Semi-braces, affine structures on groups, and set-theoretic solutions of the Yang-Baxter equation

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AGTA Workshop
Reinhold Baer Prize 2022

Caserta, 21-23 September 2022

Aims

This talk aims to:

- Introduce the algebraic structure of the *semi-brace* and its connection with set-theoretic solutions of the Yang-Baxter equation;
- Introduce *affine structures on groups* as a new method for constructing semi-braces.

Set-theoretic solutions of the Yang-Baxter equation

The quantum Yang-Baxter equation is a basic equation of the statistical mechanics that arose from Yang's work in 1967 and Baxter's one in 1972.

In 1992, Drinfel'd suggested to study the special class of set-theoretic solutions.

G. Drinfel'd, *On some unsolved problems in quantum group theory, in: Quantum Groups*, Leningrad, 1990, in: Lecture Notes in Math. vol.**1510**(2) Springer, Berlin,(1992), 1–8.

If B is a non-empty set, a *set-theoretic solution* of the Yang-Baxter equation is a map $r : B \times B \to B \times B$ that satisfies the *braid equation*, i.e.,

$$(r \times \mathrm{id}_B)(\mathrm{id}_B \times r)(r \times \mathrm{id}_B) = (\mathrm{id}_B \times r)(r \times \mathrm{id}_B)(\mathrm{id}_B \times r).$$

Open question

Determine all the set-theoretic solutions of the Yang-Baxter equation.

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Notation and terminology

Hereinafter, we will shortly call *solution* any set-theoretic solution to the Yang-Baxter equation. Moreover, if r is a solution on B, for all $a, b \in B$, we define the maps $\lambda_a, \rho_b : B \to B$ and write the map r as

$$r(a, b) = (\lambda_a(b), \rho_b(a)).$$

A solution r is said to be

- *left non-degenerate* if λ_a is bijective, for every $a \in B$.
- *right non-degenerate* if ρ_b is bijective, for every $b \in B$.
- *non-degenerate* if r is both left and right non-degenerate.
- *involutive* if $r^2 = \mathrm{id}_{B \times B}$.
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A novel approach

In 2007, W. Rump traced a novel research direction in the study of solutions.

We recall that if $(B, +, \cdot)$ is a ring and $\circ$ the *adjoint operation* on B, i.e.,

$$\forall\ a, b \in B \quad a \circ b = a + b + a \cdot b,$$

then B is said to be *Jacobson radical* if $(B, \circ)$ is a group (with identity 0).

Rump's unexpected observation (2007)

Any Jacobson radical ring $(B, +, \cdot)$ determines a solution r on B that is the map $r : B \times B \to B \times B$ defined by

$$r(a, b) := \left(\lambda_a(b),\ \lambda^{-1}_{\lambda_a(b)}(a)\right)$$

where $\lambda_a(b) := a \cdot b + b$, for all $a, b \in B$. In particular, r is non-degenerate and involutive, i.e., $r^2 = \mathrm{id}_{B \times B}$.

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"More fitting" structures for solutions

Bijective solutions can be produced through *skew braces*.

Definition (Rump - 2007; Guarnieri, Vendramin - 2017)

A triple $(B, +, \circ)$ is said to be a *skew brace* if $(B, +)$ and $(B, \circ)$ are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds, for all $a, b, c \in B$. If $(B, +)$ is abelian then B is a *brace*.

The groups $(B, +)$ and $(B, \circ)$ have the same identity that we denote by 0.

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Solutions associated to skew braces

Theorem (Rump - 2007; Guarnieri, Vendramin - 2017)

If B is a skew brace, then the map $r_B : B \times B \to B \times B$ defined by

$$r_B(a,\ b) := \left(-a + a \circ b,\ (-a + a \circ b)^{-} \circ a \circ b\right)$$

is a non-degenerate bijective solution.

Remark: The solution associated to a skew brace $(B, +, \circ)$

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Theorem (Smoktunowicz, Vendramin - 2018)

If B is a finite skew brace, then the solution associated to B is such that

$$r_B^{2n} = \mathrm{id}_B$$

where $n \in \mathbb{N}$ is the exponent of the additive quotient group $B/Z(B)$.

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Semi-braces

What happens if $(B,+)$ is a left cancellative semigroup?

Definition (Catino, Colazzo, S. - 2017)

A triple $(B,+,\circ)$ is said to be a *cancellative semi-brace* if $(B,+)$ is a left cancellative semigroup, $(B,\circ)$ is a group and

$$a\circ(b+c)=a\circ b+a\circ\left(a^{-}+c\right)$$

holds, for all $a,b,c\in B$, where a^{-} is the inverse of a with respect to $\circ$.

Denoted by 0 the identity of $(B,\circ)$, then 0 is a left identity of $(B,+)$.

- Skew braces are cancellative semi-braces;
- If $(B,\circ)$ is a group and f is an endomorphism of $(B,\circ)$ such that $f^2=f$, by setting $a+b:=b\circ f(a)$, $(B,+,\circ)$ is a cancellative semi-brace.

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for all $a, b \in B$, is a left non-degenerate solution.

Remark: $\forall\ b \in B \quad \rho_b$ is bijective $\iff$ $(B, +, \circ)$ is a skew brace.

If $(B, +, \circ)$ is the *trivial semi-brace* on a group $(B, \circ)$, i.e., $a + b := b$, for all $a, b \in B$, then the solution associated to B is

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Other semi-braces

Under mild assumptions, the more general structure of *semi-brace* gives rise to degenerate solutions.

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Constructions of (semi-)braces

Until now, many authors have been provided several methods for constructing semi-braces starting from groups and by using techniques that are usual in group theory (e.g., *semidirect product*, *matched product*, etc.).

In 2019, Rump provided a description of all braces in terms of *affine structures on groups*.

Definition (Rump - 2020)

Let $G = (B, \circ)$ be a group and $\sigma : B \to \mathrm{Sym}_B$ an anti-homomorphism from G to Sym_B. Then, σ is said to be an *affine structure* on G if the following identity holds:

$$\forall\ a, b \in B \quad a \circ \sigma_a(b) = b \circ \sigma_b(a).$$

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In 2019, Rump provided a description of all braces in terms of *affine structures on groups*.

Definition (Rump - 2020)

Let $G = (B, \circ)$ be a group and $\sigma : B \to \mathrm{Sym}_B$ an anti-homomorphism from G to Sym_B. Then, σ is said to be an *affine structure* on G if the following identity holds:

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Correspondence between affine structures and braces

Theorem (Rump - 2019)

There exists a one-to-one correspondence between affine structures and braces.

- If $(B, +, \circ)$ is a brace, set $\sigma_a := \lambda_a^{-1}$, then we obtain an affine structure $\sigma : (B, \circ) \to \mathrm{Sym}_B$ on the group $G = (B, \circ)$.
- If σ is an affine structure on a group $G = (B, \circ)$, considered the operation

$$\forall\, a, b \in B \quad a + b := a \circ \sigma_a(b),$$

we have that $(B, +)$ is an abelian group and $(B, +, \circ)$ is a brace.

Affine structures has led to new advances in the brace theory:

- In 2019, Rump provided constructions of finite braces by proving that all affine structures can be obtained by those on p, q - groups.
- In 2020, Rump obtained instances of braces in terms of affine structures of decomposable solvable groups.

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Affine structures

Can we deduce a description of semi-braces with a suitable notion of affine structure?

Definition (S. - Preprint 2022)

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In particular, if $\sigma\left(B\right) \subseteq \mathrm{Sym}_B$ we say that σ is a *cancellative affine structure*. If in addition, $\sigma_a\left(0\right) = 0$, for every $a \in B$, we say that σ is a *groupal affine structure*.

Affine structures introduced by Rump are special cases of groupal affine structures. For our purposes, we call them *abelian affine structures.*

There are no intermediate affine structures between the general case and the cancellative one.

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Affine structures and semi-braces - I

Theorem (S. - Preprint 2022)

Let $G = (B, \circ)$ be a group, $\mathcal{S}$ the class of semi-braces having G as multiplicative group, and $\mathcal{A}$ the class of affine structures on G. Then, the map

$$\psi : \mathcal{S} \to \mathcal{A}, \ B \mapsto \sigma^B$$

is a bijection, where $\sigma^B_a(b) := \lambda_{a^-}(b)$, for all $a, b \in B$.

In particular, if σ is an affine structure on the group $G = (B, \circ)$, considered the operation

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Affine structures and semi-braces - II

Definition

Let $G = (B, \circ), H = (C, \circ)$ be groups and σ, φ affine structures on G and H, respectively. If there exists a homomorphism f from G to H such that

$$\forall\, a \in B \qquad \varphi_{f(a)} f = f \sigma_a$$

we say that σ and φ are *homomorphic via f*. If in addition f is bijective, we say that σ and φ are *equivalent via f*.

Let us denote by $\mathcal{A}ff$ the *category of affine structures* and by $\mathcal{SB}$ the *category of semi-braces*.

Theorem (S. - Preprint 2022)

The categories $\mathcal{A}ff$ and $\mathcal{SB}$ are equivalent.

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A construction of affine structures

To obtain semi-braces, we can deal with directly affine structures.

Theorem (S. - Preprint 2022)

Let $G = (B, \circ)$ be a group and $\varphi, \omega : B \to B^B$ affine structures on G. If the following conditions are satisfied

$$\varphi_a \omega_b = \omega_b \varphi_a \tag{c1}$$

$$\varphi_{b \circ \omega_a(b)^-} = \omega_{\varphi_a \omega_a(b) \circ \omega_a(b)^-}, \tag{c2}$$

for all $a, b \in B$, then the map $\sigma : B \to B^B$ given by

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The matched product system of groups

Definition

Let $H := (S, \circ)$, $K := (T, \circ)$ be groups, $\alpha : H \to \mathrm{Sym}_S$ and $\beta : K \to \mathrm{Sym}_T$ homomorphisms

$$\alpha_u\left(\alpha_u^{-1}(a) \circ b\right) = a \circ \alpha_{\beta_a^{-1}(u)}(b)$$
$$\beta_a\left(\beta_a^{-1}(u) \circ v\right) = u \circ \beta_{\alpha_u^{-1}(a)}(v)$$

are satisfied, for all $a, b \in S$ and $u, v \in T$. Let us call the quadruple (H, K, α, β) a *matched product system of groups*.

The binary operation given by

$$(a, u) \circ (b, v) := \left(a \circ \alpha_{\beta_a^{-1}(u)}(b),\ u \circ \beta_{\alpha_u^{-1}(a)}(v)\right),$$

for all $(a, u), (b, v) \in S \times T$, makes $S \times T$ into a group which we denote by $H \bowtie_{\alpha,\beta} K$.

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Affine structures on the matched product of groups

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Let (H, K, α, β) be a matched product system of groups and σ^S and σ^T affine structures on the groups $H = (S, \circ)$, $K = (T, \circ)$, respectively. If the following conditions

$$\sigma_0 \alpha_u = \alpha_u \sigma_0 \qquad \sigma_0 \beta_a = \beta_a \sigma_0 \tag{I}$$

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$$\alpha_{\bar{v}} \sigma_{\rho_b(a^-)} = \sigma_{\rho_b(a^-)} \alpha_{\overline{V}} \qquad \beta_{\bar{b}} \sigma_{\rho_v(u^-)} = \sigma_{\rho_v(u^-)} \beta_{\overline{B}} \tag{III}$$

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$$\sigma := \sigma^S \times \sigma^T$$

is an affine structure on the group $H \bowtie_{\alpha,\beta} K$.

In particular, the sum of the semi-brace B obtained through σ is given by

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Thanks for your attention!

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Bibliography

F. Catino, I. Colazzo, P.S., *Semi-braces and the Yang-Baxter equation*, J. Algebra **483** (2017) 163–187.

L. Guarnieri, L. Vendramin, *Skew braces and the Yang-Baxter equation*, Math. Comput. **86**(307) (2017), 2519–2534.

E. Jespers, A. Van Antwerpen, *Left semi-braces and solutions of the Yang-Baxter equation*, Forum Math. **21**(1) (2019), 241–263.

W. Rump, *Braces, radical rings, and the quantum Yang-Baxter equation*, J. Algebra **307**(1) (2007) 153–170.

W. Rump, *Construction of finite braces,*, Ann. Comb. **23** (2) (2019) 391–416.

W. Rump, *Affine structures of decomposable solvable groups*, J. Algebra **556** (2020) 725–749.

W. Rump, *Classification of the affine structures of a generalized quaternion group of order* ≥ 32, J. Group Theory **23** (2020) 847–869.

P.S., *Affine structures on groups and semi-braces*, Preprint ArXiv:2207.05000 (2022)