if it has the same theory as some ultraproduct of finite structures.
- ***o*-minimality.** A totally ordered structure is *o*-minimal if every definable subset in one variable is a finite union of intervals and points.
- **Combinatorial tameness.** There are a variety of conditions, each characterized by the exclusion of certain configurations of definable sets.

All of the specific theories studied in the previous slide also satisfy one of the above tameness conditions.

Minimality

If G is minimal and $H \leq G$ infinite definable, then $H = G$.

Theorem (Reinecke)

A minimal group is abelian.

Proof.

The centralizer of any non-central element g must be finite by minimality, so the conjugacy class g^G is infinite, whence co-finite. Thus all non-central elements are conjugate, and $G/Z(G)$ has just one non-trivial conjugacy class. $G/Z(G)$ is not abelian, so $gZ(G)$ cannot have order two. Hence there is h with $g^h = g^{-1}$, and $C_G(h^2) > C_G(h)$. But $hZ(G)$ does not have order two, so h and h^2 are conjugate, contradicting finiteness of $C_G(h)$. □

This theorem and its generalizations provide most of the abelian subgroups in model-theoretic group theory.

Conjecture (Podewski)

A minimal field is algebraically closed.

Theorem

A minimal field K of positive characteristic is algebraically closed.

Proof.

The maps $x \mapsto x^n$ and $x \mapsto x^p - x$ have finite kernel, whence infinite image. It follows that K has no cyclic extension, and the subfield K^{alg} of absolutely algebraic numbers is algebraically closed. But it is easy to see that $K^{alg} \prec K$, so K is algebraically closed. □

Definition

A structure $\mathfrak{M}$ is *strongly minimal* if all its elementary extensions are minimal.

Theorem (Macintyre)

A strongly minimal field is algebraically closed.

$\aleph_0$-categoricity

Theorem (Ryll-Nardzewski, Svenonius, Engeler)

A theory is $\aleph_0$-categorical iff for every n there are only finitely many inequivalent formulas in n free variables.

Thus an $\aleph_0$-categorical group has a finite characteristic definable series with characteristically simple quotients.

Theorem (Wilson)

An infinite countable $\aleph_0$-categorical characteristically simple group is

(i) *an elementary abelian p-group, for some prime p, or*

(ii) *$B(F)$ or $B^-(F)$ for some non-abelian finite simple group F, where $B(F) = \mathcal{C}^0(C, F)$ (C the Cantor space), and $B^-(F) = \{f \in B(F) : f(x_0) = e\}$ for a fixed $x_0 \in C$, or*

(iii) *a perfect p-group for some prime number p.*

Wilson conjectured that Case (iii) does not exist; this is open.

Meta-Conjecture

- *A tame $\aleph_0$-categorical group or ring is virtually nilpotent.*
- *A supertame $\aleph_0$-categorical group is virtually finite-by-abelian; a supertame $\aleph_0$-categorical ring is virtually finite-by-null.*

Recall that a group/ring is *virtually P* if it has a finite index subgroup/-ring which is P; it is *finite-by-P* if it has a finite normal subgroup/ideal I such that it is P modulo I; a ring is *null* if multiplication is trivial.

Of course, one has to specify the precise meaning of tame.

This has been shown for various notions of tame by Felgner, Baldwin-Rose, Baur-Cherlin-Macintyre, Evans-W, Macpherson, Krupiński, Kaplan-Levi-Simon and Dobrowolski-W.

Note that extraspecial p-groups yield an example showing that the finite normal subgroup cannot be avoided in general.

$\aleph_1$-categoricity

Uncountable categoricity is very different from $\aleph_0$-categoricity.

Strongly minimal structures are $\aleph_1$-categorical.

Theorem (Morley)

A countable theory is $\aleph_1$-categorical iff it is κ-categorical for some/any uncountable cardinal κ.

Moreover, there is a notion of dimension, called Morley rank.

Theorem (Macintyre, Cherlin, Shelah)

A division ring with Morley rank is an algebraically closed field.

Algebraicity Conjecture (Cherlin, Zilber)

An $\aleph_1$-categorical simple group is (definably) an algebraic group over an algebraically closed field.

This conjecture led to the development of the theory of groups of finite Morley rank.

Theorem (Borovik, Poizat)

In a group with Morley rank, maximal 2-subgroups are nilpotent-by finite and conjugate.

Dependening on the Sylow-2-subgroups, there are four cases:

- **degenerate** They are finite.
- **even** They are infinite of bounded exponent.
- **odd** They are infinite without infinite subgroups of exponent 2.
- **mixed** None of the above.

While initially the study of groups of finite Morley rank was inspired by algebraic group theory, Borovik formulated a programme to prove the Algebraicity Conjecture modelled on the CFSG.

The biggest obstacle is the absence of the notion of a complete variety (for the former), and of character theory (for the latter). In particular, there is no finite Morley rank version of the Feit-Thompson Theorem.

Theorem (Altınel, Borovik, Cherlin)

The algebraicity conjecture holds in the even case. No mixed case.

Despite initial progress, the odd case is still open.

It is believed that (non-algebraic) degenerate groups exist.

Attention has shifted to *permutation* groups of finite Morley rank.

Theorem

A split strictly 2-transitive group of finite Morley rank of characteristic $\neq 2$ is definably the group of affine transformations of an algebraically closed field.

Theorem (Borovik, Cherlin)

If G acts faithfully and definably primitively on a set X, the Morley rank of G is bounded in terms of the rank of X.

Conjecture (Borovik)

A connected group acting transitively and generically $(n+2)$-transitively on a set of Morley rank n is the projective group acting on projective space over an algebraically closed field.

Pseudofiniteness

Theorem (Ax)

A field F is pseudofinite if and only if all of the following hold:

- *F is perfect;*
- *F has a unique extension of each finite degree (quasifinite);*
- *F is pseudo-algebraically closed (every absolutely irreducible variety over F has an F-rational point).*

Moreover, $Th(F_1) = Th(F_2)$ (pseudofinite) iff $F_1^{alg} \cong F_2^{alg}$.

This is probably the first deep result in algebraic model theory.

Theorem (Wilson, Point)

A pseudofinite group is simple if and only if it is a simple group of Lie type (possibly twisted) over a pseudofinite field. Moreover

$$\prod_{\mathcal{U}} G(F_i) \cong G(\prod_{\mathcal{U}} F_i),$$

where G is a simple group of Lie type (possibly twisted) and the F_i are increasing finite fields.

In an ultraproduct of finite structures, the counting measure gives us a way to compare the size of definable sets:

Definition

Let $(\mathfrak{M}_i : i \in I)$ be finite structures, and $\mathfrak{M} = \prod_{\mathcal{U}} \mathfrak{M}_i$. For a definable set X in $\mathfrak{M}$ let X_i be the corresponding set in $\mathfrak{M}_i$ (defined by the same formula), similarly for Y. Put

$$|X| = \prod_{\mathcal{U}} |X_i| \in \prod_{\mathcal{U}} \mathbb{N} = \mathbb{N}^* \quad \text{and} \quad \mu_X(Y) = \mathrm{st}\Big(\frac{|Y|}{|X|}\Big) \in \mathbb{R}_+,$$

the pseudofinite counting measure relative to X.

Hrushovski has realized that the logarithm gives a notion of dimension:

$$d_X(Y) = \frac{\log |Y|}{\log |X|} + \bar{\mathbb{Z}} \in \mathbb{R}^*/\bar{\mathbb{Z}},$$

where $\bar{\mathbb{Z}}$ is the convex hull of $\mathbb{Z}$ in $\mathbb{R}^*$.

This has in particular been used to prove a Lie Model Theorem for approximate subgroups, leading to their asymptotic classification by Breuillard, Green and Tao.

Pseudofinite permutation groups of finite dimension

Theorem (Elwes, Jaligot, Macpherson, Ryten, Zou)

Let G be a pseudofinite definably primitive permutation group on X with $\dim(X) = 1$ and $\dim(G)$ finite.

- *If $\dim(G) = 1$ then G has a definable normal abelian subgroup A of dimension 1 acting regularly on X.*
- *If $\dim(G) = 2$ and definable sections satisfy the chain condition on centralizers up to finite index, then there is a definable subgroup $H \trianglelefteq G$ of dimension 2 isomorphic to $Aff(F)$ for some pseudofinite field F of dimension 1.*
- *If $\dim(G) \geq 3$, definable sections satisfy the chain condition on centralizers up to finite index, and X contains no infinite set of 1-dimensional equivalence classes, then $\dim(G) = 3$ and there is a definable subgroup $D \leq G$ of dimension 3 isomorphic to $PSL_2(F)$ for some pseudofinite field F of dimension 1.*

o-minimality

o-minimality is an essentially 1-dimensional notion. We shall hence consider groups definable in an *o*-minimal structure, i.e. whose domain and graph of multiplication are definable sets.

Theorem

An infinite field definable in an o-minimal structure is real closed or algebraically closed of characteristic 0*.*

Theorem (Wilkie, Macintyre, van den Dries)

The real field $\mathbb{R}$ *with exponentiation and all restricted analytic functions is o-minimal.*

There is much work on *o*-minimal expansions of the real field $\mathbb{R}$, with applications to diophantine geometry, in particular the Zilber-Pink conjecture, which states roughly that atypical or unlikely intersections of an algebraic variety with certain special varieties are accounted for by finitely many special varieties.

Groups in *o*-minimal structures

Theorem (Pillay)

A group G definable in an o-minimal structure carries a definable manifold structure making it into a topological group.

Corollary

- *If G is infinite, it has an infinite definable abelian sugbroup.*
- *The definably connected component of the identity, G^0, is the smallest subgroup of finite index and is normal.*
- *G has the descending chain condition on definable subgroups.*
- *A definable subgroup H is closed; moreover H is open iff H has finite index iff $\dim H = \dim G$.*

Groups of small dimension

Theorem (Razen)

Let G be definably connected 1-dimensional. Then G is abelian, and either G is torsion-free, or G is definably compact and the torsion subgroups $G[m]$ are isomorphic to $\mathbb{Z}/m\mathbb{Z}$ for each $m > 0$.

Theorem (Nesin, Pillay, Razen)

Let G be definably connected 2-dimensional. Then either G is abelian, or definably isomorphic to $Aff(R)$ for some real closed field R.

Theorem (Nesin, Pillay, Razen)

Let G be definably connected nonsolvable 3-dimensional. Then $G/Z(G)$ is definably isomorphic to either $PSL_2(R)$ or $SO_3(R)$ for some definable real closed field R.

o-minimal Algebraicity Conjecture

Theorem (Peterzil, Pillay, Starchenko)

Let G be an infinite definably simple (centreless) group. Then there is a real closed field R such that one and only one of the following holds:

- *G and the field $R(\sqrt{-1})$ are bi-interpretable, and G is definably isomorphic to $H(R(\sqrt{-1}))$, where H is a linear algebraic group defined over $R(\sqrt{-1})$.*
- *G and the field R are bi-interpretable, and G is definably isomorphic to the semialgebraic connected component of a group $H(R)$, where H is an R-simple algebraic group defined over R.*

Pillay's conjectures

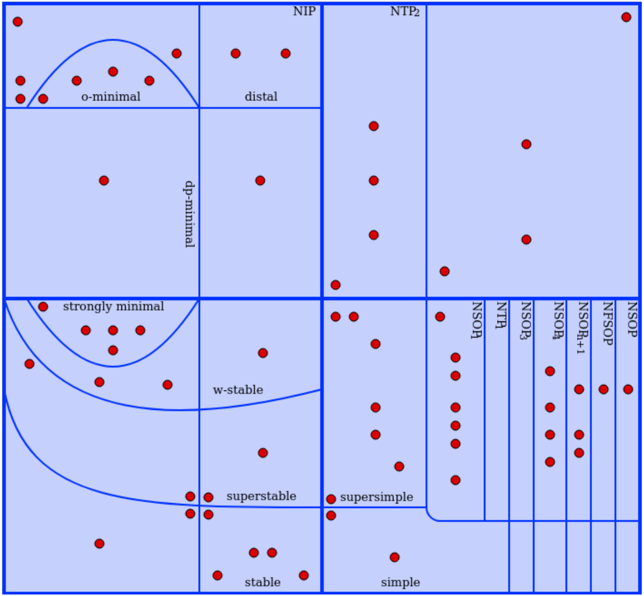
There is a close connection between groups definable in an *o*-minimal structure and Lie groups.

Theorem (Hrushovski, Peterzil, Pillay)

Let G be definably connected. Then:

- *G has a smallest type-definable subgroup of bounded index, G^{00}.*
- *G/G^{00} is a compact connected Lie group, when equipped with the logic topology.*
- *If G is definably compact then the Lie dimension of G/G^{00} is equal to the (o-minimal) dimension of G.*
- *If G is definably compact and abelian, then G^{00} is divisible and torsion-free.*

Combinatorial tameness (map by G. Conant)



Combinatorial tameness

There are a plethora of combinatorial tameness conditions. They are usually defined by prohibiting certain configurations.

For instance, a theory is NIP (not the independence property) if there is a formula $\varphi(x, y)$ such that for sufficiently large n no subset A of size n of some model $\mathfrak{M}$ has all of its subsets arise in the form $\{a \in A : \mathfrak{M} \models \varphi(a, b)\}$ for some b.

In the best of cases, this allows a notion of rank (Morley rank, Lascar rank, SU-rank,…), which may have finite or ordinal values, and properties resembling that of a dimension (supertameness).

In the second best case, this allows a (combinatorially defined) notion of independence. If there is a rank, a and b are independent over c if the rank of a over bc equals the rank of a over c.

But there are weaker tameness conditions which still have meaningful algebraic consequences.

Algebraic consequences

- In a NIP theory, every type-definable group has a smallest type-definable normal subgroup of bounded index, its *connected component*.
- If there is an ordinal valued rank, we have the descending chain condition on type-definable subgroups, up to bounded index.
- If there is a finitely valued rank, we have the ascending chain condition on type-definable subgroups, up to bounded index.
- In a NIP theory, every finite intersection of uniformy definable subgroups (defined by instances of the same formula) is equal to a subintersection of size depending only on the formula.
- Other consequences include the chain condition on centralizers, possibly up to finite index.

Aims

- Characterize definable fields.
- Characterize the $\aleph_0$-categorical groups.
- Characterize the simple groups.
- Prove the existence of infinite abelian subgroups.
- Show that abelian/nilpotent/soluble subgroups are always contained in definable ones.
- Show that type-definable groups/fields are intersections of definable ones.
- Prove the existence and definability of certain radicals (Fitting subgroup, soluble radical,...).
- Characterize groups and permutation groups of small rank.
- Develop a Sylow theory.

Thank you !