# Some topics on finite $p$-groups and pro-$p$ groups

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# Word problems in finite $p$-groups

# Derived subgroups with $\leq 3$ generators

We generalised and improved Guralnick's results on the width of the commutator word in finite $p$-groups, removing the condition that the derived subgroup is abelian.

**Theorem (I.H., G. Fernández-Alcober)**

*Let $G$ be a finite $p$-group with 2-generator derived subgroup. Then there exists $x \in G$ such that*

$$G' = \{[x, g] \mid g \in G\}.$$

**Theorem (I.H.)**

*Let $G$ be a finite $p$-group with $p \geq 5$. If $G'$ is 3-generator, then*

$$G' = \{[x, y] \mid x, y \in G\}.$$

# Further related results

If $G'$ has more than 3 generators, then the equality need not hold. Adding some extra conditions, we obtain the following.

## Theorem (I.H.)

*Let $G$ be a finite $p$-group and write $d = \log_p |G' : (G')^p|$. If $d \leq p - 1$ and the action of $G$ on $G'$ is uniserial modulo $(G')^p$, then there exists $x \in G$ such that $G' = \{[x, g] \mid g \in G\}$.*

If we consider lower central words, Guralnick's result can also be generalised by removing the assumption that $\gamma_r(G)$ is abelian.

## Theorem (I.H., M. Morigi)

*Let $G$ be a finite $p$-group with $p \geq 3$ and suppose $\gamma_r(G)$ is 2-generator. Then, there exist $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r \in G$ such that*

$$\gamma_r(G) = \{[x_1, \ldots, x_{j-1}, g, x_{j+1}, \ldots, x_r] \mid g \in G\}.$$

# Cyclic verbal subgroups of outer commutator words

For general outer commutator words, the problem becomes much harder. We solved the problem for the metabelian word.

## Theorem (I.H.)

*Let $G$ be a finite $p$-group. If $G''$ is cyclic, then*

$$G'' = \{ [[x_1, x_2], [x_3, x_4]] \mid x_1, x_2, x_3, x_4 \in G \}.$$

## Open Problem

*Let $w$ be an outer commutator word. Is it true that if the verbal subgroup $w(G)$ is cyclic, then $w(G)$ consists only of $w$-values?*

# Hausdorff dimension in profinite groups

# Normal Hausdorff spectrum

For a countably based profinite group $G$ and for a filtration $\mathcal{S}$ of $G$, we denote the Hausdorff dimension of a subset $X \subseteq G$ with respect to $\mathcal{S}$ by $\mathrm{hdim}_G^{\mathcal{S}}(X)$.

## Definition

Let $G$ be a countably based profinite group and let $\mathcal{S}$ be a filtration of $G$. The *normal Hausdorff spectrum* of $G$ with respect to the filtration $\mathcal{S}$ is

$$\mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}}(G) = \{\mathrm{hdim}_G^{\mathcal{S}}(H) \mid H \trianglelefteq_c G\} \subseteq [0, 1].$$

Does there exist a finitely generated pro-$p$ group with full normal Hausdorff spectra with respect to some of the standard filtrations?

(Standard filtrations: The lower $p$-series, the dimension subgroup series, the $p$-power series, the iterated $p$-power series and the Frattini series.)

# Full normal Hausdorff spectra

We answered the previous question in the positive for all primes $p$.

**Theorem (I.H., B. Klopsch)**

*There exists a 2-generator pro-p group $G$, with $p$ odd, such that*

$$\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(G) = [0,1],$$

*where $\mathcal{S}$ is any of the five standard filtrations series.*

**Theorem (I.H., A. Thillaisundaram)**

*There exists a 2-generator pro-2 group $G$ such that*

$$\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(G) = [0,1],$$

*where $\mathcal{S}$ is any of the five standard filtrations series.*

# Powerfully solvable and powerfully simple groups

# Powerfully solvable groups

## Definition

A powerful group $G$ is said to be *powerfully solvable* if there exists a chain

$$G = G_n \geq G_{n-1} \geq \cdots \geq G_1 \geq G_0 = 1$$

such that $[G_i, G_i] \leq G_{i-1}^p$ for $1 \leq i \leq n$.

## Theorem (I.H., G. Traustason)

*Let $G$ be a finite $p$-group. If $|G| \leq p^5$, then $G$ is powerfully solvable. There are $22 + 2p$ such groups.*

## Theorem (I.H., G. Traustason)

*The number of powerfully solvable groups of exponent $p^2$ and order $p^n$ is $p^{\alpha n^3 + o(n^3)}$, where $\alpha = \frac{-1+\sqrt{2}}{6}$.*

# Groups of type $(2, \dots, 2)$

**Definition**

A powerful group is said to be of type $(2, \overset{r}{\dots}, 2)$ if it has a basis $\{g_1, \dots, g_r\}$ such that $o(g_i) = p^2$ for all $1 \leq i \leq r$.

Let $\mathcal{P}$ be the category of groups of type $(2, \dots, 2)$.

We write $N \trianglelefteq_{\mathcal{P}} G$ for $N \in \mathcal{P}$ and $N$ powerfully embedded in $G$.

**Definition**

Let $G \in \mathcal{P}$. We say that $G$ is powerfully simple if $G \neq 1$ and if $N \trianglelefteq_{\mathcal{P}} G$ implies $N = 1$ or $N = G$.

**Theorem (I.H., G. Traustason)**

*The category $\mathcal{P}$ and the category of finite dimensional alternating algebras over $\mathbb{F}_p$ are isomorphic*

# A Jordan-Hölder type theorem

## Definition

Let $G \in \mathcal{P}$. A chain $1 = H_0 \triangleleft_{\mathcal{P}} H_1 \triangleleft_{\mathcal{P}} \cdots \triangleleft_{\mathcal{P}} H_n = G$ is a *powerful composition series* of $G$ if all the factors $H_1/H_0, \ldots, H_n/H_{n-1}$ are powerfully simple.

## Theorem (I.H., G. Traustason)

*Let $G$ be a group in $\mathcal{P}$ with two powerful composition series, say*

$$1 = H_0 \triangleleft_{\mathcal{P}} H_1 \triangleleft_{\mathcal{P}} \cdots \triangleleft_{\mathcal{P}} H_n = G$$

*and*

$$1 = K_0 \triangleleft_{\mathcal{P}} K_1 \triangleleft_{\mathcal{P}} \cdots \triangleleft_{\mathcal{P}} K_m = G.$$

*Then $m = n$ and the powerfully simple factors $H_1/H_0, \ldots, H_n/H_{n-1}$ are isomorphic to $K_1/K_0, \ldots, K_n/K_{n-1}$ (in some order).*

Grazie mille!
Eskerrik asko!