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ON SOME CLASSES OF GENERALIZED NILPOTENT GROUPS

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CASERTA 2022

The classes of nilpotent and distinct generalized nilpotent groups occupy a special place among all other classes of groups. They are densely saturated with normal subgroups, and this makes it possible to elucidate many details of their structure. Many of them do not include subgroups that are peculiar antipodes of normal subgroups. And on the other hand, quite often the absence of such antipodes leads to some of the classes of locally nilpotent subgroups. For example:

Recall that a subgroup S of a group G is called **selfnormalizing** in G if $N_G(S) = S$. Every nilpotent group does not include selfnormalizing subgroups.

On the other hand,

If a group G does not include selfnormalizing subgroups, then G is locally nilpotent.

This result has been proved by B.I. Plotkin in the paper

PB1951. Plotkin B.I. *To the theory of locally nilpotent groups.* Doklady AN USSR – 76 (1951), 639 – 641.

Consider now another antipode to normal subgroups.

If H is a selfnormalizing subgroup of G and S is a subgroup including H, then S is not always selfnormalizing. This fact leads us to the following important types of selfnormalizing subgroups.

A subgroup H is called **weakly abnormal** in G if every subgroup including H is selfnormalizing in G. In those connections, it should be pointed out the following characterization of weakly abnormal subgroups

The subgroup S is weakly abnormal in a group G if and only if $x \in S^{\langle x \rangle}$ for each element x of a group G.

This result was obtained in the paper

BBZ1988. Ba M.S., Borevich Z.I. *On arrangement of intermediate subgroups.* Rings and Linear Groups, Kubanskij Univ., Krasnodar, 1988, 14–41.

A slight strengthening leads us to the following important type of subgroups.

A subgroup S of a group G is called **abnormal** in G if $g \in \langle S, S^g \rangle$ for each element g of G.

Abnormal subgroups have appeared in the paper

HP1937. Hall P. *On the system normalizers of a soluble groups.* Proc. London Math. Soc. – 43(1937), 507–528.

The term "an abnormal subgroup" belongs to R. Carter

CR1961. Carter R.W. *Nilpotent self – normalizing subgroups of soluble groups.* Math. Z. 75(1961), 136–139.

Every nilpotent group does not include abnormal and weakly abnormal subgroups. Moreover, *Let G be a Ñ – groups. Then G does not include weakly abnormal subgroups.*

This result was obtained in the paper

KKS2011. Kirichenko V.V., Kurdachenko L.A., Subbotin I.Ya. ***Some related to pronormality subgroup families and properties of groups.*** **Algebra and Discrete Mathematics 2011, Volume 11, number 1, 75 – 108**

Recall that a group G is called an **Ñ – group** if G satisfies the following condition:

If M, L are subgroup of G such that M is maximal in L, then M is a normal subgroup of L.

The Ñ – groups are generalization of nilpotent groups, because every (locally) nilpotent group is an Ñ – group.

In particular, every locally nilpotent group does not include proper abnormal subgroups. This result was obtained in the paper

KuSU1988. Kuzennyi, N.F., Subbotin I.Ya. *New characterization of locally nilpotent IH – groups.* Ukranian. Mat. Journal, 40 (1988), 274-277.

We have the following characterization of finite nilpotent groups

Finite group G is nilpotent if and only if G does not include proper abnormal subgroups.

Therefore the following question natural appears:

What we can say about the groups, which do not include proper abnormal subgroups?
In particular, in what case the groups without proper abnormal subgroups are locally nilpotent?

In other words, it would be interesting to obtain some criteria of locally nilpotency in terms of abnormality.

We note that there are groups without proper abnormal subgroups which are not locally nilpotent. One of these groups is the group constructed by R,I, Grigorchuk in the paper

GR1980 Grigorchuk R.I. *On Burnside's problem on periodic groups.* **Funk. Analis 14 (1980), number 1, 53 – 54.**

Indeed, from the definition we can see that the normal closure of abnormal subgroup coincides with entire group. Every proper subgroup of a Grigorchuk group can be extend to a proper maximal subgroup. But every maximal subgroup of a Grigorchuk group is normal of prime index. It was proved in the paper

PE2005 E. L. PERVOVA E.L. *Maximal subgroups of some non locally finite p – groups.* **International Journal of Algebra and Computation 2005, Vol. 15, No. 05 no 6, pp. 1129-1150**

Let G be a group and A be a normal subgroup of G. A subgroup A is called **G – minimax**, if A has a series G – invariant subgroups, every infinite factor of which is abelian and satisfies maximal or minimal condition for G – invariant subgroups.

A group G is called ***generalized minimax***, if G is itself G – minimax.

The following result gives a criterion of hypercentrality

Let G be a generalized minimax group. If G does not include proper abnormal subgroups, then G is hypercentral.

This result has been proved in the paper

KOS2002. Kurdachenko L.A., Otal J., Subbotin I.Ya. *On some criteria of nilpotency.* Comm. Algebra, 30 (2002), no 8, 3755 - 3776.

Let G be a group. If x is an element of G then put $x^G = \{ x^g = g^{-1} x g \mid g \in G \}$. The **FC – center of G** is the set $FC(G) = \{ x \mid x^G$ is finite $\}$. It is possible to prove that the FC – center of G is its characteristic subgroup.

A group G is called an **FC – group** if G = FC(G).

Starting from the FC – center we construct the **upper FC – central series** of a group G

$$\langle 1 \rangle = FC_0(G) \le FC_1(G) \le \ldots FC_\alpha(G) \le FC_{\alpha+1}(G) \le \ldots FC_\gamma(G)$$

by the following rule: $FC_1(G) = FC(G)$ and recursively $FC_{\alpha+1}(G)/FC_\alpha(G) = FC(G/FC_\alpha(G))$ for all ordinal α, and $FC_\lambda(G) = \cup_{\beta<\lambda} FC_\beta(G)$ for all limit ordinals λ.

The last term $FC_\gamma(G)$ is called the **upper FC – hypercenter of G** and we have $FC(G/FC_\gamma(G)) = \langle 1 \rangle$.

A group G is said to be **FC – hypercentral** if the upper FC – hypercenter of G coincides with G. A group G is said to be **FC – nilpotent** if it is FC – hypercentral and has finite FC – central series.

The following result gives us a criterion of hypercentrality for FC – nilpotent groups

Let G be a group without proper abnormal subgroups.

(i) Every term of the upper FC – central series of G, having finite number of G, is hypercentral. In particular, if G is FC – nilpotent, then G is hypercentral.

(ii) Let C be the term of the upper FC – central series of G, having finite number, and S be a normal subgroup of G including C. If the factor S/C is hypercentral, then S is hypercentral.

This result has been proved in the paper

KRV2006. Kurdachenko L.A., Russo A., Vincenzi G. *Groups without proper abnormal subgroups.* Journal Group Theory 9 (2006), no 4, 507 – 518.

And here is the last similar result

Let G be a periodic group and suppose that H is a locally nilpotent normal subgroup of G. Assume further that G/H is hypercentral. If G has no proper abnormal subgroups then G is locally nilpotent.

This result has been proved in the paper

KRV2006. Kurdachenko L.A., Otal J., Russo A., Vincenzi G. *Abnormal subgroups and Carter subgroups in some classes of infinite groups.* Journal of Algebra 297 (2006) 273–291

Another criterion of nilpotency are connected with the following type of subgroups.

A subgroup H of a group G is called **contranormal** in G if $G = H^G$.

The term «a contranormal subgroup» has been introduced by J.S. Rose in his paper

RJ1968. J.S. Rose. *Nilpotent subgroups of finite soluble groups.* Math. Zeitschrift – 106(1968), 97 – 112.

We note that every abnormal subgroup is contranormal, but converse is not true. For example, let $G = D \rtimes \langle b \rangle$ where D is a divisible abelian 2 – subgroup and $d^b = d^{-1}$ for each element $d \in D$. It is not hard to see that the subgroup $\langle b \rangle$ is contranormal. But $\langle b \rangle$ cannot be abnormal, because hypercentral groups does not include proper abnormal subgroups.

We note that every maximal subgroup of a group G is conormal in G, the subgroup generated by conormal subgroups of G is conormal in G. On the other hand, the intersection of conormal subgroups can be not conormal.

For the finite groups we have the following criteria of nilpotency.

Finite group G is nilpotent if and only if G does not include proper contranormal subgroups.

In this connection it is natural to consider the structure of infinite groups which does not include proper contranormal subgroups,

Note that in this situation we cannot speak of the nilpotency of these group. Moreover, they could be very far from being nilpotent.

Indeed, the groups whose subgroups are subnormal do not include proper contranormal subgroups. Such groups are locally nilpotent but can be not nilpotent. In this connection, it is suitable to recall the example constructed by H. Heineken and A. Mohamed in the paper

HM 1968. Heineken H., Mohamed A. *A group with trivial centre satisfying the normalizer condition.* J. Algebra 10(3) (1968), 368 – 376.

This is a p – group H with the following properties:

H includes a normal elementary abelian p – subgroup A such that H/A is a Prüfer p – group; every proper subgroup of H is subnormal in G, ζ(H) = <1>.

We say that a group G is **contranormal – free** if G does not include proper contranormal subgroups.

We show here some established and some newer results about contranormal – free groups.

A periodic group G is said to be **Sylow – nilpotent** if G is locally nilpotent and a Sylow p – subgroup of G is nilpotent for each prime p.

It is not hard to see that every Sylow – nilpotent group does not include proper contranormal subgroups.

Let G be a Chernikov group. If G is contranormal – free, then G is nilpotent.

Let G be a locally finite group, every Sylow p – subgroup of which is Chernikov for every prime p. If G is contranormal – free, then G is Sylow – nilpotent.

These results have been obtained in the paper

KS2003. Kurdachenko L.A., Subbotin I.Ya. *Pronormality, contranormality and generalized nilpotency in infinite groups.* Publicacions Matemàtiques, 2003, 47, number 2, 389 – 414

The following results were obtained for the periodic contranormal – free groups.

Let G be a group, H be a locally nilpotent normal subgroups of G such that G/H is hyperfinite. If G is contranormal – free, then G is locally nilpotent.

As corollaries we obtain

Let G be a periodic group, H be a normal locally nilpotent subgroups of G such that G/H is nilpotent. If G is contranormal – free, then G is locally nilpotent.

Let G be a periodic group and H be a normal locally nilpotent subgroup such that G/H is a Chernikov group. If G is contranormal – free, then G is locally nilpotent.

Let G be a locally finite group and H be a normal locally nilpotent subgroup such that the Sylow p – subgroups of G/H are Chernikov for all prime p. If G is contranormal – free, then G is locally nilpotent.

Let G be a hyperfinite group. If G is contranormal – free, then G is hypercentral.

Let G be a periodic group, H be a normal nilpotent subgroups of G such that G/H is nilpotent and $\Pi(H) \cap \Pi(G/H) = \varnothing$. If G is contranormal – free, then G is nilpotent.

These results has been obtained in the paper

KLM2022. A. Kurdachenko L.A., Longobardi P., Maj M., *On the structure of some locally nilpotent groups without contranormal subgroups*, J. Group Theory, 25 (2022) 75 – 90.

If G/H is a Chernikov group, then we obtain assertion (iii) of the paper

WB2020. Wehrfritz B.A.F. *Groups with no proper contranormal subgroups.* **Publicacions Matemàtiques, 2020, 64**, 183 – 194

In the paper [**WB2020**] B.A.F. Wehrfritz proved the following result

Let G be a nilpotent – by – finite group. If G is contranormal – free, then G is nilpotent.

In this connection it is interesting to consider nilpotent – by – (finitely generated) groups. For these groups the following result has been obtained

Let G be a group and H be a nilpotent normal subgroup of G such that G/H is finitely generated and soluble – by – finite. If G is contranormal – free, then G is hypercentral.

This result has been obtained in the paper

DKS2021. Dixon M.R., Kurdachenko L.A., Subbotin I.Ya. *On the structure of some contranormal – free groups* – Comm. Algebra, 49 (11) (2021) 4940 – 4046.

If G is a group and H, K, H ≤ K, are the normal subgroups of G, then, as usual, the factor – group K/H is called the **factor of a group G**. A factor K/H is called **perfect** (more precisely, **G – perfect**), if [K/H, G] = K/H. Otherwise we will say that a factor is not perfect. This means that [K/H, G] ≠ K/H.

A factor K/H is called **central** (more precisely, **G – central**), if [K, G] ≤ H.

We have the following characterization of finite nilpotent groups

Finite group G is nilpotent if and only if every factor of G is not G – perfect.

Therefore the following question arose:

What we can say about the structure of groups, whose non – trivial factors are not G – perfect?

And, in particular,

When the group, whose non – trivial factors are not G – perfect, is nilpotent ?

If G is a group, whose non – trivial factors are not G – perfect, then clearly every chief factor of G is central.

Recall that a group G is called **$\overline{Z}$ – group** is every chief factor of G is G – central. The class of the $\overline{Z}$ – groups is very wide. We note that there exist the $\overline{Z}$ – groups (moreover hypercentral groups) which have perfect factors. For example.

Let D be a divisible abelian 2 – subgroup. Then D has an automorphism ι such that $\iota(d) = d^{-1}$ for each element $d \in D$. Define a semidirect product $G = D \rtimes \langle b \rangle$ such that $d^b = \iota(d) = d^{-1}$ for each element $d \in D$. Let a be an arbitrary element of D. Since D is divisible, there exists an element $d \in D$ such that $d^2 = a$. We have $[b, d] = b^{-1} d^{-1} b\, d = d^2 = a$. It follows that $[b, D] = D$ and therefore $[G, D] = D$. We note that a group G is not nilpotent, however a series

$$\langle 1 \rangle \leq \Omega_1(D) \leq \ldots \leq \Omega_n(D) \leq \Omega_{n+1}(D) \leq \ldots D \leq G$$

is central, so that G is hypercentral abelian – by – finite group.

Note that we cannot say that all groups, whose non – trivial factors are not G – perfect, are nilpotent. Moreover, such groups can be very far from being nilpotent groups. Indeed, H. Heineken and A. Mohamed in the paper [**HM 1968**] constructed the p – group H, p is a prime, satisfying the following conditions: H includes a normal elementary abelian p – subgroup A such that H/A is a Prüfer p – group and every proper subgroup S of H is subnormal in G and nilpotent, SA ≠ H, and ζ(H) = <1>. It is possible to prove that every non – trivial factor of H is not G – perfect.

Another example is a group, which has been constructed by R.I. Grigorchuk in the paper [**GR1980**]. This is a finitely generated infinite p – group, whose non – trivial normal subgroups have finite index. It is almost obvious to see that every non – trivial factor of this group is not G – perfect.

It is not hard to see that every Sylow – nilpotent group has no non – trivial factor G – perfect factor.

Recall that a group G is called ***generalized radical*** if G has an ascending series whose factors are locally nilpotent or locally finite.

It should be noted that every generalized radical group has an ascending series of normal, indeed characteristic, subgroups with locally nilpotent or locally finite factors.

Let p be a prime. We say that a group G has ***finite section p – rank*** $\mathbf{sr}_p(G) = \mathbf{r}$ if every elementary abelian p – section of G is finite of order at most p^r and there is an elementary abelian p – section A/B of G such that |A/B| = p^r.

The group G is said to have ***finite section rank*** if the section p – rank of G is finite for all primes p.

The following results have been obtained recently together with P. Longobardi and M. Maj.

Let G be a locally generalized radical group, having finite section rank. If G has no nontrivial G – perfect factors, then G satisfies the following conditions:

(i) for every prime p there exists a number s_p such that the hypercenter of G, having number s_p, includes the Sylow p – subgroup of G;

(ii) the factor – group $G/\mathbf{Tor}(G)$ is nilpotent.

In particular, G is hypercentral, moreover the hypercentral length of G is at most $\omega + k$ for some positive integer k.

Let G be a group and A be a normal nilpotent subgroup of G such that G/A is a locally finite group whose Sylow p – subgroup are Chernikov for all prime p. If G has no nontrivial G – perfect factors, then G satisfies the following conditions:

(i) for every prime p there exists a positive integer s_p such that the hypercenter of G, having number s_p, includes the Sylow p – subgroup of G;

(ii) the factor – group $G/\mathbf{Tor}(G)$ is nilpotent.

As the corollaries we obtain

Let G be a periodic group and A be a normal nilpotent subgroup of G such that G/A is a locally finite group whose Sylow p – subgroup are Chernikov for all prime p. If G has no nontrivial G – perfect factors, then G is Sylow – nilpotent.

Let G be a group and A be a normal nilpotent subgroup of G such that G/A is finite. If G has no nontrivial G – perfect factors, then G is nilpotent.

Thank

you