



A Note on Finite Groups with Given CSS-subgroups

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Abstract

For a finite group G , a subgroup H of G is called a CSS-subgroup of G if G possesses a normal subgroup K such that $G = HK$ and $H \cap K$ is an SS-quasinormal subgroup in G . In this note, we prove that for any saturated formation $\mathfrak{F}$ containing the class of all supersolvable groups, $G \in \mathfrak{F}$ if and only if G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and every non-cyclic Sylow subgroup P of N has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ or $2|D|$ (if P is a non-abelian 2-subgroup and $|P : D| > 2$) not having a supersolvable supplement in G is a CSS-subgroup of G . Moreover, we show that the above result is still true if we replace the condition “ P of N ” by “ P of $F^*(N)$ ”, where $F^*(N)$ is the generalized Fitting subgroup of N . These two results extend recent results of Diao and Liu in [3] and a series of classical and recent results in the literature.

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1 Introduction

The groups considered in this note are assumed to be finite and G always stands for a finite group. All notions and notation used here are standard and can be found in [2].

In 1962, Kegel called a subgroup H of G *S-permutable* in G if H permutes with all Sylow subgroups P of G (see [7]). Subsequently,

in 1996, Wang [11] called a subgroup H of G *c-normal* in G if G has a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the largest normal subgroup of G contained in H . In 2008, Li et al. [8] generalized the concept of *S-quasinormality* by giving the following concept: A subgroup H of G is called *SS-quasinormal* in G whenever G has a subgroup B such that $G = HB$ and H permutes with all Sylow subgroups of B . By examining the symmetric group S_4 , we can see that all *S-quasinormal* subgroups are *SS-quasinormal*, but the converse is not true in general. Recently, in 2019, Zhao et al. [12] obtained a new class of embedding subgroups as follows: A subgroup H of a group G is said to be a *CSS-subgroup* of G if G possesses a normal subgroup K such that $G = HK$ and $H \cap K$ is *SS-quasinormal* in G . Obviously, the class of all *SS-quasinormal* and *c-normal* subgroups are contained in the class of all *CSS-subgroups*. However, the converse is not true in general (see Examples 1 and 2 in [12]). Moreover, using this new concept, in 2020, Zhao et al. [13] studied the structure of finite groups under the assumption that certain subgroups of G are *CSS-subgroups* of G . Indeed, they proved that if $\mathfrak{F}$ is any saturated formation containing the class of all supersolvable groups, if G has a normal solvable subgroup N such that $G/N \in \mathfrak{F}$ and every maximal subgroup of any Sylow subgroup of $F(N)$ is a *CSS-subgroup* of G , then $G \in \mathfrak{F}$. The influence of *CSS-subgroups* on the structure of finite groups was further explored by Heliel et al. [4] in 2021. In fact, they proved that $G \in \mathfrak{F}$ if and only if G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and every subgroup of N (or of $F^*(N)$) of prime order or of order 4 (if $p = 2$) is a *CSS-subgroup* of G .

Heliel et al. [4] raises four questions concerning the structure of finite groups with certain assumptions using the notion of *CSS-subgroups*. In 2023, Diao and Liu [3] answered the first one. More precisely, they proved that, if p is the smallest prime divisor of $|G|$ and P is a Sylow p -subgroup of G , then if P has a subgroup D such that $1 < |D| < |P|$ and every subgroup of P with order $|D|$ or $2|D|$ (if P is a non-abelian 2-subgroup and $|D| = 2$) is a *CSS-subgroup* of G , then G is p -nilpotent. The previous result opened the door for us to give affirmative answer to the rest of the questions listed in [4]. More precisely, we state and prove the following two results.

Theorem 1.1 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and each non-cyclic Sylow subgroup P of N possesses a subgroup D with $1 < |D| < |P|$ such that every subgroup of P with*

order $|D|$ or $2|D|$ (if P is a non-abelian 2-subgroup and $|P : D| > 2$) not having a supersolvable supplement in G is a CSS-subgroup of G .

Theorem 1.2 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and each non-cyclic Sylow subgroup P of $F^*(N)$ possesses a subgroup D with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ or $2|D|$ (if P is a non-abelian 2-subgroup and $|P : D| > 2$) is a CSS-subgroup of G .*

2 Preliminaries

For the sake of convenience, we summarize in this section some basic results that will be used repeatedly in the proofs.

Lemma 2.1 (see [8], Lemma 2.2) *Let H be a nilpotent subgroup of G . Then H is S -permutable in G if and only if $H \leq F(G)$ and H is SS -quasinormal in G .*

Lemma 2.2 *Let H be a CSS-subgroup of G .*

- (1) *If $H \leq M \leq G$, then H is a CSS-subgroup of M .*
- (2) *If N is a normal subgroup of G , then HN/N is a CSS-subgroup of G/N provided that $(|H|, |N|) = 1$.*
- (3) *If N is a normal subgroup of G such that $N \leq H$, then H/N is a CSS-subgroup of G/N .*
- (4) *If H is a p -subgroup of G such that H is not SS -quasinormal in G , then G possesses a normal subgroup M such that*

$$G = HM \quad \text{and} \quad |G : M| = p.$$

PROOF — (1)–(3) can be found in [12, Lemma 2.3]. For (4), the hypothesis guarantees that G possesses a normal subgroup K such that

$$G = HK \quad \text{and} \quad H \cap K < H.$$

Since G/K is a p -group, we can pick a maximal subgroup M of G with $O^p(G) \leq K \leq M < G$ satisfying the properties in (4). $\square$

Recall that a subgroup H of G is called *weakly S-permutable in G* if G possesses a subnormal subgroup T such that

$$G = HT \quad \text{and} \quad T \cap H \leq H_{sG},$$

where H_{sG} is the subgroup of G generated by all those subgroups of H which are S -permutable in G (see [10]).

The result below highlights an important connection between the two concepts of CSS-subgroups and weakly S -permutable subgroups.

Lemma 2.3 *Let P be a normal p -subgroup of G . If a subgroup H of P is a CSS-subgroup of G , then H is also weakly S -permutable in G .*

PROOF — By hypothesis, G possesses a normal subgroup K such that $G = HK$ and $H \cap K$ is SS -quasinormal in G . It remains to show that $H \cap K \leq H_{sG}$. Besides, Lemma 2.1 implies that $H \cap K$ is an S -permutable subgroup in G as

$$H \cap K \leq P \leq F(G).$$

Consequently, H is weakly S -permutable in G , as required. $\square$

Lemma 2.4 (see [10], Lemma 2.16) *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. If N is a normal cyclic subgroup of G such that $G/N \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

Lemma 2.5 (see [6], Chapter X, p. 128) *If $F^*(G)$ is a solvable subgroup, then $F^*(G) = F(G)$.*

3 Proofs

Let us first prove the following result.

Theorem 3.1 *Assume that every maximal subgroup of each non-cyclic Sylow subgroup P of G , which does not have a supersolvable supplement in G , is a CSS-subgroup of G . Then G is supersolvable.*

PROOF — Suppose, for the sake of contradiction, that the theorem is false, and let G be a counterexample for which $|G|$ is of minimal order. As an application of the special case of [3, Theorem 1.3], G possesses

a Sylow q -subgroup Q such that $Q \trianglelefteq G$, where q is the largest prime divisor of $|G|$.

To establish the result, we now show that G/Q is supersolvable. To this end, let M/Q be a maximal subgroup of any non-cyclic Sylow subgroup PQ/Q of G/Q . Since M can be written in the form

$$M = M_1 Q$$

for some maximal subgroup M_1 of P , we can see that $M_1 Q/Q$ has a supersolvable supplement in G/Q whenever M_1 has a supersolvable supplement in G and $M_1 Q/Q$ is a CSS-subgroup of G/Q whenever M_1 is a CSS-subgroup of G by using Lemma 2.2 (2). By the minimal choice of G , we have that G/Q is supersolvable as required.

Finally, by virtue of Lemma 2.3, it is straightforward that every maximal subgroup of the Sylow q -subgroup Q , not having a supersolvable supplement in G , is weakly S -permutable in G . As a direct consequence of Skiba's result (see [10, Theorem 1.4]), we have that G is a supersolvable group; a contradiction to our choice of G . This completes the proof of the theorem. $\square$

We now proceed to establish the first main result.

PROOF OF THEOREM 1.1 — “ $\implies$ ” It is evident by considering the identity normal subgroup $N = 1$.

“ $\impliedby$ ” For this implication, we may assume that G is not in the formation $\mathfrak{F}$ and consider the pair (G, N) as a counterexample for which $|G| + |N|$ is of minimal order. The proof is presented in five distinct steps.

- (1) The hypothesis is still true for the pair (E, E) providing E is a Hall subgroup of N , as well as for the pair $(G/E, N/E)$, where E is a normal Hall subgroup of N .

Let us initially assume that E is a Hall subgroup of N and P is a non-cyclic Sylow subgroup of E . The hypothesis of the theorem yields that P possesses a subgroup D with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ or $2|D|$ (if P is a non-abelian 2-subgroup and $|P : D| > 2$) either has a supersolvable supplement B in G or is a CSS-subgroup of G . If the latter holds, then H is a CSS-subgroup of E by using Lemma 2.2 (1). If the former case holds, then

$$E = G \cap E = HB \cap E = H(B \cap E)$$

and so $B \cap E$ is a supersolvable supplement of H in E . Thus, the hypothesis is true for the pair (E, E) . For the second part of (1), we know that $E \trianglelefteq G$ as $E \text{ char } N \trianglelefteq G$. Furthermore,

$$(G/E)/(N/E) \simeq G/N \in \mathfrak{F}.$$

Let us consider any non-cyclic Sylow p -subgroup P_1/E of N/E , where p divides the order of N/E . Hence, we obtain that N has a non-cyclic Sylow p -subgroup P with

$$P = P_1 E.$$

We may take any subgroup H_1/E of PE/E with order $|D|$ or $2|D|$ (if P is a non-abelian 2-subgroup and $|P : D| > 2$). Therefore, the Schur–Zassenhaus theorem shows that H_1 can be written in the form

$$H_1 = H \rtimes E,$$

where H is a Sylow p -subgroup of H_1 . This implies that

$$|H| = |D| \quad \text{or} \quad |H| = 2|D|,$$

and hence either HE/E has a supersolvable supplement BE/E in G/E or it is a CSS-subgroup of G/E by using Lemma 2.2 (2). Accordingly, the hypothesis is true for the pair $(G/E, N/E)$ as well.

- (2) *If E is a non-identity normal Hall subgroup of N , then $E = N$.*

Our choice of G together with step (1) imply that $G/E \in \mathfrak{F}$ which means that the hypothesis is true for the pair (G, E) . The minimal choice of (G, N) assures that $E = N$.

- (3) *For a Sylow p -subgroup P of N , P cannot be a cyclic subgroup provided that p is the smallest prime divisor of $|N|$.*

Assume that P is a cyclic subgroup. As a direct consequence of [5, Chapter IV, Section 2.8], N has a normal Hall subgroup E such that

$$N = P \rtimes E.$$

In view of step (2), we can see that $N = P$ is a cyclic subgroup. Hence, $G \in \mathfrak{F}$ by using Lemma 2.4. This contradiction shows that P cannot be cyclic.

For the rest of the proof, P is assumed to be some fixed Sylow p -subgroup of N , where p is the smallest prime divisor of G . By step (3) and the hypothesis of the theorem, P possesses a subgroup D with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ or $2|D|$ (if P is a non-abelian 2-subgroup and $|P : D| > 2$) not having supersolvable supplement in G is a CSS-subgroup of G .

(4) N is a proper normal subgroup of G .

Assume that $N = G$. In this case, we know that $|P : D| > p$ (otherwise, G is supersolvable by Theorem 3.1, a contradiction). We claim that every subgroup H of P with order $|D|$ not having a supersolvable supplement in G is SS-quasinormal in G . Let H be any subgroup of P with order $|D|$ such that H neither has a supersolvable supplement in G nor is SS-quasinormal in G . In view of Lemma 2.2 (4), we can see that G possesses some normal subgroup M such that

$$G = HM \quad \text{and} \quad |G : M| = p.$$

Assume that $M \cap P$ is a cyclic Sylow p -subgroup of M . Hence, by using [5, Chapter IV, Section 2.8], it follows that

$$M = (M \cap P) \rtimes E,$$

where E is a normal Hall p' -subgroup of M . Therefore, E is also a normal Hall subgroup of G because

$$(|G : E|, |E|) = 1 \quad \text{and} \quad E \text{ char } M \trianglelefteq G.$$

Hence, by step (2), $E = 1$ and so $G = H(M \cap P)$ is a p -group, a contradiction to our assumption on G . Thus, we may assume that $M \cap P$ is non-cyclic. It follows that

$$1 < |D| < |M \cap P| < |P|$$

as $|P : D| > p$. Since $G/M \in \mathfrak{F}$, then the hypothesis is still true for the pair (G, M) . But

$$|G| + |M| < |G| + |N|,$$

which contradicts our choice of (G, N) . Thus, it can be assumed

that every subgroup H of P with order $|D|$ not having a supersolvable supplement in G is SS-quasinormal in G . Using similar arguments, one can show that every subgroup H of P with order $2|D|$ (if P is a non-abelian 2-subgroup) not having a supersolvable supplement in G is SS-quasinormal in G as well. As an application of [9, Theorem 3.1], we have $G \in \mathfrak{F}$, a contradiction to our assumption on G . Hence, N is a proper normal subgroup of G as required.

- (5) In view of steps (1) and (4), we can see that the hypothesis of the theorem is true for the pair (N, N) and N is supersolvable by the minimal choice of G . If q is the largest prime divisor of $|N|$ and Q is a Sylow q -subgroup of N , then, by using step (2), we have $Q \trianglelefteq G$ as $Q \text{ char } N \trianglelefteq G$ and so

$$Q = N = P.$$

Now, from Lemma 2.3, it is straightforward that every subgroup of N with order $|D|$ or $2|D|$ (if N is a non-abelian 2-subgroup and $|N : D| > 2$) not having a supersolvable supplement in G is weakly S -permutable in G . Therefore, by Skiba's result (see [10, Theorem 1.4]), we conclude that $G \in \mathfrak{F}$, a final contradiction.

Therefore, the claim follows. $\square$

We now state several corollaries which are immediate consequences of Theorem 1.1.

Corollary 3.2 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and every maximal subgroup of each non-cyclic Sylow subgroup P of N not having a supersolvable supplement in G is a CSS-subgroup of G .*

Corollary 3.3 (see [4], Theorem 2) *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and every subgroup of N of prime order or of order 4 (if $p = 2$) not having a supersolvable supplement in G is a CSS-subgroup of G .*

Now, we are in a position to prove Theorem 1.2.

PROOF OF THEOREM 1.2 — “ $\implies$ ” It is evident by considering the identity normal subgroup $N = 1$.

“ $\impliedby$ ” For this implication, it is clear that the pair $(F^*(N), F^*(N))$ satisfies the hypothesis of the theorem by using Lemma 2.2 (1). Consequently, $F^*(N)$ is supersolvable by Theorem 1.1. By virtue of Lemma 2.5, we have that

$$F^*(N) = F(N).$$

Now, let P be any non-cyclic Sylow subgroup of $F(N)$. Clearly, $P \trianglelefteq G$ as

$$P \text{ char } F(N) \text{ char } N \trianglelefteq G.$$

Accordingly, Lemma 2.3 assures that every subgroup of P with order $|D|$ or $2|D|$ (if P is a non-abelian 2-subgroup and $|P : D| > 2$) is weakly S -permutable in G . Therefore, we obtain that $G \in \mathfrak{F}$ by Skiba's result (see [10, Theorem 1.3]). This completes the proof of the theorem. $\square$

The following corollaries are immediate consequences of Theorem 1.2.

Corollary 3.4 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. Then $G \in \mathfrak{F}$ if and only if G has a normal solvable subgroup N such that $G/N \in \mathfrak{F}$ and each non-cyclic Sylow subgroup P of $F(N)$ possesses a subgroup D with $1 < |D| < |P|$ such that every subgroup of P with order $|D|$ or $2|D|$ (if P is a non-abelian 2-subgroup and $|P : D| > 2$) is a CSS-subgroup of G .*

Corollary 3.5 (see [13], Theorem 1.1) *If G has a normal solvable subgroup N such that G/N is supersolvable and every maximal subgroup of each Sylow subgroup P of $F(N)$ is a CSS-subgroup of G , then G is supersolvable.*

Corollary 3.6 (see [13], Theorem 1.2) *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. Then $G \in \mathfrak{F}$ if and only if G has a normal solvable subgroup N such that $G/N \in \mathfrak{F}$ and every maximal subgroup of each Sylow subgroup P of $F(N)$ is a CSS-subgroup of G .*

Corollary 3.7 *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and every maximal subgroup of each non-cyclic Sylow subgroup P of $F^*(N)$ is a CSS-subgroup of G .*

Corollary 3.8 (see [4], Corollary 7) *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. Then $G \in \mathfrak{F}$ if and only if G has a normal solvable subgroup N such that $G/N \in \mathfrak{F}$ and every subgroup of $F(N)$ of prime order or of order 4 (if $p = 2$) is a CSS-subgroup of G .*

Corollary 3.9 (see [4], Theorem 3) *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and every subgroup of $F^*(N)$ of prime order or of order 4 (if $p = 2$) is a CSS-subgroup of G .*

Remark 3.10 Our main results supersedes some recent and earlier results in the literature such as Theorems 3.6 and 3.7 in [1], Theorems 3.1, 3.2, 3.3, 1.3 in [9] and Theorems 4.1, 4.2 in [11].

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