Università
degli Studi
della Campania
Luigi Vanvitelli

Dipartimento di Matematica e Fisica

On Groups Factorized by Mutually Permutable Subgroups

AGTA Workshop-Reinhold Baer Prize 2022
September 21-23, Caserta

Maria Ferrara

Introduction

If $G=AB$ and A and B are two abelian subgroups of G

$G=AB=\{ab : a\in A, b\in B\}$

"G is factorized by A and B"

Introduction

If $G=AB$ and A and B are two abelian subgroups of G

$\Downarrow$ N. Itô 1955

G is metabelian!

Università degli Studi della Campania *Luigi Vanvitelli*

Dipartimento di Matematica e Fisica

Introduction

If $G=AB$ and A and B are two abelian subgroups of G

$\Downarrow$ N. Itô 1955

G is metabelian!

$\hookrightarrow$ $G'=[G,G]$ is abelian

Introduction

$G = AB$ with $A, B \leq G$

If $A \in \mathfrak{X}$
$B \in \mathfrak{X}$ $\overset{?}{\Longrightarrow}$ $G \in \mathfrak{X}$

Introduction

$G = AB$ with $A, B \trianglelefteq G$

If $A \in \mathfrak{X}$
$B \in \mathfrak{X}$ $\overset{?}{\Longrightarrow} G \in \mathfrak{X}$

Università
degli Studi
della Campania
Luigi Vanvitelli

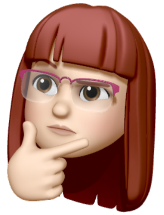
Dipartimento di Matematica e Fisica

Introduction

$G = AB$ with $A, B \trianglelefteq G$

If $A \in \mathfrak{X}$

$B \in \mathfrak{X}$ $\quad \overset{?}{\Longrightarrow} G \in \mathfrak{X}$



supersoluble?!

Introduction

A group G is said to be supersoluble if it has a normal series all of whose factors are cyclic.

Introduction

A group G is said to be supersoluble if it has a normal series all of whose factors are cyclic.

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$$

$G_i \trianglelefteq G$ $\qquad$ G_i / G_{i-1} is cyclic

Examples

Any group of prime power order is supersoluble.

Examples

Any group of prime power order is supersoluble.

$|G| = p^n \to G$ is nilpotent $\to Z(G) \neq \{1\}$

Examples

Any group of prime power order is supersoluble.

$|G| = p^n \Rightarrow G$ is nilpotent $\Rightarrow Z(G) \neq \{1\}$

$x \in Z(G)$, $x \neq 1$, $o(x) = p \rightarrow N = \langle x \rangle \trianglelefteq G$

Examples

Any group of prime power order is supersoluble.

$|G|=p^n \Rightarrow G$ is nilpotent $\Rightarrow Z(G)\neq\{1\}$

$x\in Z(G),\ x\neq 1,\ o(x)=p \rightarrow N=\langle x\rangle \trianglelefteq G$

$$\begin{tikzcd}
G \arrow[d, dash, "\text{by induction}"] \\
N=\langle x\rangle \arrow[d, dash] \\
1
\end{tikzcd}$$

Examples

Any group of prime power order is supersoluble.

The smallest example of a finite non-supersoluble group is the alternating group of order 4.

why?

A_4 has no non-trivial cyclic normal subgroup!

Supersoluble groups

The class of supersoluble groups is closed with respect to forming subgroups, images, and finite direct products.

Supersoluble groups

The class of supersoluble groups is closed with respect to forming subgroups, images, and finite direct products.

The product of two normal supersoluble subgroups need not be supersoluble!

Problem

Under which conditions, will the product of two (normal) supersoluble subgroups still be supersoluble?

Problem

Under which conditions, will the product of two (normal) supersoluble subgroups still be supersoluble?

Theorem (R. Baer 1957)

If a finite group G is the product of two supersoluble normal subgroups

Problem

Under which conditions, will the product of two (normal) supersoluble subgroups still be supersoluble?

Theorem (R. Baer 1957)

If a finite group G is the product of two supersoluble normal subgroups and G' is nilpotent, then G is supersoluble.

Problem

Under which conditions, will the product of two (normal) supersoluble subgroups still be supersoluble?

Theorem (D. R. Friesen 1971)

If a finite group G is the product of two normal supersoluble subgroups of coprime indices, then G is supersoluble.

If G is the product of two nilpotent subgroups, then G is not necessarily supersoluble.

Theorem (O.H. Kegel 1965)

If G is a finite group such that $G=HK=HL=KL$ where H and K are nilpotent subgroups and L is supersoluble, then G is supersoluble.

Problem

Under which conditions, will the product of two (normal) supersoluble subgroups still be supersoluble?

Theorem (R. Baer 1957)

If a finite group G is the product of two supersoluble normal subgroups and G' is nilpotent, then G is supersoluble.

Generalizations

Theorem (M. Asaad and A. Shaalan, 1989)

- Suppose that A and B are supersoluble subgroups of a finite group G, G' is nilpotent and $G=AB$.

Generalizations

Theorem (M. Asaad and A. Shaalan, 1989)

- Suppose that A and B are supersoluble subgroups of a finite group G, G' is nilpotent and $G=AB$.

- Suppose further that A is permutable with every subgroup of B

Generalizations

Theorem (M. Asaad and A. Shaalan, 1989)

- Suppose that A and B are supersoluble subgroups of a finite group G, G' is nilpotent and $G=AB$.

- Suppose further that A is permutable with every subgroup of B

Two subgroups H and K of a group G are said to permute if $HK=KH$

Generalizations

Theorem (M. Asaad and A. Shaalan, 1989)

- Suppose that A and B are supersoluble subgroups of a finite group G, G′ is nilpotent and G=AB.

- Suppose further that A is permutable with every subgroup of B and B is permutable with every subgroup of A.

Generalizations

Theorem (M. Asaad and A. Shaalan, 1989)

- Suppose that A and B are supersoluble subgroups of a finite group G, G' is nilpotent and $G=AB$.
- Suppose further that A is permutable with every subgroup of B and B is permutable with every subgroup of A.

Then G is supersoluble.

Let G be a group.

Two subgroups A and B of G are said to be **mutually permutable** if

Let G be a group.

Two subgroups A and B of G are said to be mutually permutable if

$$AY = YA \text{ and } XB = BX$$

for all subgroups Y of B and X of A.

Let G be a group.

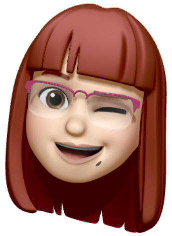
Two subgroups A and B of G are said to be mutually permutable if

$$AY = YA \text{ and } XB = BX$$

for all subgroups Y of B and X of A.

Examples

Any two normal subgroups are mutually permutable.



Generalizations

Theorem (M. Asaad and A. Shaalan, 1989)

- Suppose that A and B are supersoluble subgroups of a finite group G, G' is nilpotent and $G=AB$.
- Suppose further that A and B are mutually permutable.

Then G is supersoluble.

Groups which are products of two mutually permutable subgroups have been recently investigated by several authors, especially in the finite case.

Groups which are products of two mutually permutable subgroups have been recently investigated by several authors, especially in the finite case.

M. Asaad, A. Ballester-Bolinches, R. Esteban-Romero,
Products of Finite Groups
de Gruyter, Berlin (2010)

For the case of infinite groups see

J.C. Beidleman, H. Heineken,
Totally permutable torsion groups
J.Group Theory 2 (1999), 377–392

J.C. Beidleman, H. Heineken,
Mutually permutable subgroups and groups classes
Arch. Math. (Basel) 85 (2005), 18–30

For the case of infinite groups see

M. De Falco, F. de Giovanni, C. Musella,
Locally finite products of totally permutable nilpotent groups
Algebra Colloq. 16 (2009), 535–540

F. de Giovanni, R. Ialenti,
Groups with finite abelian section rank factorized by mutually permutable subgroups
Comm. Algebra 44 (2016), 118-124

For the case of infinite groups see

J.C. Beidleman, H. Heineken,
Totally permutable torsion groups
J.Group Theory 2 (1999), 377–392

J.C. Beidleman, H. Heineken,
Mutually permutable subgroups and groups classes
Arch. Math. (Basel) 85 (2005), 18–30

Theorem (J.C. Beidleman and H. Heineken, 2005)

If $G=AB$ is a finite group which is factorized by two mutually permutable subgroups A and B, then

Theorem (J.C. Beidleman and H. Heineken, 2005)

If $G=AB$ is a finite group which is factorized by two mutually permutable subgroups A and B, then A', B' are subnormal subgroups of G.

In general, H is subnormal in G if there is a series $H=H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_n = G$

Theorem (J.C. Beidleman and H. Heineken, 2005)

If $G=AB$ is a finite group which is factorized by two mutually permutable subgroups A and B, then A', B' are subnormal subgroups of G.

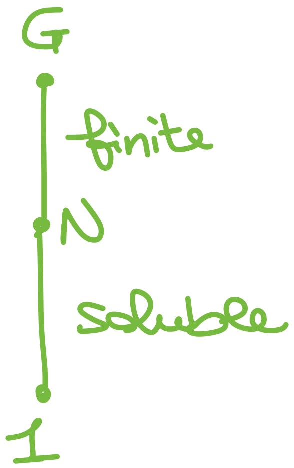
If A' and B' are nilpotent

$\Downarrow$

$\langle A', B'\rangle^G$ is nilpotent

Theorem (F. de Giovanni and R. Ialenti, 2016)

Let $G = AB$ be a soluble-by-finite group



G has $N \trianglelefteq G$, N soluble such that G/N is finite

Theorem (F. de Giovanni and R. Ialenti, 2016)

Let $G = AB$ be a soluble-by-finite group with finite abelian section rank

Finite abelian section rank

Let G be an abelian group and let S be a non-empty subset of G.

Finite abelian section rank

Let G be an abelian group and let S be a non-empty subset of G.

Then S is called linearly independent, if $0 \notin S$ and, given distinct elements $s_1, \ldots, s_r$ of S and integers $m_1, \ldots, m_r$, the relation

$m_1 s_1 + \ldots + m_r s_r = 0$ implies that $m_i s_i = 0$ for all i.

Finite abelian section rank

If p is a prime and G an abelian group, the p-rank of G, $r_p(G)$ is defined as the cardinality of a maximal independent subset of elements of p-power order.

Similarly the 0-rank $r_0(G)$ is the cardinality of a maximal independent subset of elements of infinite order.

Finite abelian section rank

A group **has finite abelian subgroup rank** if each abelian subgroup has finite 0-rank and finite p-rank for all primes p.

A group G has **finite abelian section rank** if every abelian section of G has finite 0-rank and finite p-rank for all primes p.

Theorem (F. de Giovanni and R. Ialenti, 2016)

Let $G = AB$ be a soluble-by-finite group with finite abelian section rank which is factorized by two mutually permutable finite-by-nilpotent subgroups A and B.

Theorem (F. de Giovanni and R. Ialenti, 2016)

Let $G = AB$ be a soluble-by-finite group with finite abelian section rank which is factorized by two mutually permutable finite-by-nilpotent subgroups A and B.

If A' and B' are locally nilpotent, then also the normal closure $\langle A', B' \rangle^G$ is locally nilpotent.

Università
degli Studi
della Campania
Luigi Vanvitelli

Dipartimento di Matematica e Fisica

Maria Ferrara and Marco Trombetti

On groups factorized by mutually permutable subgroups

Results in Mathematics, to appear

Theorem (F. de Giovanni and R. Ialenti, 2016)

Let $G = AB$ be a soluble-by-finite group with finite abelian section rank which is factorized by two mutually permutable finite-by-nilpotent subgroups A and B.

If A' and B' are locally nilpotent, then also the normal closure $\langle A', B' \rangle^G$ is locally nilpotent.

Theorem (F. de Giovanni and R. Ialenti, 2016)

Let $G = AB$ be a soluble-by-finite group with finite abelian section rank which is factorized by two mutually permutable ~~finite-by-nilpotent~~ subgroups A and B.

If A' and B' are locally nilpotent, then also the normal closure $\langle A', B'\rangle^G$ is locally nilpotent.

Theorem (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ be a soluble-by-finite group with finite abelian section rank which is factorized by two mutually permutable subgroups A and B.

If A' and B' are locally nilpotent, then also the normal closure $\langle A', B' \rangle^G$ is locally nilpotent.

Theorem (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ be a soluble-by-finite group with finite abelian section rank which is factorized by two mutually permutable subgroups A and B.

If A' and B' are locally nilpotent, then also the normal closure $\langle A', B' \rangle^G$ is locally nilpotent.

A group is called a minimax group if it has a series of finite length whose factors satisfy Max or Min.

A group is called a minimax group if it has a series of finite length whose factors satisfy Max or Min.

In the following, $\mathscr{M}$ denotes the class of minimax groups containing a soluble subgroup of finite index.

For any group G, $\rho_{\mathscr{L}\mathscr{N}}(G)$ denotes the Hirsch-Plotkin radical of G.

For any group G, $\rho_{\mathscr{L}\mathscr{N}}(G)$ denotes the Hirsch-Plotkin radical of G.

In a group G there is a unique maximal normal locally nilpotent subgroup $\rho_{\mathscr{L}\mathscr{N}}(G)$ (called the **Hirsch-Plotkin radical**) containing all normal locally nilpotent subgroups of G.

Theorem (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ is a locally-$\mathcal{M}$ group

Theorem (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ is a locally-$\mathcal{M}$ group which is factorized by two mutually permutable subgroups A and B

Università
degli Studi
della Campania
Luigi Vanvitelli

Dipartimento di Matematica e Fisica

Theorem (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ is a locally-$\mathscr{M}$ group which is factorized by two mutually permutable subgroups A and B, then

$$\langle \rho_{\mathscr{L}\mathscr{N}}(A'), \rho_{\mathscr{L}\mathscr{N}}(B') \rangle^G \leq \rho_{\mathscr{L}\mathscr{N}}(G')$$

If G is a group, we denote by $\rho_{0,\mathscr{L}\mathscr{N}}(G) = \{1\}$.

If G is a group, we denote by $\rho_{0,\mathscr{L}\mathscr{N}}(G) = \{1\}$.

If we have defined the subgroup $\rho_{i,\mathscr{L}\mathscr{N}}(G)$ for some positive integer i, then we put

$$\rho_{i+1,\mathscr{L}\mathscr{N}}(G)/\rho_{i,\mathscr{L}\mathscr{N}}(G) = \rho_{\mathscr{L}\mathscr{N}}(G/\rho_{i,\mathscr{L}\mathscr{N}}(G))$$

If G is a group, we denote by $\rho_{0,\mathscr{L}\mathscr{N}}(G) = \{1\}$.

If we have defined the subgroup $\rho_{i,\mathscr{L}\mathscr{N}}(G)$ for some positive integer i, then we put

$$\rho_{i+1,\mathscr{L}\mathscr{N}}(G)/\rho_{i,\mathscr{L}\mathscr{N}}(G) = \rho_{\mathscr{L}\mathscr{N}}(G/\rho_{i,\mathscr{L}\mathscr{N}}(G))$$

If G is a group, we denote by $\rho_{0,\mathscr{L}\mathscr{N}}(G) = \{1\}$.

If we have defined the subgroup $\rho_{i,\mathscr{L}\mathscr{N}}(G)$ for some positive integer i, then we put

$$\rho_{i+1,\mathscr{L}\mathscr{N}}(G)/\rho_{i,\mathscr{L}\mathscr{N}}(G) = \rho_{\mathscr{L}\mathscr{N}}(G/\rho_{i,\mathscr{L}\mathscr{N}}(G))$$

This defines the subgroups $\rho_{n,\mathscr{L}\mathscr{N}}(G)$ for any integer $n \geq 0$.

Clearly, $\rho_{\mathscr{L}\mathscr{N}}(G) = \rho_{1,\mathscr{L}\mathscr{N}}(G)$.

Corollary 1 (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ be a locally-$\mathcal{M}$ group

Corollary 1 (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ be a locally-$\mathscr{M}$ group which is factorized by two mutually permutable (non-trivial) soluble subgroups A and B.

Corollary 1 (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ be a locally-$\mathscr{M}$ group which is factorized by two mutually permutable (non-trivial) soluble subgroups A and B.

If the derived length of A is c and the derived length of B is d, then $\rho_{k,\mathscr{L}\mathscr{N}}(G'') = G''$,

Corollary 1 (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ be a locally-$\mathscr{M}$ group which is factorized by two mutually permutable (non-trivial) soluble subgroups A and B. If the derived length of A is c and the derived length of B is d, then $\rho_{k,\mathscr{L}\mathscr{N}}(G'') = G''$, where k denotes the maximum between $c-1$ and $d-1$.

Theorem (J.C. Beidleman - H. Heineken, 2005)

Let $G = AB$ be a finite group which is factorized by two mutually permutable soluble subgroups A and B.

Theorem (J.C. Beidleman - H. Heineken, 2005)

Let $G = AB$ be a finite group which is factorized by two mutually permutable soluble subgroups A and B.

Then G is soluble.

Corollary 2 (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ be a locally (soluble-by-finite) group of finite rank which is factorized by two mutually permutable soluble subgroups A and B.

Corollary 2 (M. Ferrara and M. Trombetti, 2022)

Let $G = AB$ be a locally (soluble-by-finite) group of finite rank which is factorized by two mutually permutable soluble subgroups A and B.

Then G is hyperabelian.

A group G is said hyperabelian if it has an ascending normal series with abelian factor.

Università degli Studi della Campania *Luigi Vanvitelli*

Dipartimento di Matematica e Fisica

Thank you for listening!