Equivariant cohomology, lattices, and trees

Sam Hughes

Reinhold Baer Prize 2022

Lattices in locally compact groups

Let $H = \mathrm{Isom}(X)$ be a locally compact group with Haar measure μ. A discrete subgroup $\Gamma \leq H$ is:

- a *lattice* if X/Γ has finite covolume;
- a *uniform lattice* if X/Γ is compact.

For a lattice Γ in a product $\prod_{i=1}^{n} H_i$ we say Γ is:

- *irreducible* Γ does not have a finite index subgroup which splits as a direct product of two infinite groups;
- *reducible* otherwise.

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On $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices

- It was thought every $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice was reducible i.e. virtually $\mathbb{Z}^n \times F_m$.
- **Leary-Minasyan 2019:** Constructed groups $\mathrm{LM}(A)$ using a graph with a single vertex and edge which are irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices.
- These groups are not linear, residually finite, or biautomatic answering a 25 year old question about whether every $\mathrm{CAT}(0)$ group is biautomatic.

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Graphs of lattices

Definition (Graph of lattices)

Let H be a locally compact group with Haar measure μ. A graph of H-lattices $(A, \mathcal{A}, \psi)$ is a graph of groups $(A, \mathcal{A})$ equipped with a morphism $\psi : \mathcal{A} \to H$ such that:

1. *Each local group $A_\sigma \in \mathcal{A}$ is covirtually an H-lattice and the image $\psi(A_\sigma)$ is an H-lattice;*
2. *The local groups are commensurable in $\Gamma = \pi_1(\mathcal{A})$ and their images are commensurable in H;*
3. *For each $e \in EA$ the element t_e of the path group $\pi(\mathcal{A})$ is mapped under ψ to an element of $\mathrm{Comm}_H(\psi_e(A_e))$.*

A rough classification

Let $(A, \mathcal{A}, \psi)$ be a graph of $\mathrm{Isom}(\mathbb{E}^n)$-lattices with Bass-Serre tree $\mathcal{T}$ and fundamental group Γ. Let $T = \mathrm{Aut}(\mathcal{T})$.

Theorem (H. 2021)

Assume A is finite. If for each local group A_σ, the kernel $\mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$, then Γ is a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice and hence a $\mathrm{CAT}(0)$ group. Conversely, if Λ is a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice, then Λ splits as a finite graph of uniform H-lattices with Bass-Serre tree $\mathcal{T}$.

Generic properties?

Theorem (H. 2021)

Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{E}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. The following are equivalent:

1. Γ *is an irreducible* $(\mathrm{Isom}(\mathbb{E}^n) \times T)$*-lattice;*
   (topological)
2. Γ *acts on* $\mathcal{T}$ *faithfully;*
   (geometric)
3. Γ *does not virtually fibre;*
   (homological)
4. Γ *is* C^**-simple;*
   (analytic)
5. *and if* $n = 2$, Γ *is not virtually biautomatic.*
   (language theoretic)

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Cohomology computations

In my thesis I did a number of cohomology and K-theoretic calculations:

- The ring $H^*(\Gamma;\mathbb{Z})$ for Γ a lattice in $\mathrm{PSL}_2(\mathbb{R})$;
- The groups $H^n(\Gamma;\mathbb{Z})$ for Γ a lattice in $\mathrm{PGL}_2(\mathbb{R})$;
- The groups $K_n^\Gamma(\underline{E}\Gamma)$ for Γ isomorphic to $\mathrm{PSL}_2(\mathbb{Z}[1/p])$ or $\mathrm{SL}_2(\mathbb{Z}[1/p])$;
- The groups $KO_n^\Gamma(\underline{E}\Gamma)$ for Γ isomorphic to $\mathrm{SL}_3(\mathbb{Z})$ and $\mathrm{GL}_3(\mathbb{Z})$;
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Relevant preprints and publications

7. Sam Hughes and Motiejus Valiunas. Commensurating HNN extensions: hierarchical hyperbolicity and biautomaticity, 2022, submitted. arXiv: 2203.11996 [Math.GR]
6. Sam Hughes. Irreducible lattices fibring over the circle, 2022, submitted. arXiv: 2201.06525 [Math.GR]
5. Sam Hughes. Graphs and complexes of lattices, 2021, submitted. arXiv: 2104.13728 [Math.GR]
4. Sam Hughes. Lattices in a product of trees, hierarchically hyperbolic groups, and virtual torsion-freeness. *Bulletin of the London Mathematical Society* **54**(4), 1413–1419, 2022.
3. Indira Chatterji, Sam Hughes, and Peter Kropholler. Groups acting on trees and the first ℓ^2-Betti number. *Proceedings of the Edinburgh Mathematical Society*, **64**(4), 916–923, 2021.
2. Sam Hughes. On the equivariant K- and KO-homology of some special linear groups. *Algebraic and Geometric Topology*, **21**(7), 3483–3512, 2021.
1. Sam Hughes. Cohomology of Fuchsian and non-Euclidean crystallographic groups, 2019, to appear in *Manuscripta Mathematica*.

Note that 6 and 7 grew out of work from my thesis.

Thank you!

Thank you for listening!