Quotients of the free product of groups and their exponent

Carmine Monetta

University of Salerno

AGTA Workshop - Reinhold Baer Prize 2022

September 23, 2022

Raimundo Bastos

UnB

Emerson de Melo

Ricardo de Oliveira UFG

This work was partially supported by a grant of the Università della Campania "Luigi Vanvitelli", in the framework of the project GoAL (V:ALERE 2019)

The group we will deal with

Let G be a group and let $G * G$ the free product of G with its-self.

Aim: to find a normal subgroup N of $G * G$ such that in $(G * G)/N$ some generalized permutability holds.

This is equivalent to find relations between the elements of the first and the second factor in $G * G$ ensuring the generalized permutability.

The group we will deal with

Let G be a group and let $G * G$ the free product of G with its-self.

Aim: to find a normal subgroup N of $G * G$ such that in $(G * G)/N$ some generalized permutability holds.

This is equivalent to find relations between the elements of the first and the second factor in $G * G$ ensuring the generalized permutability.

The group we will deal with

Let G be a group and let $G * G$ the free product of G with its-self.

Aim: to find a normal subgroup N of $G * G$ such that in $(G * G)/N$ some generalized permutability holds.

This is equivalent to find relations between the elements of the first and the second factor in $G * G$ ensuring the generalized permutability.

Motivation

Definition

Let G be a group and let H and K be subgroup of G.
Then H and K permutes if there exist two functions

$$\alpha : H \times K \to K \qquad \text{and} \qquad \beta : H \times K \to H$$

such that

$$hk = \alpha(h,k)\beta(h,k)$$

for every $h \in H$, $k \in K$.

Exercise

If a group G is generated by finite subgroups H and K, with H and K permuting, then G is finite.

Motivation

Definition

Let G be a group and let H and K be subgroup of G.
Then H and K permutes if there exist two functions

$$\alpha : H \times K \to K \qquad \text{and} \qquad \beta : H \times K \to H$$

such that

$$hk = \alpha(h, k)\beta(h, k)$$

for every $h \in H$, $k \in K$.

Exercise

If a group G is generated by finite subgroups H and K, with H and K permuting, then G is finite.

Motivation

Definition

Let G be a group and let H and K be subgroup of G.
Then H and K permutes if there exist two functions

$$\alpha : H \times K \to K \qquad \text{and} \qquad \beta : H \times K \to H$$

such that

$$hk = \alpha(h,k)\beta(h,k)$$

for every $h \in H$, $k \in K$.

Exercise

If a group G is generated by finite subgroups H and K, with H and K permuting, then G is finite.

Theorem (Coxeter)

Let $m \geq 3$, $p_i \geq 2$ for $i = 1, \ldots, \lfloor \frac{m}{2} \rfloor$. Then the group

$$\mathcal{C}(m; p_1, \ldots, p_{\lfloor \frac{m}{2} \rfloor}) = \left\langle a_1, a_2 \mid a_1^m = a_2^m = (a_1^i a_2^i)^{p_i} = 1 \text{ for } 1 \leq i \leq \lfloor \frac{m}{2} \rfloor \right\rangle .$$

is finite if and only if $p_i = 2$ for every $i = 1, \ldots, \lfloor \frac{m}{2} \rfloor$.

H. S. M. Coxeter,
The abstract groups $R^m = S^m = (R^i S^i)^{p_i} = 1$, $S^m = T^2 = (S^j T)^{2p_j} = 1$, and $S^m = T^2 = (S^{-j} T S^j T)^{p_j} = 1$,
Proceedings of The London Mathematical Society, **2s-41** (1936), pp. 278–301.

The group $G = \mathcal{C}(m; 2, \ldots, 2)$ is isomorphic to $C_2^{m-1} \rtimes C_m$.

★ G is generated by $H_1 = \langle a_1 \rangle$ and $H_2 = \langle a_2 \rangle$ such that

$$\{h\gamma(h) \mid h \in H_1\} \subset H_1 H_2 \cap H_2 H_1$$

where the map $\gamma : H_1 \to H_2$ is defined by $\gamma(a_1^i) = a_2^i$.

Even though H_1 and H_2 do not permute, sufficient "permutability" relations still hold to ensure the finiteness of the group $\langle H_1, H_2 \rangle$.

The group $G = \mathcal{C}(m; 2, \ldots, 2)$ is isomorphic to $C_2^{m-1} \rtimes C_m$.

★ G is generated by $H_1 = \langle a_1 \rangle$ and $H_2 = \langle a_2 \rangle$ such that

$$\{h\gamma(h) \mid h \in H_1\} \subset H_1 H_2 \cap H_2 H_1$$

where the map $\gamma : H_1 \to H_2$ is defined by $\gamma(a_1^i) = a_2^i$.

Even though H_1 and H_2 do not permute, sufficient "permutability" relations still hold to ensure the finiteness of the group $\langle H_1, H_2 \rangle$.

The group $G = \mathcal{C}(m; 2, \ldots, 2)$ is isomorphic to $C_2^{m-1} \rtimes C_m$.

★ G is generated by $H_1 = \langle a_1 \rangle$ and $H_2 = \langle a_2 \rangle$ such that

$$\{h\gamma(h) \mid h \in H_1\} \subset H_1 H_2 \cap H_2 H_1$$

where the map $\gamma : H_1 \to H_2$ is defined by $\gamma(a_1^i) = a_2^i$.

Even though H_1 and H_2 do not permute, sufficient "permutability" relations still hold to ensure the finiteness of the group $\langle H_1, H_2 \rangle$.

The weak permutability group

Starting from this idea, in 1980 **Said Sidki** came up with a group construction which he called the weak permutability group.

S. N. Sidki,
On weak permutability between groups, J. Algebra, **63**, (1980) pp. 186–225.

Let G be a group, and $G^{\varphi} = \{g^{\varphi} \mid g \in G\}$ be an isomorphic copy of G.

$$\chi(G) := \langle G, G^{\varphi} \mid [g, g^{\varphi}] = 1, \ \ g \in G \rangle.$$

The weak permutability group

Starting from this idea, in 1980 **Said Sidki** came up with a group construction which he called the weak permutability group.

S. N. Sidki,
On weak permutability between groups, J. Algebra, **63**, (1980) pp. 186–225.

Let G be a group, and $G^\varphi = \{g^\varphi \mid g \in G\}$ be an isomorphic copy of G.

$$\chi(G) := \langle G, G^\varphi \mid [g, g^\varphi] = 1, \quad g \in G \rangle.$$

The weak permutability group

Starting from this idea, in 1980 **Said Sidki** came up with a group construction which he called the weak permutability group.

S. N. Sidki,
On weak permutability between groups, J. Algebra, **63**, (1980) pp. 186–225.

Let G be a group, and $G^{\varphi} = \{g^{\varphi} \mid g \in G\}$ be an isomorphic copy of G.

$$\chi(G) := \langle G, G^{\varphi} \mid [g, g^{\varphi}] = 1, \;\; g \in G \rangle.$$

Properties

$G \times G$ is a homomorphic image of $\chi(G)$ via

$$\alpha : \chi(G) \to G \times G$$

which sends $g \to (g, 1)$ and $g^{\varphi} \to (1, g)$ is an epimorphism.

Theorem (Sidki)

Let $\mathcal{P}$ be a one of the following properties:

- *finite π-group, with π a set of primes;*
- *finite nilpotent;*
- *soluble;*
- *perfect.*

If G is a $\mathcal{P}$-group $\Rightarrow \chi(G)$ is a $\mathcal{P}$-group.

Properties

$G \times G$ is a homomorphic image of $\chi(G)$ via

$$\alpha : \chi(G) \to G \times G$$

which sends $g \to (g, 1)$ and $g^{\varphi} \to (1, g)$ is an epimorphism.

Theorem (Sidki)

Let $\mathcal{P}$ be a one of the following properties:

- *finite π-group, with π a set of primes;*
- *finite nilpotent;*
- *soluble;*
- *perfect.*

If G is a $\mathcal{P}$-group $\Rightarrow \chi(G)$ is a $\mathcal{P}$-group.

Properties

$G \times G$ is a homomorphic image of $\chi(G)$ via

$$\alpha : \chi(G) \to G \times G$$

which sends $g \to (g, 1)$ and $g^{\varphi} \to (1, g)$ is an epimorphism.

Theorem (Sidki)

Let $\mathcal{P}$ be a one of the following properties:

- *finite π-group, with π a set of primes;*
- *finite nilpotent;*
- *soluble;*
- *perfect.*

If G is a $\mathcal{P}$-group $\Rightarrow \chi(G)$ is a $\mathcal{P}$-group.

Properties

$G \times G$ is a homomorphic image of $\chi(G)$ via

$$\alpha : \chi(G) \to G \times G$$

which sends $g \to (g, 1)$ and $g^{\varphi} \to (1, g)$ is an epimorphism.

Theorem (Sidki)

Let $\mathcal{P}$ be a one of the following properties:
- *finite π-group, with π a set of primes;*
- *finite nilpotent;*
- *soluble;*
- *perfect.*

If G is a $\mathcal{P}$-group $\Rightarrow \chi(G)$ is a $\mathcal{P}$-group.

Properties

$G \times G$ is a homomorphic image of $\chi(G)$ via

$$\alpha : \chi(G) \to G \times G$$

which sends $g \to (g, 1)$ and $g^{\varphi} \to (1, g)$ is an epimorphism.

Theorem (Sidki)

Let $\mathcal{P}$ be a one of the following properties:

- *finite π-group, with π a set of primes;*
- *finite nilpotent;*
- *soluble;*
- *perfect.*

If G is a $\mathcal{P}$-group $\Rightarrow \chi(G)$ is a $\mathcal{P}$-group.

Properties

$G \times G$ is a homomorphic image of $\chi(G)$ via

$$\alpha : \chi(G) \to G \times G$$

which sends $g \to (g, 1)$ and $g^{\varphi} \to (1, g)$ is an epimorphism.

Theorem (Sidki)

Let $\mathcal{P}$ be a one of the following properties:
- *finite π-group, with π a set of primes;*
- *finite nilpotent;*
- *soluble;*
- *perfect.*

If G is a $\mathcal{P}$-group $\Rightarrow$ $\chi(G)$ is a $\mathcal{P}$-group.

Properties

$G \times G$ is a homomorphic image of $\chi(G)$ via

$$\alpha : \chi(G) \to G \times G$$

which sends $g \to (g, 1)$ and $g^{\varphi} \to (1, g)$ is an epimorphism.

Theorem (Sidki)

Let $\mathcal{P}$ be a one of the following properties:
- *finite π-group, with π a set of primes;*
- *finite nilpotent;*
- *soluble;*
- *perfect.*

If G is a $\mathcal{P}$-group $\Rightarrow \chi(G)$ is a $\mathcal{P}$-group.

If G is an infinite group, $\chi(G)$ is given by infinitely many relations.

Theorem (Bridson, Kochloukova)

A group G is finitely presented if and only if $\chi(G)$ is finitely presented.

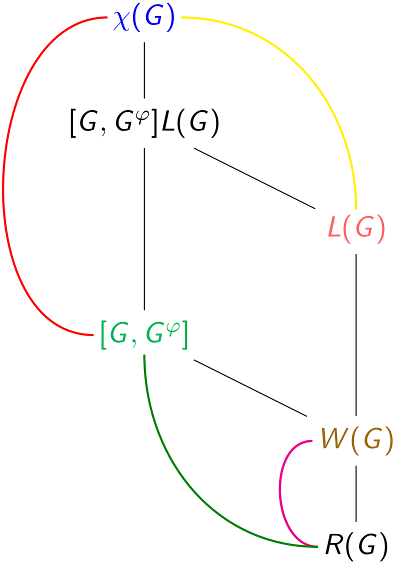
M. R. Bridson and D. Kochloukova, *Weak commutativity and finiteness properties of groups*, Bull. Lond. Math. Soc., **51**, (2019) pp. 168–180.

If G is an infinite group, $\chi(G)$ is given by infinitely many relations.

Theorem (Bridson, Kochloukova)

A group G is finitely presented if and only if $\chi(G)$ is finitely presented.

M. R. Bridson and D. Kochloukova, *Weak commutativity and finiteness properties of groups*, Bull. Lond. Math. Soc., **51**, (2019) pp. 168–180.



$\simeq G \times G$

$\simeq G$

$\simeq M(G) = H_2(\mathbb{Z}, G)$ the **Schur Multiplier of** G

$\simeq G \wedge G$ the **non-abelian exterior square of** G

$\chi(G)$ behaves... but!

Even though χ behaves with respect to many finiteness conditions, the structure of the group $\chi(G)$ remains quite unclear also when G is a finite group.

Exponent problem

To find information about the exponent $\exp(\chi(G))$, when G is a finite p-group.

The study of $\chi(G)$ when G is a p-group reveals basic differences between the case $p = 2$ and p odd.

Proposition (Sidki)

Let G be an elementary abelian p-group of order p^n. Then

$$|\chi(G)| = \begin{cases} p^{2n} p^{\frac{n(n-1)}{2}}, & p > 2 \\ 2^n 2^{2^n - 1}, & p = 2 \end{cases}.$$

The study of $\chi(G)$ when G is a p-group reveals basic differences between the case $p=2$ and p odd.

Proposition (Sidki)

Let G be an elementary abelian p-group of order p^n. Then

$$|\chi(G)| = \begin{cases} p^{2n} p^{\frac{n(n-1)}{2}}, & p > 2 \\ 2^n 2^{2^n - 1}, & p = 2 \end{cases} .$$

The study of $\chi(G)$ when G is a p-group reveals basic differences between the case $p = 2$ and p odd.

Proposition (Sidki)

Let G be an elementary abelian p-group of order p^n. Then

$$|\chi(G)| = \begin{cases} p^{2n} p^{\frac{n(n-1)}{2}}, & p > 2 \\ 2^n 2^{2^n - 1}, & p = 2 \end{cases}.$$

Bound for $|\chi(G)|$, p odd

Theorem (Rocco)

Let G be a p-group of order p^n, p odd. Then

$$|\chi(G)| \mid p^{2n} p^{\frac{n(n-1)}{2}}.$$

Moreover, if G has nilpotency class c, then $\chi(G)$ has nilpotency class at most $2c$.

N. R. Rocco,
On weak commutativity between finite p-groups, p odd, J. Algebra, **76** (1982) pp. 471–488.

Bound for $|\chi(G)|$, p odd

Theorem (Rocco)

Let G be a p-group of order p^n, p odd. Then

$$|\chi(G)| \mid p^{2n} p^{\frac{n(n-1)}{2}}.$$

Moreover, if G has nilpotency class c, then $\chi(G)$ has nilpotency class at most $2c$.

N. R. Rocco,
On weak commutativity between finite p-groups, p odd, J. Algebra, **76** (1982) pp. 471–488.

Bound for $|\chi(G)|$, p odd

Theorem (Rocco)

Let G be a p-group of order p^n, p odd. Then

$$|\chi(G)| \mid p^{2n} p^{\frac{n(n-1)}{2}}.$$

Moreover, if G has nilpotency class c, then $\chi(G)$ has nilpotency class at most $2c$.

N. R. Rocco,
On weak commutativity between finite p-groups, p odd, J. Algebra, **76** (1982) pp. 471–488.

Bound for $|\chi(G)|$, p odd

Theorem (Rocco)

Let G be a p-group of order p^n, p odd. Then

$$|\chi(G)| \mid p^{2n} p^{\frac{n(n-1)}{2}}.$$

Moreover, if G has nilpotency class c, then $\chi(G)$ has nilpotency class at most $2c$.

N. R. Rocco,
On weak commutativity between finite p-groups, p odd, J. Algebra, **76** (1982) pp. 471–488.

Exponent strategy

The sequence $1 \to [G, G^{\varphi}] \to \chi(G) \to G \times G \to 1$ is exact.

Idea: to look at the generators of $[G, G^{\varphi}]$, i.e. $[x, y^{\varphi}]$, and work in $\langle H, H^{\varphi} \rangle$, where $H = \langle x, y \rangle$, because the situation is different if the group is 2-generated.

Lemma (Bastos, de Melo, de Oliveira)

Let H be a 2-generator p-subgroup of a group G. If H has class c, then $\langle H, H^{\varphi} \rangle \leq \chi(G)$ is a p-group of class at most $c + 1$.

R. Bastos, R. de Melo and R. de Oliveira,
On the exponent of the Weak commutativity group $\chi(G)$, Mediterr. J. Math. **18** (2021), pp. 1–9.

Exponent strategy

The sequence $1 \to [G, G^{\varphi}] \to \chi(G) \to G \times G \to 1$ is exact.

Idea: to look at the generators of $[G, G^{\varphi}]$, i.e. $[x, y^{\varphi}]$, and work in $\langle H, H^{\varphi} \rangle$, where $H = \langle x, y \rangle$, because the situation is different if the group is 2-generated.

Lemma (Bastos, de Melo, de Oliveira)

Let H be a 2-generator p-subgroup of a group G. If H has class c, then $\langle H, H^{\varphi} \rangle \leq \chi(G)$ is a p-group of class at most $c + 1$.

R. Bastos, R. de Melo and R. de Oliveira,
On the exponent of the Weak commutativity group $\chi(G)$, Mediterr. J. Math. **18** (2021), pp. 1–9.

Exponent strategy

The sequence $1 \to [G, G^\varphi] \to \chi(G) \to G \times G \to 1$ is exact.

Idea: to look at the generators of $[G, G^\varphi]$, i.e. $[x, y^\varphi]$, and work in $\langle H, H^\varphi \rangle$, where $H = \langle x, y \rangle$, because the situation is different if the group is 2-generated.

Lemma (Bastos, de Melo, de Oliveira)

Let H be a 2-generator p-subgroup of a group G. If H has class c, then $\langle H, H^\varphi \rangle \leq \chi(G)$ is a p-group of class at most $c + 1$.

R. Bastos, R. de Melo and R. de Oliveira,
On the exponent of the Weak commutativity group $\chi(G)$, Mediterr. J. Math. **18** (2021), pp. 1–9.

Exponent strategy

The sequence $1 \to [G, G^{\varphi}] \to \chi(G) \to G \times G \to 1$ is exact.

Idea: to look at the generators of $[G, G^{\varphi}]$, i.e. $[x, y^{\varphi}]$, and work in $\langle H, H^{\varphi} \rangle$, where $H = \langle x, y \rangle$, because the situation is different if the group is 2-generated.

Lemma (Bastos, de Melo, de Oliveira)

Let H be a 2-generator p-subgroup of a group G. If H has class c, then $\langle H, H^{\varphi} \rangle \leq \chi(G)$ is a p-group of class at most $c + 1$.

R. Bastos, R. de Melo and R. de Oliveira,
On the exponent of the Weak commutativity group $\chi(G)$, Mediterr. J. Math. **18** (2021), pp. 1–9.

Our contribution

We analyzed some classes of p-groups.

1. p-groups of nilpotency class at most $2p-3$, with p odd;
2. p-groups of nilpotency class c (best for $c \geq 2p-2$);
3. 2-groups of nilpotency class at most 3;
4. p-groups of maximal class.

R. Bastos, R. de Melo, R. de Oliveira and C. Monetta, *On weak commutativity in p-groups*, submitted.

Our contribution

We analyzed some classes of p-groups.

1. p-groups of nilpotency class at most $2p-3$, with p odd;
2. p-groups of nilpotency class c (best for $c \geq 2p-2$);
3. 2-groups of nilpotency class at most 3;
4. p-groups of maximal class.

R. Bastos, R. de Melo, R. de Oliveira and C. Monetta, *On weak commutativity in p-groups*, submitted.

Our contribution

We analyzed some classes of p-groups.

1. p-groups of nilpotency class at most $2p-3$, with p odd;
2. p-groups of nilpotency class c (best for $c \geq 2p-2$);
3. 2-groups of nilpotency class at most 3;
4. p-groups of maximal class.

R. Bastos, R. de Melo, R. de Oliveira and C. Monetta,
On weak commutativity in p-groups, submitted.

Our contribution

We analyzed some classes of p-groups.

1. p-groups of nilpotency class at most $2p-3$, with p odd;
2. p-groups of nilpotency class c (best for $c \geq 2p-2$);
3. 2-groups of nilpotency class at most 3;
4. p-groups of maximal class.

R. Bastos, R. de Melo, R. de Oliveira and C. Monetta, *On weak commutativity in p-groups*, submitted.

Our contribution

We analyzed some classes of p-groups.

1. p-groups of nilpotency class at most $2p-3$, with p odd;
2. p-groups of nilpotency class c (best for $c \geq 2p-2$);
3. 2-groups of nilpotency class at most 3;
4. p-groups of maximal class.

R. Bastos, R. de Melo, R. de Oliveira and C. Monetta, *On weak commutativity in p-groups*, submitted.

Powerful p-groups

Let G be a p-group and N be a subgroup of G.

We define $\mathbf{p} = 4$ if $p = 2$ and $\mathbf{p} = p$ if $p > 2$.

- ⋆ G is **powerful** if $G' \leq G^{\mathbf{p}}$.
- ⋆ N is **powerfully embedded in** G if $[N, G] \leq N^{\mathbf{p}}$.

Theorem (Lubotzky, Mann)

If G is a powerful p-group, then $\gamma_i(G)$ is powerfully embedded in G, i.e., $\gamma_{i+1}(G) \leq \gamma_i(G)^{\mathbf{p}}$, for every $i \geq 1$.

A. Lubotzky and A. Mann, *Powerful p-groups. i. finite group*, J. Algebra **105** (1987), 484–505.

Powerful p-groups

Let G be a p-group and N be a subgroup of G.

We define $\mathbf{p} = 4$ if $p = 2$ and $\mathbf{p} = p$ if $p > 2$.

- ⋆ G is **powerful** if $G' \leq G^{\mathbf{p}}$.
- ⋆ N is **powerfully embedded in** G if $[N, G] \leq N^{\mathbf{p}}$.

Theorem (Lubotzky, Mann)

If G is a powerful p-group, then $\gamma_i(G)$ is powerfully embedded in G, i.e., $\gamma_{i+1}(G) \leq \gamma_i(G)^{\mathbf{p}}$, for every $i \geq 1$.

A. Lubotzky and A. Mann, *Powerful p-groups. i. finite group*, J. Algebra **105** (1987), 484–505.

Powerful p-groups

Let G be a p-group and N be a subgroup of G.

We define $\mathbf{p} = 4$ if $p = 2$ and $\mathbf{p} = p$ if $p > 2$.

- ⋆ G is **powerful** if $G' \leq G^{\mathbf{p}}$.
- ⋆ N is **powerfully embedded in** G if $[N, G] \leq N^{\mathbf{p}}$.

Theorem (Lubotzky, Mann)

If G is a powerful p-group, then $\gamma_i(G)$ is powerfully embedded in G, i.e., $\gamma_{i+1}(G) \leq \gamma_i(G)^{\mathbf{p}}$, for every $i \geq 1$.

A. Lubotzky and A. Mann, *Powerful p-groups. i. finite group*, J. Algebra **105** (1987), 484–505.

Theorem (Bastos, de Melo, Gonçalves, Nunes)

Let G be a powerful p-group, p odd. Then

- *If $k \geq 2$, then the k-th term of the lower central series $\gamma_k(\chi(G))$ and $[G, G^{\varphi}]$ are powerfully embedded in $\chi(G)$.*
- *If $p = 3$, then $\exp(\chi(G))$ divides $3 \cdot \exp(G)$.*
- *If $p \geq 5$, then $\exp(\chi(G)) = \exp(G)$.*

R. Bastos, E. de Melo, N. Gonçalves and R. Nunes, *Non-abelian tensor square and related constructions of p-groups*, Arch. Math. **114** (2020) pp. 481–490.

Remark

If $p = 2$, and $G = C_2 \times C_2 \times C_2$, then $[G, G^{\varphi}]$ is not powerfully embedded in $\chi(G)$.

Theorem (Bastos, de Melo, Gonçalves, Nunes)

Let G be a powerful p-group, p odd. Then

- *If $k \geq 2$, then the k-th term of the lower central series $\gamma_k(\chi(G))$ and $[G, G^{\varphi}]$ are powerfully embedded in $\chi(G)$.*
- *If $p = 3$, then $\exp(\chi(G))$ divides $3 \cdot \exp(G)$.*
- *If $p \geq 5$, then $\exp(\chi(G)) = \exp(G)$.*

R. Bastos, E. de Melo, N. Gonçalves and R. Nunes, *Non-abelian tensor square and related constructions of p-groups*, Arch. Math. **114** (2020) pp. 481–490.

Remark

If $p = 2$, and $G = C_2 \times C_2 \times C_2$, then $[G, G^{\varphi}]$ is not powerfully embedded in $\chi(G)$.

Theorem (Bastos, de Melo, Gonçalves, Nunes)

Let G be a powerful p-group, p odd. Then

- *If $k \geq 2$, then the k-th term of the lower central series $\gamma_k(\chi(G))$ and $[G, G^{\varphi}]$ are powerfully embedded in $\chi(G)$.*
- *If $p = 3$, then $\exp(\chi(G))$ divides $3 \cdot \exp(G)$.*
- *If $p \geq 5$, then $\exp(\chi(G)) = \exp(G)$.*

R. Bastos, E. de Melo, N. Gonçalves and R. Nunes, *Non-abelian tensor square and related constructions of p-groups*, Arch. Math. **114** (2020) pp. 481–490.

Remark

If $p = 2$, and $G = C_2 \times C_2 \times C_2$, then $[G, G^{\varphi}]$ is not powerfully embedded in $\chi(G)$.

Theorem (Bastos, de Melo, Gonçalves, Nunes)

Let G be a powerful p-group, p odd. Then

- *If $k \geq 2$, then the k-th term of the lower central series $\gamma_k(\chi(G))$ and $[G, G^{\varphi}]$ are powerfully embedded in $\chi(G)$.*
- *If $p = 3$, then $\exp(\chi(G))$ divides $3 \cdot \exp(G)$.*
- *If $p \geq 5$, then $\exp(\chi(G)) = \exp(G)$.*

R. Bastos, E. de Melo, N. Gonçalves and R. Nunes, *Non-abelian tensor square and related constructions of p-groups*, Arch. Math. **114** (2020) pp. 481–490.

Remark

If $p = 2$, and $G = C_2 \times C_2 \times C_2$, then $[G, G^{\varphi}]$ is not powerfully embedded in $\chi(G)$.

Potent p-group

Let G be a p-group and N be a subgroup of G.

⋆ G is **potent** if $\begin{cases} G' \leq G^4 & \text{if } p = 2 \\ \gamma_{p-1}(G) \leq G^p & \text{if } p > 2 \end{cases}$

POWERFUL $\Rightarrow$ POTENT

Open problem

Find an upper bound for $\exp(\chi(G))$ when G is a potent p-group, p odd.

Potent p-group

Let G be a p-group and N be a subgroup of G.

⋆ G is **potent** if $\begin{cases} G' \leq G^4 & \text{if } p = 2 \\ \gamma_{p-1}(G) \leq G^p & \text{if } p > 2 \end{cases}$

POWERFUL $\Rightarrow$ POTENT

Open problem

Find an upper bound for $\exp(\chi(G))$ when G is a potent p-group, p odd.

Nilpotency class

Theorem (Bastos, de Melo, de Oliveira)

∘ *If G is a p-group of class at most $p-1$, then* $\exp(\chi(G))$ *divides* $\exp(G)^2$.

∘ *If G is nilpotent of class c, then* $\exp(\chi(G))$ *divides* $\exp(G)^{n+1}$, *where* $n = \lceil \log_{p-1}(c+1) \rceil$.

R. Bastos, R. de Melo and R. de Oliveira,
On the exponent of the Weak commutativity group $\chi(G)$, Mediterr. J. Math. **18** (2021), pp. 1–9.

Nilpotency class

Theorem (Bastos, de Melo, de Oliveira)

∘ *If G is a p-group of class at most $p-1$, then* $\exp(\chi(G))$ *divides* $\exp(G)^2$.

∘ *If G is nilpotent of class c, then* $\exp(\chi(G))$ *divides* $\exp(G)^{n+1}$, *where* $n = \lceil \log_{p-1}(c+1) \rceil$.

R. Bastos, R. de Melo and R. de Oliveira,
On the exponent of the Weak commutativity group $\chi(G)$, Mediterr. J. Math. **18** (2021), pp. 1–9.

Groups with $c < 2p - 2$

Theorem (Bastos, de Melo, de Oliveira, M.)

Let p be an odd prime and G a p-group of nilpotency class at most $2p-3$. Then

- $\exp([G, G^{\varphi}])$ *divides* $p \cdot \exp(G)$
- $\exp(\chi(G))$ *divides* $p \cdot \exp(G)^2$.

⋆ The improvement occurs when $p - 1 < c < 2p - 2$.

In the proof we use the concept of regular p-group because if H is a regular p-group generated by a set X, then $\exp(H) = \max\{|x| \mid x \in X\}$.

Groups with $c < 2p - 2$

Theorem (Bastos, de Melo, de Oliveira, M.)

Let p be an odd prime and G a p-group of nilpotency class at most $2p-3$. Then

- $\exp([G, G^{\varphi}])$ *divides* $p \cdot \exp(G)$
- $\exp(\chi(G))$ *divides* $p \cdot \exp(G)^2$.

⋆ The improvement occurs when $p - 1 < c < 2p - 2$.

In the proof we use the concept of regular p-group because if H is a regular p-group generated by a set X, then $\exp(H) = \max\{|x| \mid x \in X\}$.

Groups with $c < 2p - 2$

Theorem (Bastos, de Melo, de Oliveira, M.)

Let p be an odd prime and G a p-group of nilpotency class at most $2p-3$. Then

- $\exp([G, G^{\varphi}])$ *divides* $p \cdot \exp(G)$
- $\exp(\chi(G))$ *divides* $p \cdot \exp(G)^2$.

⋆ The improvement occurs when $p - 1 < c < 2p - 2$.

In the proof we use the concept of regular p-group because if H is a regular p-group generated by a set X, then $\exp(H) = \max\{|x| \mid x \in X\}$.

For $c \geq 2p - 2$

Theorem

Let p be an odd prime and G a p-group of nilpotency class c. Then

- $\exp([G, G^{\varphi}])$ *divides* $\exp(G)^{n+1}$;
- $\exp(\chi(G))$ *divides* $\exp(G)^{n+2}$

where $n = \left\lceil \log_{p-1}\left(\frac{c+1}{p}\right)\right\rceil$.

We prove it by induction on c showing that there exists a term of the lower central series of $[G, G^{\varphi}]$ which fits into a short exact sequence.

For $c \geq 2p - 2$

Theorem

Let p be an odd prime and G a p-group of nilpotency class c. Then

- ∘ $\exp([G, G^{\varphi}])$ *divides* $\exp(G)^{n+1}$;
- ∘ $\exp(\chi(G))$ *divides* $\exp(G)^{n+2}$

where $n = \left\lceil \log_{p-1}\left(\frac{c+1}{p}\right)\right\rceil$.

We prove it by induction on c showing that there exists a term of the lower central series of $[G, G^{\varphi}]$ which fits into a short exact sequence.

A fundamental tool

Denote by $\Omega_i(G)$ the subgroup $\langle g \in G \mid g^{p^i} = 1 \rangle$.

Theorem (Fernández-Alcober, González-Sánches, Jaikin-Zapirain)

Let G be a finite p-group and $k \geqslant 1$. Assume that $\gamma_{k(p-1)}(G) \leq \gamma_r(G)^{p^s}$ for some r and s such that $k(p-1) < r + s(p-1)$. Then the exponent $\exp(\Omega_i(G))$ is at most p^{i+k-1} for all i.

G. A. Fernández-Alcober, J. González-Sánches and A. Jaikin-Zapirain, *Omega subgroups of pro-p groups*, Isr. J. Math., **166** (2008) pp. 393–412.

The case $p = 2$

Theorem

Let G be a 2-group of class $c \in \{2, 3\}$. Then

- $\exp([G, G^{\varphi}])$ *divides* $2^{c-1} \cdot \exp(G)$
- $\exp(\chi(G))$ *divides* $2^{c-1} \cdot \exp(G)^2$.

The proof is mainly based on commutator calculus and on the existence of a regular subgroup.

The case $p = 2$

Theorem

Let G be a 2-group of class $c \in \{2, 3\}$. Then

- $\exp([G, G^{\varphi}])$ *divides* $2^{c-1} \cdot \exp(G)$
- $\exp(\chi(G))$ *divides* $2^{c-1} \cdot \exp(G)^2$.

The proof is mainly based on commutator calculus and on the existence of a regular subgroup.

p-groups of maximal class

A p-group of order p^n is said to be of maximal class if it has nilpotency class $n-1$.

Theorem (Bastos, de Melo, de Oliveira, M.)

Let p be a prime and G a p-group of maximal class.

1. *If $p = 2$, then $\exp(\chi(G))$ divides $2 \cdot \exp(G)^2$.*
2. *If p is odd, then $\exp(\chi(G))$ divides $p^2 \cdot \exp(G)$.*

p-groups of maximal class

A p-group of order p^n is said to be of maximal class if it has nilpotency class $n-1$.

Theorem (Bastos, de Melo, de Oliveira, M.)

Let p be a prime and G a p-group of maximal class.

1. *If $p = 2$, then $\exp(\chi(G))$ divides $2 \cdot \exp(G)^2$.*
2. *If p is odd, then $\exp(\chi(G))$ divides $p^2 \cdot \exp(G)$.*

Thank you for the attention!