Seminormal, Non-Normal Maximal Subgroups and Soluble PST-Groups

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Abstract

All groups in this paper are finite. Let G be a group. Maximal subgroups of G are used to establish several new characterisations of soluble PST-groups.

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1 Introduction and statement of results

All groups in this paper are finite.

There are many articles in the literature (for instance, [1],[5],[3],[6] to name just the four classical ones) where global information about a group G is obtained by assuming that some members of relevant families of subgroups of G are either normal or satisfy a sufficiently strongly embedding property extending normality. In many cases, the subgroups are the normal subgroups of G, and the embedding assumptions are that they are permutable or S-permutable in G.

Recall that a subgroup H of a group G is said to permute with a subgroup K of G if HK is a subgroup of G. H is said to be permutable (respectively, S-permutable) in G if H permutes with all subgroups
(respectively, Sylow subgroups) of $G$. Examples of permutably subgroups include the normal subgroups of $G$. Non-Dedekind modular groups and non-modular nilpotent groups show that $S$-permutability, permutability and normality are quite different subgroup embedding properties. However, according to a result of Kegel [12], every $S$-permutable subgroup of $G$ is always subnormal.

A group $G$ is a $\text{PST-group}$ if every subnormal subgroup of $G$ is $S$-permutable in $G$. In the same way classes of $\text{PT-groups}$ and $\text{T-groups}$ are defined, in which every subnormal subgroup is permutable or normal respectively. Since normal subgroups are permutable and obviously permutable subgroups are $S$-permutable then it follows that $T$ is a proper subclass of $PT$ and $PT$ is a proper subclass of $\text{PST}$. Soluble $\text{PST}$, $\text{PT}$ and $\text{T}$-groups were studied and characterised by Agrawal [1], Zacher [15] and Gaschütz [10] respectively.

**Theorem 1**

1. A soluble group $G$ is a $\text{PST-group}$ if and only if the nilpotent residual $L$ of $G$ is an abelian Hall subgroup of $G$ on which $G$ acts by conjugation as power automorphisms.

2. A soluble $\text{PST-group}$ $G$ is a $\text{PT}$-group (respectively $\text{T}$-group) if and only if $G/L$ is a modular (respectively Dedekind) group.

Note that if $G$ is a soluble $T$, $PT$ or $PST$-group then every subgroup and every quotient of $G$ inherits the same properties.

We mention that in [5, Chapter 2] many of the beautiful results on these classes of groups are presented.

Subgroup embedding properties closely related to permutability and $S$-permutability are semipermutability and $S$-semipermutability introduced by Chen in [8]: a subgroup $X$ of a group $G$ is said to be $\text{semipermutable}$ (respectively, $S$-$\text{semipermutable}$) in $G$ provided that it permutes with every subgroup (respectively, Sylow subgroup) $K$ of $G$ such that $\gcd(|X|,|K|) = 1$. A semipermutable subgroup of a group need not be subnormal. For example a 2-Sylow subgroup of the non-abelian group of order 6 is semipermutable but not subnormal.

Note that a subnormal semipermutable (respectively, $S$-semipermutable) subgroup of a group $G$ must be normalised by every
subgroup (respectively, Sylow subgroup) $P$ of $G$ such that $\gcd(|X|,|P|) = 1$. This observation was the basis for Beidleman and Ragland [7] to introduce the following subgroup embedding properties.

A subgroup $X$ of a group $G$ is said to be *seminormal* (respectively, *S*-seminormal)\(^1\) in $G$ if it is normalised by every subgroup (respectively, Sylow subgroup) $K$ of $G$ such that $\gcd(|X|,|K|) = 1$.

By [7, Theorem 1.2], a subgroup of a group is seminormal if and only if it is $S$-seminormal. Furthermore, seminormal subgroups are not necessarily subnormal: it is enough to consider a non-subnormal subgroup $H$ of a group $G$ such that $\pi(H) = \pi(G)$. The following result is an interesting characterisation of soluble PST-groups.

**Theorem 2** ([7]) *Let $G$ be a soluble group. Then the following statements are pairwise equivalent:*

1. $G$ is a PST-group.
2. All the subnormal subgroups of $G$ are seminormal in $G$.
3. All the subnormal subgroups of $G$ are semipermutable in $G$.
4. All the subnormal subgroups of $G$ are $S$-sempemutable in $G$.

**Definition 3** *Let $G$ be a group. Then*

1. $G$ is called an $S(n)$NM-group if every non-seminormal subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.
2. $G$ is called an $S(s)$NM-group if every non-$S$-sempemutable subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.
3. $G$ is called a $P(s)$NM-group if every non-sempemutable subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.

The following three theorems provide some new and different characterisations of soluble PST-groups.

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\(^1\) Note that the term *seminormal* has different meanings in the literature
**Theorem A** A group $G$ is a soluble PST-group if and only if every subgroup of $G$ is an $S(s)NM$-group.

**Theorem B** A group $G$ is a soluble PST-group if and only if every subgroup of $G$ is an $S(n)NM$-group.

**Theorem C** A group $G$ is a soluble PST-group if and only if every subgroup of $G$ is a $P(s)NM$-group.

Robinson [13] introduced classes of groups in which cyclic subnormal subgroups are $S$-permutable, permutable or normal.

**Definition 4** A group $G$ is called a $PST_c$-group if every cyclic subnormal subgroup of $G$ is $S$-permutable in $G$.

Similarly, classes $PT_c$ and $T_c$ are defined, by requiring cyclic subnormal subgroups to be permutable or normal respectively. Robinson [13] provided characterisations for both soluble and insoluble cases. Here we mention only the soluble case.

**Theorem 5** ([13]) Let $G$ be a group and $F = F(G)$, the Fitting subgroup of $G$.

1. G is a soluble $PST_c$-group if and only if there is a normal subgroup $L$ such that,
   a) $L$ is abelian and $G/L$ is nilpotent.
   b) $p'$-elements of $G$ induce power automorphisms in the Sylow $p$-subgroup $L_p$ of $L$ for all primes $p$.
   c) $\pi(L) \cap \pi(F/L) = \emptyset$

2. A soluble $PST_c$-group is supersoluble.

3. A soluble group $G$ is a $PT_c$ ($T_c$)-group if and only if $G$ is a soluble $PST_c$-group such that all the elements of $G$ induce power automorphisms in $L$ and $F/L$ is a modular (Dedekind) group, where $L$ is the subgroup described in 1.

Note that the important distinction between soluble PST-groups and soluble $PST_c$-groups is that the nilpotent residual is a Hall subgroup of the Fitting subgroup whereas the nilpotent residual of a soluble PST-group is a Hall subgroup of the entire group. In fact, Robinson in [13] showed that the sets of primes $\pi(L)$ and $\pi(G/L)$ can have
a large intersection, even when $G$ is a soluble $T_c$-group. It is clear that a soluble PST$_c$-group such that the nilpotent residual is a Hall subgroup of $G$ is a PST-group. Also, note that the class of all soluble PST$_c$-group is much different than the class of soluble PST-group as the following theorem shows.

**Theorem 6 ([13])** Let $G$ be a group. Then

1. If every subgroup of $G$ is a PST$_c$-group, then $G$ is a soluble PST-group.

2. If every quotient of $G$ is a soluble PST$_c$-group, then $G$ is a soluble PST-group.

In addition, a PST$_c$-group is a PT$_c$ ($T_c$)-group if all of its Sylow subgroups are modular (Dedekind) respectively [13].

There are similar connections as in Theorems 2 and 5 with classes PST$_c$, PT$_c$ and $T_c$ as seen in the next two theorems.

**Theorem 7 ([4])** Let $G$ be a soluble group. Then the following statements are pairwise equivalent:

1. $G$ is a PST$_c$-group.

2. All the cyclic subnormal subgroups of $G$ are seminormal in $G$.

3. All the cyclic subnormal subgroups of $G$ are semipermutable in $G$.

4. All the cyclic subnormal subgroups of $G$ are $S$-semipermutable in $G$.

**Theorem 8 ([4])** Let $G$ be a soluble group with abelian nilpotent residual $L$. Then:

1. $G$ is a PT$_c$ ($T_c$)-group if and only if every cyclic subnormal subgroup of $G$ is seminormal in $G$, all the elements of $G$ induce power automorphisms in $L$, and $F/L$ is a modular (Dedekind) group.

2. $G$ is a PT$_c$ ($T_c$)-group if and only if every cyclic subnormal subgroup of $G$ is semipermutable in $G$, all the elements of $G$ induce power automorphisms in $L$, and $F/L$ is a modular (Dedekind) group.

3. $G$ is a PT$_c$ ($T_c$)-group if and only if every cyclic subnormal subgroup of $G$ is $S$-semipermutable in $G$, all the elements of $G$ induce power automorphisms in $L$, and $F/L$ is a modular (Dedekind) group.
4. $G$ is a PT$_c$ ($T_c$)-group if and only if $G$ is an PST$_c$-group such that all the elements of $G$ induce power automorphisms in $L$, and $F/L$ is a modular (Dedekind) group.

**Definition 9**  Let $G$ be a group.

1. $G$ is called a $S(n)NM_c$-group if every cyclic non-seminormal subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.

2. $G$ is called an $S(s)NM_c$-group if every cyclic non-$S$-semipermutable subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.

3. $G$ is called a $P(s)NM_c$-group if every cyclic non-semipermutable subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.

We now list three theorems that are similar to Theorems A, B and C; however we only consider certain subgroups of a group which are contained in non-normal maximal subgroups.

**Theorem D**  Let $G$ be a group. Then

1. If every subgroup of $G$ is an $S(n)NM_c$-group, then $G$ is a soluble PST$_c$-group and so $G$ is a soluble PST-group.

2. If every subgroup of $G$ is a PST$_c$-group, then $G$ is an $S(n)NM_c$-group and hence a soluble PST-group.

**Theorem E**  Let $G$ be a group. Then

1. If every subgroup of $G$ is an $S(s)NM_c$-group, then $G$ is a soluble PST$_c$-group and so $G$ is a soluble PST-group.

2. If every subgroup of $G$ is a PST$_c$-group, then $G$ is an $S(s)NM_c$-group and a soluble PST-group.

**Theorem F**  Let $G$ be a group. Then

1. If every subgroup of $G$ is a $P(s)NM_c$-group, then $G$ is a soluble PST$_c$-group and so is a soluble PST-group.

2. If every subgroup of $G$ is a PST$_c$-group, then $G$ is a $P(s)NM_c$-group and a soluble PST-group.
2 Preliminaries

The lemmas encountered here are used in the proofs of the main theorems of this paper.

Lemma 10 ([5, Theorem 2.1.8, p. 57])

1. Let $G$ be a soluble group and let $L$ be the nilpotent residual of $G$. Then $G$ is a PST-group if and only if $L$ is an abelian Hall subgroup of $G$ and $G$ acts by conjugation on $L$ as a group of power automorphisms.

2. A soluble group is a PST-group if and only if every subnormal subgroup of $G$ is S-permutable (seminormal, semipermutable in $G$).

Lemma 11 ([14, Theorem 13.3.7, p. 399]) Let $N$ be a minimal normal subgroup of a group $G$. Then $N$ normalizes all the subnormal subgroups of $G$.

Lemma 12 ([9, Theorem 5.9, p. 238; 14, Theorem 9.2.9, p. 265]) A finite soluble group is generated by its system normalizers.

Lemma 13 ([14, Theorem 9.2.7, p. 264]) Let $G$ be a finite soluble group and let $L$ be the nilpotent residual of $G$. If $L$ is abelian and $D$ is a system normalizer of $G$, then $G = L \rtimes D$, that is, $G$ is a semidirect product of $L$ by $D$.

Lemma 14 ([2, Corollary 1.3.3, p. 9]) Let the finite group $G = AB$ be the product of two subgroups $A$ and $B$. Then for each prime $p$ there exist Sylow $p$-groups $A_0$ of $A$ and $B_0$ of $B$ such that $A_0B_0$ is a Sylow $p$-subgroup of $G$.

3 Proof of the theorems

Proof of Theorem A — Let $G$ be a group. Assume that $G$ is a soluble PST-group, let $L$ be the nilpotent residual of $G$, and let $D$ be a system normalizer of $G$. By Lemma 10 $L$ is an abelian Hall subgroup of $G$ and $G$ acts by conjugation on $L$ as a group of power automorphisms. Moreover, by Lemma 13 $G = L \rtimes D$, the semidirect product of $L$ by $D$. We prove that $G$ is an S(s)NM-group by induction on $|G|$. Let $A$ be a non-$S$-semipermutable subgroup of $G$. Then $L \neq 1$ and $A \cap L \lhd G$. Also $A/A \cap L$ is a non-$S$-semipermutable subgroup
of $G/A \cap L$. Now $A \cap L = 1$, for otherwise, by induction, $A/A \cap L$ would be contained in a non-normal maximal subgroup $M/A \cap L$ of $G/A \cap L$. Then $M$ would be a non-normal maximal subgroup of $G$ containing $A$. This would mean that $G$ is an $S(s)NM$-group. Hence $A \cap L = 1$. Since $L$ and $D$ are Hall subgroups we may assume $A \leq D$. Let $M$ be a maximal subgroup of $G$ containing $D$. Assume that $M \triangleleft G$, then $D^g \leq M$ for all $g \in G$ and so $D^G \leq M$. But $D^G = G$ by Lemma 12 so that $M$ is non-normal. Thus $A \leq M$ and hence, $G$ is an $S(s)NM$-group. Now applying [5, 2.1.9] we have every subgroup $H$ of $G$ is a soluble PST-group. Hence, by the argument above $H$ is an $S(s)NM$-group.

Now assume that every subgroup of $G$ is an $S(s)NM$-group but $G$ is not a soluble PST-group. Let $G$ be the counterexample of least order. Then every proper subgroup of $G$ is a soluble PST-group. Thus every proper subgroup of $G$ is supersoluble and hence $G$ is a soluble group. Since $G$ is not a PST-group there is a subnormal subgroup $H$ which is not $S$-semipermutable in $G$. Let $M$ be a maximal normal subgroup of $G$ such that $H \leq M$. Now $G$ is an $S(s)NM$-group so there is a non-normal maximal subgroup $L$ of $G$ such that $H \leq L$. Note that $G = LM$ and both $L$ and $M$ are soluble PST-subgroups of $G$. There is a Sylow $p$-subgroup $P$ of $G$ such that the gcd $(p, |H|) = 1$ and $H$ does not permute with $P$.

By Lemma 14 there is a Sylow $p$-subgroup $A$ of $L$ and a Sylow $p$-subgroup $B$ of $M$ such that $AB$ is a subgroup of $G$ and $AB \in Syl_p(G)$. Note that $H$ permutes with $A$ and $B$ so $H$ permutes with $AB = Q$. There is an element $x \in G$ such that $P^x = Q$. The properties of $G$ as stated in the Theorem are inherited by quotients, so if $N$ is a minimal normal subgroup of $G$ contained in $M$, then $(HN)P/N = P(HN)/N$ is a subgroup of $G/N$. Hence $P$ permutes with $HN$.

If $(HN)P$ is a proper subgroup of $G$, then, by the hypothesis of the theorem, $HP = PH$, which is a contradiction. Hence, $G = (HN)P$. By Lemma 11 $N$ normalizes $H$ and so $H \triangleleft HN$. Since $G = HNP$, there is an element $a \in P$ and $b \in HN$ such that $x = ab$. Thus, $H^b = H$ or $H^{b^{-1}} = H$ and $H$ permutes with $P^b$ so $HP = PH$, a final contradiction. $\Box$

**Proof of Theorem B** — First assume that $G$ is a soluble PST-group. As in the proof of Theorem A we prove that $G$ is an $S(n)NM$-group in the same way we showed that $G$ is an $S(s)NM$-group in the proof of Theorem A. As in that proof, we use the fact that every subgroup
of G is a soluble PST-group to prove that every subgroup of G is an S(n)NM-group.
Conversely, assume that every subgroup of G is an S(n)NM-group but G is not a soluble PST-group and let G be such a group of smallest order. As in the proof of Theorem A, G is soluble and by Lemma 10, Part 2, there is a subnormal subgroup H of G which is not seminormal in G. There is a normal maximal subgroup M of G and a maximal subgroup L of G such that $G = LM$ and $H \leq L \cap M$. Now L and M are soluble PST-groups so that L (respectively, M) contains a Sylow $p$-subgroup A (respectively, B) such that $AB$ is a Sylow subgroup of G. (Note this proof follows that of the proof of Theorem A). There is a Sylow $p$-subgroup P of G which does not normalize H but H is normalized by $AB$. So there is an $x \in G$ such that $P^x = Q = AB$. As in the proof of Theorem A a minimal normal subgroup N of G normalizes H and P normalizes $HN$ in G. Also $G = HNP$.

Then there is an element $a \in P$ and an element $b \in HN$ such that $x = ab$ and $H^b^{-1} = H$ is normalized by $P^b$. This is the final contradiction. □

**Proof of Theorem C** — To obtain a proof of Theorem C just replace S-semipermutable in the proof of Theorem A by semipermutable and we obtain the desired proof. □

**Proof of Theorem D** — Suppose that every subgroup of the group G is an $S(n)NM_c$-group but G is not a soluble PST$_c$-group and we assume G is a counterexample of least order to the result. Then G is not a soluble PST$_c$-group but every proper subgroup of G is a soluble PST$_c$-group. By Theorem 5 (2) every proper subgroup of G is supersoluble and hence G is soluble.

By Theorem 7 (2) there is a cyclic subnormal subgroup H which is not seminormal in G. Hence there is a Sylow $p$-subgroup P such that P does not normalize H. As in the proof of Theorem A there exists a normal maximal subgroup L of G and a non-normal maximal subgroup M of G such that $G = LM$ and $H \leq L \cap M$. Since L and M are $S(n)NM_c$-subgroups of G, it follows from Lemma 13 that there are Sylow $p$-subgroups A of L and B of M such that $AB$ is a Sylow $p$-subgroup of G and both A and B normalize H. There is an element $x \in G$ such that $P^x = AB$. Let $Q = AB$.

Consider a minimal normal subgroup N of G with $N \leq L$. We now consider the quotient $G/N$ of G. Since the properties of G, as enunciated in the statement of the theorem, are inherited by quotients of $S(n)NM_c$, the minimality of G implies $HN/N$ in $G/N$ is normal-
ized by PN/N. Hence, P normalizes HN. Also by Lemma 11 N normalizes H in HN. If HNP is a proper subgroup of G, then P normalizes H, which is a contradiction. Thus HNP = G. Let \( x = ab \) where \( b \in HN \) and \( a \in P \), then \( H^b = H \) and \( P^x = P^b \). Hence

\[ H^{b^{-1}} = H \text{ and } (p^b)^{b^{-1}} = P \]

normalizes H, a final contradiction.

Hence, G is a soluble PST\(_c\)-group. Now let X be a subgroup of G. Then every subgroup of X is an S(\( n \))NM\(_c\)-group so that our proof can be applied to X to show that X is a soluble PST\(_c\)-group. By Theorem 6 (1) G is a soluble PST-group. This completes the proof of part (1) of Theorem D.

If every subgroup of G is a PST\(_c\)-group, then by Theorem 6 (1) G is a soluble PST-group. To show that every subgroup of G is an S(\( n \))NM\(_c\)-group follows from the necessity part of the proof of Theorem A.

\( \square \)

**Proof of Theorem E** — In the proof of Theorem E replace in Theorem D seminormal subgroup with S-semipermutable subgroup. Also replace S(\( n \))NM\(_c\) by S(\( s \))NM\(_c\). \( \square \)

**Proof of Theorem F** — In the proof of Theorem F replace in Theorem D seminormal subgroup with semipermutable subgroup. Also replace S(\( n \))NM\(_c\) by P(\( s \))NM\(_c\). \( \square \)

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