

THE UPPER AND LOWER CENTRAL SERIES IN LINEAR GROUPS

Università degli Studi



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degli Studi
della Campania
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The upper and lower central series in linear groups

Q.J. Math. **73** (2022), 261–275.

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Groups whose proper subgroups are linear

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Subnormality in linear groups

J. Pure Appl. Algebra (2023), no. 2, Paper No. 107185.

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Motivation and History

I. Schur (1902)

If G is a group such that $G/\zeta_1(G)$ is finite, then $G' = [G, G]$ is finite.

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Let n be a positive integer.

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P. Hall (1956)

Let n be a positive integer.

If G is a group such that $\gamma_{n+1}(G)$ is finite, then $G/\zeta_{2n}(G)$ is finite.

Find those natural classes of groups \mathfrak{X} such that the following statements hold for every group \mathbf{G} and $n \in \mathbb{N}$:

- 1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}$.
- 2) $\mathbf{G}/\zeta_n(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{n+1}(\mathbf{G}) \in \mathfrak{X}$.
- 3) $\gamma_{n+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_n(\mathbf{G}) \in \mathfrak{X}$.

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Finite	✓	✓

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Finite	✓	✓
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	<i>Baer</i>	<i>Hall</i>
Finite	✓	✓
Černikov	✓	✗
Polycyclic-by-finite	✓	✓

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Finite	✓	✓
Černikov	✓	✗
Polycyclic-by-finite	✓	✓
Residually finite + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	✗

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Finite	✓	✓
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Minimax + $\mathfrak{S}\mathfrak{F}$	✓	✗

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Finite	✓	✓
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Polycyclic-by-finite	✓	✓
Residually finite + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	✗
Minimax + $\mathfrak{S}\mathfrak{F}$	✓	✗
Reduced Minimax + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	?

	<i>Baer</i>	<i>Hall</i>
Finite	✓	✓
Černikov	✓	✗
Polycyclic-by-finite	✓	✓
Residually finite + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	✗
Minimax + $\mathfrak{S}\mathfrak{F}$	✓	✗
Reduced Minimax + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	?
Finite Rank + $\mathfrak{S}\mathfrak{F}$	✓	✗

Ju.I. Merzljakov (1967)

If G is a linear group and n is a non-negative integer, then $\gamma_{n+1}(G)$ is finite if and only if $G/\zeta_n(G)$ is finite.

	<i>Baer</i>	<i>Hall</i>	<i>Merzljakov</i>
Finite	✓	✓	✓
Černikov	✓	✗	
Polycyclic-by-finite	✓	✓	
Residually finite + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	✗	
Minimax + $\mathfrak{S}\mathfrak{F}$	✓	✗	
Reduced Minimax + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	?	
Finite Rank + $\mathfrak{S}\mathfrak{F}$	✓	✗	

	<i>Baer</i>	<i>Hall</i>	<i>Merzljakov</i>
Finite	✓	✓	✓
Černikov	✓	✗	✓
Polycyclic-by-finite	✓	✓	✓
Residually finite + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	✗	✗
Minimax + $\mathfrak{S}\mathfrak{F}$	✓	✗	✓
Reduced Minimax + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	?	✓
Finite Rank + $\mathfrak{S}\mathfrak{F}$	✓	✗	

	<i>Baer</i>	<i>Hall</i>	<i>Merzljakov</i>
Finite	✓	✓	✓
Černikov	✓	✗	✓
Polycyclic-by-finite	✓	✓	✓
Residually finite + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	✗	✗
Minimax + $\mathfrak{S}\mathfrak{F}$	✓	✗	✓
Reduced Minimax + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	?	✓
Finite Rank + $\mathfrak{S}\mathfrak{F}$	✓	✗	<i>char</i> > 0: ✓ <i>char</i> = 0: ✗

	<i>Baer</i>	<i>Hall</i>	<i>Merzljakov</i>
Finite	✓	✓	✓
Černikov	✓	✗	✓
Polycyclic-by-finite	✓	✓	✓
Residually finite + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	✗	✗
Minimax + $\mathfrak{S}\mathfrak{F}$	✓	✗	✓
Reduced Minimax + $\mathfrak{S}\mathfrak{F}$	<i>Schur</i> : ✗	✓	✓
Finite Rank + $\mathfrak{S}\mathfrak{F}$	✓	✗	<i>char</i> > 0: ✓ <i>char</i> = 0: ✗

The image features a central black rectangle with the words "Thank You" written in a white, elegant cursive font. This central element is surrounded by a complex, multi-layered border composed of numerous overlapping horizontal and vertical bars in a wide variety of colors, including shades of purple, yellow, blue, red, green, brown, and pink. The layers create a sense of depth and a vibrant, celebratory atmosphere.

Thank You