Right nilpotency for skew left braces: an approach by means of a Jordan-Hölder like theorem

Vicent Pérez Calabuig

with Adolfo Ballester Bolinches and Ramon Esteban Romero

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Skew left braces and YBE

Set-theoretic solution

A set-theoretic solution of the Yang–Baxter equation is a pair (X, r), where X is a set and

$$r\colon X \times X \to X \times X, \quad r(x, y) = (\sigma_x(y), \tau_y(x)), \quad x, y \in X$$

is a bijective map such that
$(r \times id)(id \times r)(r \times id) = (id \times r)(r \times id)(id \times r)$.

(X, r) is said to be *non-degenerate*, if σ_x and τ_x are bijective, for all $x \in X$.

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Skew left braces and YBE

L. Guarnieri and L. Vendramin (2017), *Skew left braces and the Yang-Baxter equation*, Math. Comput., 86(307), 2519–2534.

Every non-degenerate set-theoretic solution of the Yang-Baxter equation leads to a skew left brace, and, viceversa, every skew left brace provides a non-degenerate set-theoretic solution of the YBE.

Definition

A *skew left brace* $(B, +, \cdot)$ is defined to be a set B endowed with two group structures $(B, +)$ (*the additive group*) and $(B, \cdot)$ (*the multiplicative group*) satisfying the following property:

$$a \cdot (b + c) = a \cdot b - a + a \cdot c, \quad \text{for every } a, b, c \in B. \tag{1}$$

Skew left braces and YBE

- A set-theoretic solution of the Yang–Baxter equation is said to be *involutive*, if $r^2 = id_{X \times X}$.
- Non-degenerate involutive solutions are associated with skew left braces $(B, +, \cdot)$ such that $(B, +)$ is abelian: *left braces*

W. Rump (2007), *Braces, radical rings and the quantum Yang–Baxter equation*, J. Algebra, 307, 153–170.

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Multipermutation solutions

P. Etingof, T. Schedler and A. Soloviev (1999), *Set theoretical solutions to the Yang-Baxter Equation*, Duke Math. J, 100(2), 169–209.

Let (X, r) is an involutive non-degenerate solution of the YBE, with $r(x, y) = (\sigma_x(y), \tau_y(x))$, for all $x, y \in X$.

- the *retraction relation* $\sim$ on X: $x \sim y$ if $\sigma_x = \sigma_y$.
- the *retraction solution* $\mathrm{Ret}(X, r) = (X/\sim, \bar{r})$ is defined by

$$\bar{r}([x], [y]) = ([\sigma_x(y)], [\tau_y(x)]), \quad \text{for all } [x], [y] \in X/\sim$$

- Then, we can iterate this process and define inductively

$$\mathrm{Ret}^1(X, r) = \mathrm{Ret}(X, r),$$
$$\mathrm{Ret}^n(X, r) = \mathrm{Ret}(\mathrm{Ret}^{n-1}(X, r)), \quad \text{for all } n > 1$$

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Multipermutation solutions

Definition

An involutive non-degenerate solution (X, r) is said to be a *multipermutation solution of level m*, if m is the smallest natural such that $\mathrm{Ret}^m(X, r)$ has cardinality 1.

Right nilpotency (Rump, 2007): characterises multipermutation solutions in the involutive non-degenerate case associated with left braces

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Non-involutive case

V. Lebed and L. Vendramin (2019), *On structure groups of set-theoretic solutions to the Yang-Baxter equation*, Proc. Edinb. Math. Soc., 62, 683–717.

Let (X, r) is a non-degenerate solution of the YBE, with $r(x, y) = (\sigma_x(y), \tau_y(x))$, for all $x, y \in X$.

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- $\mathrm{Ret}^{n+1}(X, r) = \mathrm{Ret}(\mathrm{Ret}^n(X, r))$, for all $n \geq 1$.

Multipermutation solutions and right nilpotency

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General case: Is it possible to characterise multipermutation solutions for the general non-degenerate case by means of right nilpotency?

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Preliminaries

Throughout this presentation B will denote an arbitrary finite skew left brace

Given $\mathfrak{X}$ a class of groups, B is said to be of *of $\mathfrak{X}$-type* if $(B, +)$ belongs to $\mathfrak{X}$.

Rump's left braces are skew left braces of abelian type.

- We denote by

$$\lambda\colon (B, \cdot) \to \operatorname{Aut}(B, +), \quad a \mapsto \lambda_a,$$

  with $\lambda_a(b) = -a + ab$, for all $a, b \in B$.

- We will also consider the *star operation* $*\colon B \times B \to B$, defined as

$$a * b = \lambda_a(b) - b, \quad \text{for every } a, b \in B$$

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Ideals

- If X and Y are subsets of B,

$$X * Y := \langle x * y : x \in X, y \in Y \rangle_+$$

The star operation can be considered as a commutator in the semidirect product $G = [K]_\lambda C$, where $K = (B, +)$ and $C = (B, \cdot)$.

Definition

A non-empty subset I of B is a *left ideal*, if $(I, +)$ is a subgroup of $(B, +)$ and $B * I \subseteq I$, or equivalently $\lambda_b(I) \subseteq I$, for every $b \in B$.

A left ideal I is an *ideal* if $(I, +)$ is a normal subgroup and $I * B \subseteq I$ also holds.

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Ideals

Proposition (Guarnieri, Vendramin)

Let B be a skew left brace and let I be an ideal of B. Then

1. $(I, \cdot)$ *is a normal subgroup of* $(B, \cdot)$.
2. $bI = b + I$, *for every* $b \in B$.
3. I *is a subbrace of* B *and* B/I *is also a skew left brace.*

Example

- $\mathrm{Fix}(B) = \{a \in B \mid \lambda_b(a) = a, \text{ for every } b \in B\}$ is a left ideal.
- $\mathrm{Soc}(B) = \ker \lambda \cap Z(B, +)$ is an ideal of B called the *socle* of B.

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Definition

Let I, J be ideals of B with $J \leq I$. We say that the quotient I/J is

- a *chief factor*, if I/J is a minimal ideal of B/J.
- an *s-factor*, if $I/J \subseteq \mathrm{Soc}(B/J)$.
- an *f-factor*, if $I/J \subseteq \mathrm{Fix}(B/J)$.

An increasing sequence of subsets $0 = I_1 \leq I_2 \leq \ldots \leq I_n = B$, such that I_i is an ideal of B is called an *ideal series* of B

We call an ideal series of B

$$0 = I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n = B,$$

a *chief series*, an *s-series* or an *f-series*, if each factor I_{k+1}/I_k is, respectively, a chief factor, an s-factor or an f-factor.

Jordan-Hölder theorem

Theorem

For any two chief series of B,

$$0 = I_1 < I_2 < \ldots < I_n = B$$
$$0 = J_1 < J_2 < \ldots < J_m = B$$

it holds that $n = m$ *and there exists a permutation* $\sigma \in \mathrm{Sym}(n)$ *such that* I_{i+1}/I_i *is isomorphic to* $J_{\sigma(i+1)}/J_{\sigma(i)}$ *and* I_{i+1}/I_i *is an s-factor (f-factor) if, and only if,* $J_{\sigma(i+1)}/J_{\sigma(i)}$ *is an s-factor (respectively, f-factor).*
The length $n =: \mathrm{chief\,len}(B)$ *of any chief series is called the chief length of B.*

Let X, Y be subsets of B. We can define

$$L_1(X,Y) = Y \quad L_{n+1} = X * L_n(X,Y) \quad \text{for every } n \geq 1$$
$$R_1(X,Y) = X \quad R_{n+1} = R_n(X,Y) * Y \quad \text{for every } n \geq 1$$

- For the case $X = Y = B$, Rump denotes $B^n := L_n(B,B)$ and $B^{(n)} := R_n(B,B)$.
- B^i and $B^{(i)}$ are, respectively, a left ideal and an ideal of B.
- $\{B^n\}_{n\in\mathbb{N}}$ and $\{B^{(n)}\}_{n\in\mathbb{N}}$ are called, respectively, the *left and right series* of B.

W. Rump (2007), *Braces, radical rings, and the quantum Yang-Baxter equation*, J. Algebra, 307(1), 153–170.

Nilpotency of skew left braces

Definition

Then, B is said to be right (left) nilpotent of class m, if the right (left) series reaches the trivial subbrace 0 and $m = \min\{n \in \mathbb{N} : B^{(n)} = 0\ (B^n = 0)\} =: \operatorname{nil\,class}_r(B)\ (\operatorname{nil\,class}_l(B))$.

Left nilpotency

Proposition

B is left nilpotent if, and only if, it admits an f-series of left ideals.

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Right nilpotency

- $\text{Soc}_1(B) = \text{Soc}(B)$,
- for every $n \geq 1$, let $\text{Soc}_{n+1}(B)$ be the ideal of B such that

$$\text{Soc}_{n+1}(B)/\,\text{Soc}_n(B) = \text{Soc}\left(B/\,\text{Soc}_n(B)\right)$$

- The socle series of B is an ascending ideal s-series: $\text{Soc}_1(B) \leq \ldots \leq \text{Soc}_n(B) \leq \ldots$

Proposition (Rump)

If B is of abelian type and (B, r_B) is the involutive non-degenerate solution of the YBE associated to B, then $(B/Soc_n(B), r_n) = \text{Ret}^n(B, r_B)$, for every $n \geq 1$.

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Right nilpotency

Theorem (Cedó, Gateva-Ivanova, Smoktunowicz)

If B is of abelian type and (B, r) is the involutive non-degenerate solution of the YBE associated to B, then (B, r) has multipermutation level m if, and only if, B is right nilpotent of nilpotency class $m + 1$.

Question

Is it possible to get this result for a general skew left brace?

Main result

Proposition

If (B, r_B) is the non-degenerate solution of the YBE associated to B, $(B/Soc_n(B), r_n) = \mathrm{Ret}^n(B, r)$, for every $n \geq 1$.

Theorem

The following are equivalent:

1. *(B, r_B) is a multipermutation solution of level m.*
2. *B is of nilpotent type and right nilpotent.*
3. *Every chief factor of B is an s-factor.*
4. *B admites an s-series (Cedó, Smoktunowicz and Vendramin)*

In this case: $m \leq \mathrm{chief\,len}(B)$ and $\mathrm{nil\,class}_r(B) \leq m + 1$.

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