Group-theoretic tools into the study of set-theoretic solutions of the Yang-Baxter equation

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The quantum Yang-Baxter equation

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It is a fundamental topic of theoretical physics that takes its name from **[Yang, 1968]**, and **[Baxter, 1971]**. A purely algebraic derivation is obtained as follows.

Let V be a vector space over a field F. Any linear map $R: V \otimes V \to V \otimes V$ satisfying the relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where $R_{12} = R \otimes \mathrm{id}_V$, $R_{23} = \mathrm{id}_V \otimes R$, and

$$R_{13} = (\mathrm{id}_V \otimes \tau)R_{12}(\mathrm{id}_V \otimes \tau),$$

with $\tau(u \otimes v) = v \otimes u$, is said to be a solution of the quantum Yang-Baxter equation.

Applications

It depends on the idea that in some scattering situations, particles may preserve their momentum while changing their quantum internal states.

The discovery of the Yang–Baxter equation in theoretical physics and statistical mechanics has led to many applications also in

- quantum groups,
- quantum computing,
- knot theory,
- braided categories,
- the analysis of integrable systems,
- quantum mechanics, etc.

The interest in this equation is growing, as new properties of it are found, but finding all the solutions is a difficult task.

Set-theoretic solutions

[Drinfel'd, 1992] suggested to study a simpler case. Into the specific, fixed a basis X on a vector space V, we can find all the solutions induced by *linear extension* of maps $\mathcal{R} : X \times X \to X \times X$.

Set-theoretic solutions

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If X is a set, a map $\mathcal{R} : X \times X \to X \times X$ is a solution if and only if the map $r := \tau\mathcal{R}$, with $\tau : X \times X \to X \times X$, $(x, y) \mapsto (y, x)$, satisfies the braid equation

$$(r \times \mathrm{id}_X)\,(\mathrm{id}_X \times r)\,(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)\,(r \times \mathrm{id}_X)\,(\mathrm{id}_S \times r)\,.$$

The pair (X, r) is called a set-theoretic solution of the Yang-Baxter equation.

Some definitions

If (X, r) is a set-theoretic solution, we introduce the maps $\lambda_x : X \to X$ and $\rho_y : X \to X$ to write

$$r(x, y) = (\lambda_x(y), \rho_y(x)),$$

for all $x, y \in X$.

Moreover, (X, r) is said to be

- *bijective* if r is bijective;
- *finite* if $|X|$ is finite;
- *left (or right) non-degenerate* if λ_x (or ρ_y) is bijective, for every $x \in X$ (or $y \in X$);
- *non-degenerate* if r is both left and right non-degenerate;
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From now, on we only deal with bijective and non-degenerate set-theoretic solutions that we briefly call *solutions*.

The structure group of a solution

Definition (Etingof, Schedler, Soloviev, 1999)

Let (X, r) be a solution. The structure group of (X, r) is the group

$$\mathcal{G}(X, r) = \langle X \mid xy = \lambda_x(y)\rho_y(x) \rangle .$$

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Example

Let X be a set and consider the trivial solution (X, τ) with $\tau(x, y) = (y, x)$, for all $x, y \in X$. Then,

$$\mathcal{G}(X, r) = \mathbb{Z}^X$$

is the free abelian group generated by X.

Some properties of the structure group

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- if (X, r) is involutive:
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  3. $\mathcal{G}(X, r)$ is bi-orderable if $\mathcal{G}(X, r)$ is the free abelian group, that is if (X, r) is the trivial solution;
  4. $\mathcal{G}(X, r)$ is left-orderable if and only if $\mathcal{G}(X, r)$ is poly-$\mathbb{Z}$ if and only if (X, r) is a multipermutation solution (**[Bachiller, Cedó, Vendramin, 2018]**).

Multipermutation solutions

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$\mathcal{G}(X, r)$ is left-orderable if and only if (X, r) is a multipermutation solution.

The permutation group of a solution

Given an involutive solution (X, r), the permutation group of (X, r) is the group

$$G(X, r) = \langle \lambda_x \mid x \in X \rangle \leq \mathrm{Sym}_X .$$

If (X, r) is finite, then $G(X, r)$ is solvable.

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A useful strategy for determining all the involutive solutions consists in finding those that can not be deconstructed into other ones:

An involutive solution (X, r) is said to be decomposable if there exists a partition $\{Y, Z\}$ of X such that $r_{|Y \times Y}$ and $r_{|Z \times Z}$ are solutions; otherwise, (X, r) is called indecomposable.

Permutation groups as multiplicative groups of left braces

[Rump, 2007] introduced braces as a generalization of radical rings:

A left brace is a triple $(B, +, \circ)$ such that $(B, +)$ is an abelian group, $(B, \circ)$ is a group and

$$\forall a, b, c \in B \quad a \circ (b + c) + a = a \circ b + a \circ c.$$

Important fact: Every left brace B gives rise to an involutive solution (B, r_B).

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Important fact: Every left brace B gives rise to an involutive solution (B, r_B).

Theorem (Cedó, Jespers, Okniński, 2014)

If B is a left brace, then there exists an involutive solution (X, r) such that its retraction $\mathrm{Ret}(X, r)$ is isomorphic to the solution (B, r_B). Moreover, $G(X, r)$ is isomorphic to the multiplicative group of the left brace B.

The non-involutive case

Recently, to study not necessarily involutive solutions, we introduce the following group:

Definition (Castelli, Stefanelli, M., 2022)

Let (X, r) be a solution. We call permutation group of (X, r) the group

$$F(X, r) = \langle \lambda_x, \eta_x \mid x \in X \rangle,$$

where $\eta_x : X \to X$ is the map given by

$$\eta_x = \rho_{\lambda_y^{-1}(x)} \qquad \text{for all } x, y \in X.$$

If (X, r) is involutive, then $G(X, r) = F(X, r)$.

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This group can be rewritten in terms of q-cycle sets, special algebraic structures that are in one-to-one correspondence with (not necessarily involutive) solutions.

q-cycle sets

Definition (Rump, 2019)

A triple $(X, \cdot, :)$ is said to be a q-cycle set if the map

$$\sigma_x : X \to X, y \mapsto x \cdot y$$

is bijective, for every $x \in X$, and the following conditions

$$(x \cdot y) \cdot (x \cdot z) = (y : x) \cdot (y \cdot z)$$
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hold, for all $x, y, z \in X$. Besides, X is regular if the map

$$\delta_x : X \to X, y \mapsto x : y$$

is bijective, for every $x \in X$.

One-to-one correspondence with solutions

Theorem (Rump, 2019)

If X is a finite and regular q-cycle set, then (X, r) is a finite solution, where $r : X \times X \to X \times X$ is the map given by

$$r(x, y) = \left(\sigma_x^{-1}(y), \delta_{\sigma_x^{-1}(y)}(x)\right),$$

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for all $x, y \in X$. Vice versa, if (X, r) is a finite solution, set

$$x \cdot y := \lambda_x^{-1}(y) \qquad \text{and} \qquad x : y := \rho_{\lambda_y^{-1}(x)}(y),$$

for all $x, y \in X$, then X is a finite regular q-cycle set.

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Instead, involutive solutions are in one-to-one correspondence with cycle sets, that are q-cycle sets in which $\cdot$ and $:$ coincide (**[Rump, 2005]**).

The permutation group of a q-cycle set

Given a solution (X, r), the permutation group of (X, r) is the group

$$F(X,r) = \langle \lambda_x,\, \eta_x \mid x \in X \rangle\,,$$

with $\eta_x = \rho_{\lambda_y^{-1}(x)}$, for all $x, y \in X$.

In terms of a q-cycle set $(X, \cdot, :)$, the permutation group of $(X, \cdot, :)$ is the group

$$F(X) = \langle \sigma_x,\, \delta_x \mid x \in X \rangle\,.$$

with $\sigma_x : X \to X,\, y \mapsto x \cdot y$ and $\delta_x : X \to X,\, y \mapsto x : y$.

Indecomposable q-cycle sets

Theorem (Castelli, Stefanelli, M., 2022)

Let (X, r) be a solution. Then (X, r) is indecomposable if and only if $F(X, r)$ acts transitively on X.

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The proof is based on the fact that two elements $x, y \in X$ are in the same orbit with respect to the action of the group $F(X, r)$ if and only if they are in the same orbit with respect to the action of the group

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Remark: In general, even if the orbits coincide, these two groups are different.

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Definition (Castelli, Stefanelli, M., 2022)

A regular q-cycle set X is said to be indecomposable if $F(X)$ acts transitively on X.

The retraction of a q-cycle set

[Castelli, Catino, Stefanelli, 2021]

If X is a q-cycle set, we can define the retract relation $\sim$ on X:

$$\forall x, y \in X \qquad x \sim y \Longleftrightarrow \sigma_x = \sigma_y \quad \text{and} \quad \delta_x = \delta_y.$$

The quotient $\operatorname{Ret}(X) := X/\sim$ is still a q-cycle set, called the retraction of X.

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A regular q-cycle set X is said to be retractable if $|X| = 1$ or if there exist two distinct elements x, y of X such that $\sigma_x = \sigma_y$ and $\delta_x = \delta_y$.

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A regular q-cycle set X is said to be retractable if $|X| = 1$ or if there exist two distinct elements x, y of X such that $\sigma_x = \sigma_y$ and $\delta_x = \delta_y$.

Moreover, X is multipermutational of level n if n is the minimal non-negative integer such that $|\mathrm{Ret}^n(X)| = 1$, where

$$\mathrm{Ret}^n(X) := \mathrm{Ret}\left(\mathrm{Ret}^{n-1}(X)\right), \text{ for } n \geq 2.$$

Extensions of known results for involutive solutions to the general case

The following results are consistent with those contained in **[Castelli, Pinto, Rump, 2020]**, **[Ramirez, Vendramin, 2021]**, **[Cedó, Jespers, Okniński, 2020]** in the context of cycle sets.

Theorem (Castelli, Stefanelli, M., 2022)

If X is an indecomposable q-cycle set such that $|X| > 1$ and $F(X)$ is regular, then X is retractable.
Moreover, if $F(X)$ is abelian, then X is multipermutational.

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Theorem (Castelli, Stefanelli, M., 2022)

Let X be a retractable indecomposable q-cycle set such that $|X|$ is not a prime number. Then, $F(X)$ acts imprimitively on X.

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Thank you!