The Upper and Lower Central Series in Linear Groups

22 September

Università degli Studi di Napoli Federico II

Marco Trombetti

AGTA Workshop — Reinhold Baer Prize 2022

Università degli Studi della Campania *Luigi Vanvitelli*

F. de Giovanni • B.A.F. Wehrfritz • M.T.

The upper and lower central series in linear groups

Q.J. Math. **73** (2022), 261–275.

F. de Giovanni • B.A.F. Wehrfritz • M.T.

The upper and lower central series in linear groups

Q.J. Math. **73** (2022), 261–275.

Groups whose proper subgroups are linear

J. Algebra **592** (2022), 153–168.

F. de Giovanni • B.A.F. Wehrfritz • M.T.

The upper and lower central series in linear groups

Q.J. Math. **73** (2022), 261–275.

Groups whose proper subgroups are linear

J. Algebra **592** (2022), 153–168.

Subnormality in linear groups

J. Pure Appl. Algebra (2023), no. 2, Paper No. 107185.

F. de Giovanni • B.A.F. Wehrfritz • M.T.

The upper and lower central series in linear groups
Q.J. Math. **73** (2022), 261–275.

Groups whose proper subgroups are linear
J. Algebra **592** (2022), 153–168.

Subnormality in linear groups
J. Pure Appl. Algebra (2023), no. 2, Paper No. 107185.

• • •

I. Schur (1902)

If G *is a group such that* $G/\zeta_1(G)$ *is finite, then* $G' = [G, G]$ *is finite.*

I. Schur (1902)

If G is a group such that $G/\zeta_1(G)$ is finite, then $G' = [G, G]$ is finite.

R. Baer (1952)

Let n be a positive integer.
If G is a group such that $G/\zeta_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite.

I. Schur (1902)

If G *is a group such that* $G/\zeta_1(G)$ *is finite,*
then $G' = [G, G]$ *is finite.*

R. Baer (1952)

Let n *be a positive integer.*
If G *is a group such that* $G/\zeta_n(G)$ *is finite,*
then $\gamma_{n+1}(G)$ *is finite.*

P. Hall (1956)

Let n *be a positive integer.*
If G *is a group such that* $\gamma_{n+1}(G)$ *is finite,*
then $G/\zeta_{2n}(G)$*is finite.*

Find those natural classes of groups $\mathfrak{X}$ such that the following statements hold for every group $\mathbf{G}$ and $\mathbf{n} \in \mathbb{N}$:

1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}$.
2) $\mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X}$.
3) $\gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X}$.

1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}$.
2) $\mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X}$.
3) $\gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X}$.

	Baer	*Hall*
Finite	✓	✓

1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}.$

2) $\mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X}.$

3) $\gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X}.$

	Baer	*Hall*
Finite	✓	✓
Černikov		

1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}$.

2) $\mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X}$.

3) $\gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X}$.

	Baer	*Hall*
Finite	✓	✓
Černikov	✓	

1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}.$

2) $\mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X}.$

3) $\gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X}.$

	Baer	*Hall*
Finite	✓	✓
Černikov	✓	✗

1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}$.

2) $\mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X}$.

3) $\gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X}$.

	Baer	*Hall*
Finite	✓	✓
Černikov	✓	✗
Polycyclic-by-finite	✓	✓

1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}$.

2) $\mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X}$.

3) $\gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X}$.

	Baer	*Hall*
Finite	✓	✓
Černikov	✓	✗
Polycyclic-by-finite	✓	✓
Residually finite + $\mathfrak{S}\mathfrak{F}$	*Schur*: ✗	✗

1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}$.

2) $\mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X}$.

3) $\gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X}$.

	Baer	*Hall*
Finite	✓	✓
Černikov	✓	✗
Polycyclic-by-finite	✓	✓
Residually finite + $\mathfrak{S}\mathfrak{F}$	*Schur*: ✗	✗
Minimax + $\mathfrak{S}\mathfrak{F}$	✓	✗

1) $\mathbf{G}/\zeta_1(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}' \in \mathfrak{X}$.

2) $\mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X} \implies \gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X}$.

3) $\gamma_{\mathbf{n}+1}(\mathbf{G}) \in \mathfrak{X} \implies \mathbf{G}/\zeta_{\mathbf{n}}(\mathbf{G}) \in \mathfrak{X}$.

	Baer	*Hall*
Finite	✓	✓
Černikov	✓	✗
Polycyclic-by-finite	✓	✓
Residually finite + $\mathfrak{SF}$	*Schur*: ✗	✗
Minimax + $\mathfrak{SF}$	✓	✗
Reduced Minimax + $\mathfrak{SF}$	*Schur*: ✗	**?**

	Baer	*Hall*
Finite	✓	✓
Černikov	✓	✗
Polycyclic-by-finite	✓	✓
Residually finite + $\mathfrak{SF}$	*Schur*: ✗	✗
Minimax + $\mathfrak{SF}$	✓	✗
Reduced Minimax + $\mathfrak{SF}$	*Schur*: ✗	**?**
Finite Rank + $\mathfrak{SF}$	✓	✗

Ju.I. Merzljakov (1967)

If G is a linear group and n is a non-negative integer, then $\gamma_{n+1}(G)$ is finite if and only if $G/\zeta_n(G)$ is finite.

	Baer	*Hall*	*Merzljakov*
Finite	✓	✓	✓
Černikov	✓	✗	
Polycyclic-by-finite	✓	✓	
Residually finite + $\mathfrak{SF}$	*Schur*: ✗	✗	
Minimax + $\mathfrak{SF}$	✓	✗	
Reduced Minimax + $\mathfrak{SF}$	*Schur*: ✗	**?**	
Finite Rank + $\mathfrak{SF}$	✓	✗	

	Baer	*Hall*	*Merzljakov*
Finite	✓	✓	✓
Černikov	✓	✗	✓
Polycyclic-by-finite	✓	✓	✓
Residually finite + $\mathfrak{SF}$	*Schur*: ✗	✗	✗
Minimax + $\mathfrak{SF}$	✓	✗	✓
Reduced Minimax + $\mathfrak{SF}$	*Schur*: ✗	**?**	✓
Finite Rank + $\mathfrak{SF}$	✓	✗	

	Baer	*Hall*	*Merzljakov*
Finite	✓	✓	✓
Černikov	✓	✗	✓
Polycyclic-by-finite	✓	✓	✓
Residually finite + $\mathfrak{SF}$	*Schur*: ✗	✗	✗
Minimax + $\mathfrak{SF}$	✓	✗	✓
Reduced Minimax + $\mathfrak{SF}$	*Schur*: ✗	**?**	✓
Finite Rank + $\mathfrak{SF}$	✓	✗	$char > 0$: ✓ $char = 0$: ✗

Our Results

	Baer	*Hall*	*Merzljakov*
Finite	✓	✓	✓
Černikov	✓	✗	✓
Polycyclic-by-finite	✓	✓	✓
Residually finite + $\mathfrak{SF}$	*Schur*: ✗	✗	✗
Minimax + $\mathfrak{SF}$	✓	✗	✓
Reduced Minimax + $\mathfrak{SF}$	*Schur*: ✗	✓	✓
Finite Rank + $\mathfrak{SF}$	✓	✗	$char > 0$: ✓ $char = 0$: ✗

Thank You