Groups containing a large subgroup which satisfies a central-like embedding property

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Schur's Theorem

Theorem (I. Schur, 1904)

Let G be a group.
If $G/Z(G)$ is finite, then G' is finite.

Note that
$H \leqslant Z(G) \Leftrightarrow \forall g \in G, \langle H, g \rangle$ is abelian.

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The structure of non periodic quasihamiltonian groups

Theorem (K. Iwasawa 1943)
Let G be a nonperiodic group. Then G is quasihamiltonian if and only if the following conditions hold:

(1) The set T of all elements of finite order is a locally finite subgroup containing G';

(2) all subgroups of T are normal in G and all elements of order prime or of order 4 are central in G;

(3) if G is not abelian, then G/T has rank 1.

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The structure of periodic quasihamiltonian groups

Theorem (K. Iwasawa 1943)
Let G be a periodic group. Then G is quasihamiltonian if and only if $G = Dr_{i \in I} G_i$ where the subgroups $\{G_i \mid i \in I\}$ are primary groups pairwise coprime and each G_i satisfies one of the following conditions:

(i) all subgroups of G_i are normal in G_i;

(ii) G_i has finite exponent and contains an abelian normal subgroup A such that G_i/A is cyclic; meoreover if p^k is the exponent of A and p^m is the order of $G_i/A = \langle bA \rangle$, then there exists a positive integer s such that $s < k \leqslant s + m$ and $a^b = a^{1+p^s}$ $\forall a \in A$.

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A theorem of Schur's type

Definition. A subgroup H of a group G is said to be *permutably embedded* in G if $\langle H, g\rangle$ is quasihamiltonian for every $g \in G$.

Theorem (MDF, F. de Giovanni, C. Musella, 2008)

Let G be a group.
If G contains a permutably embedded subgroup of finite index, then there exists a finite normal subgroup N such that G/N is quasihamiltonian.

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Lattice theoretical properties of quasihamiltonian groups

Let G be a quasihamiltonian group.
Let H, K, L be subgroups of G such that $H \leqslant L$.
Then $\langle H, K\rangle \cap L = \langle H, K \cap L\rangle$.

Definition. Let L be a modular lattice. L is called *permodular* if for all elements a and b such that $b \leqslant a$ and the interval $[a/b]$ has finite length, we have that $[a/b]$ is finite.

If G is a quasihamiltonian group, then the subgroup lattice $\mathcal{L}(G)$ is permodular.

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Definition. A subgroup H of a group G is said to be *permodularly embedded* in G if $\langle H, g\rangle$ has permodular subgroup lattice for every $g \in G$.

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Central-by-Černikov groups

Theorem (S.N. Černikov, 1959)

Let G be a group.
If $G/Z(G)$ is a Černikov group, then G' is a Černikov group.

Theorem (A. Schlette, 1969)

Let G be a group.
If $G/Z(G)$ is a Černikov group, then G is abelian-by-finite.
More precisely, if $J/Z(G)$ is the largest abelian divisible subgroup of $G/Z(G)$, then J is abelian.

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Groups which are Černikov over a pemutably embedded normal subgroup

Theorem (MDF, F. de Giovanni, C. Musella, 2022)

Let G be a group.
If G contains a permutably embedded normal subgroup Q such that G/Q is a Černikov group, then G is quasihamiltonian-by-finite.
More precisely, if J/Q is the largest abelian divisible subgroup of G/Q, then J is quasihamiltonian.

Theorem (MDF, F. de Giovanni, C. Musella, 2022)

Let G be a periodic group.
If G contains a permutably embedded normal subgroup Q such that G/Q is a Černikov group, then G has a normal Černikov subgroup N such that G/N is quasihamiltonian.

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Let G be a group.
If G contains a permutably embedded subgroup Q such that the interval [G/Q] is distributive, then G is quasihamiltonian.

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Central-by-polycyclic groups

Proposition

Let G be a group.
If $G/Z(G)$ is polycyclic, then G' is polycyclic.

Proof - Put Z=Z(G) and let E be a finitely generated subgroup of G such that $G = EZ$.
Put $K = Z \cap E$. Then there exists a finitely generated subgroup F such that $K = F^E$; but then $K = F$
$\Rightarrow$ K is polycyclic $\Rightarrow$ E is polycyclic.
It follows that G' is polycyclic. $\square$

Corollary

Let G be a group.
If $G/Z(G)$ is supersoluble, then G' is supersoluble.

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Groups which are polycyclic over a pemutably embedded normal subgroup

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If G contains a non periodic permutably embedded normal subgroup Q such that G/Q is polycyclic, then G has a normal polycyclic subgroup N such that G/N is quasihamiltonian.

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Maximal subgroups of groups which are polycyclic over a pemutably embedded normal subgroup

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Proof - Let E be any finitely generated subgroup of G and put $K = E \cap Q$.
$E/K \simeq EQ/Q$ is polycyclic and K is soluble $\Rightarrow$ E is soluble.
If E/K is finite, then E is finite-by-quasihamiltonian and hence polycyclic.

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Assume that E/K is infinite $\Rightarrow$ E/K contains a torsion-free non-trivial subgroup of finite index Y/K.
There exists a finitely generated subgroup F such that $K = F^E$.
Using the structure of quasihamiltonian groups, we can prove that every subgroup of K is normalized by Y
$\Rightarrow$ K is contained in the FC-centre of E
$\Rightarrow$ K it is finitely generated and hence polycyclic
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An example

Example There exists a group G containing a permutably embedded subgroup Q such that G/Q is polycyclic, but G is not polycyclic-by-quasihamiltonian.

Let p be a prime number and let Q be a group of type p^∞
$\Rightarrow Aut(Q)$ is isomorphic to the multiplicative group of p-adic integers
$\Rightarrow Aut(Q)$ contains a free abelian subgroup X of rank 2 whose elements fix every element of Q of order p or of order 4.
Let G be the semidirect product of X and Q
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Assume for a contradiction that there exists a polycyclic normal subgroup N such that G/N is quasihamiltonian.

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$\forall g \in G,\ [Q, g]$ is a divisible subgroup of Q.

If $x \in X \cap N$, $[Q, x] \leqslant N$, so that $[Q, x] = \{1\}$.

$\Rightarrow X \cap N = \{1\} \Rightarrow G/N$ has torsion-free rank at least 2.

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Thank you!