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Minkowski t-Graph and Kendall τ-Graph of the Symmetric Group

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Abstract

This paper presents the construction and some characterisation of the Minkowski t-graph and Kendall τ-graph of the symmetric group of degree n. Further, we demonstrate some properties of this graph associated with its connectedness, bipartition, planarity, and regularity.

Mathematics Subject Classification (*2020*): 20F05, 20E65, 05C12, 05C15

Keywords: symmetric group; Minkowski distance; Kendall τ-distance; graph

1 Introduction

Recent works reveal many different ways of associating a graph with a given finite group (see [2],[4],[10]). The difference between one and others lies in the adjacency criteria used to relate two group elements constituting the set of vertices of such a graph.

One of the graphs studied in this paper is the Minkowski t-graph, or simply the t-graph, of the symmetric group of degree n. The t-graph of a finite group G was first introduced by G. Diaz et al. in [4]. In that work was investigated the t-graph on a finitely generated group G. It has to lead to an interesting combinatorial problem, establishing sufficient conditions to guarantee the existence of isolated points in the t-graph when G is a two-generator group. It also proposed an expression to determine the number of the connected components of the t-graph. Other results have to do with the conditions that must be fulfilled for the t-graphs of the dihedral groups to be a path graph or a cycle. Consequently, it was obtained that the chromatic number of the t-graph depends exclusively on the parity of t.

The second graph we investigate, the τ-Kendall graph of the symmetric group of degree n, defines its adjacency criterion based on the Kendall metric. The Kendall τ-distance is a metric used in Statistics that counts the number of pairwise disagreements between two ranking lists. The larger the distance, the more dissimilar the two lists are. Many authors first discussed it at the beginning of the 20th century and rediscovered it by M.G. Kendall (see [8]). Kendall τ-distance is also called bubble-sort distance since it is equivalent to the number of swaps that the bubble-sort algorithm would take to place one list in the same order as the other.

Recently, codes have been studied in which the codewords are elements of the symmetric group of degree n, called permutation codes. Instead of considering the Hamming distance, the Kendall τ-distance is used (see [1],[17]). One of the essential results of these works is the non-existence of single-error-correcting permutation codes when n is a prime greater than four or $4 \leqslant n \leqslant 10$.

Another interesting problem related to Kendall's distance is the reconstruction of permutations distorted by single Kendall τ-errors, which was first proposed by Levenshtein in [9] and most recently studied by X. Wang et al. (see [11],[16]). This problem consists of the following: a codeword is transmitted through multiple channels, and a decoder receives all the different outputs and reconstructs the transmitted codeword. The initial investigations studied the minimum number of transmission channels required to precisely decode the transmitted sequence.

In this paper, we use the Kendall τ-distance to define a graph over the symmetric group of degree n. The initial idea is to use the same philosophy that establishes the adjacency criterion in the Minkowski graph. First, we obtain some results on the Minkowski t-graph of the symmetric group, and then we define and characterise the Kendall τ-graph over the same group. The rest of this paper is organised as follows. Section 2 gives basics and notations concerning elementary graph theory and the symmetric group of degree n. Section 3 presents the results of the Minkowski t-graph of the symmetric group. We show that this graph is non-connected whenever t is an even natural number. We also present a table (Table 2) indicating the number of connected components for $1 \leqslant n \leqslant 7$ and $1 \leqslant t \leqslant 18$. In Section 4, we present the construction and some characterisation of the Kendall τ-graph. We demonstrate some properties of this graph associated with its connectedness, bipartition, planarity, and regularity. This section concludes the work.

2 Preliminaries

In this section, for the reader's convenience and later use, we recall some definitions, notations and results concerning elementary graph theory and the symmetric group. The reader is referred for undefined terms and concepts of graph theory to [5] or [15].

An *undirected graph* (or simply a *graph*) $\mathcal{G}$ is a pair $\mathcal{G} = (V, E)$, where V is a set whose elements are called *vertices*, and E is a (possibly empty) set of pairs of vertices, whose elements are called *edges*. In the following, $\mathcal{G}$ always denotes a graph. If x and y are vertices of $\mathcal{G}$, then we say that x is *adjacent* to y if $\{x, y\}$ is an edge. In this case, the edge's *endpoints* are x and y. A vertex x is said to be *incident* with an edge e if x is an endpoint of e. We also say that e is *incident* with x whenever x is an endpoint of e. The number of vertices in $\mathcal{G}$ is called the *order* of $\mathcal{G}$, and the number of edges in $\mathcal{G}$ is called the *size* of $\mathcal{G}$. A *subgraph* of $\mathcal{G}$ is a graph $\mathcal{H}$ whose vertices and edges form subsets of the vertices and edges of $\mathcal{G}$.

The *degree* of a vertex x in $\mathcal{G}$, denoted by $\deg(x)$, is the number of edges incident with x. A vertex of degree zero is called an *isolated vertex* or *isolated point*. A *path graph* or *linear graph* is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$ where $i = 1, \ldots, n-1$. That is, a path has two terminal vertices (vertices that have degree 1), while all others (if any) have degree 2.

A *walk* in $\mathcal{G}$ is a sequence of vertices $\{v_1, \ldots, v_n\}$ such that $\{v_i, v_{i+1}\} \in E$. The *length* of a walk is just the number of edges it has. A *trail* in $\mathcal{G}$ is a walk in G with the property that no edge is repeated. A *path* in $\mathcal{G}$ is a trail with the property that no vertex is repeated. A closed trail, also called a *circuit*, is a trail for which $v_1 = v_n$. A closed path, usually called a *cycle*, is a path whose endpoints are the same vertex. We say that two vertices, x and y in $\mathcal{G}$, are *connected* if $\mathcal{G}$ contains a path from x to y. Otherwise, they are called *disconnected*. If every pair of vertices in $\mathcal{G}$ are connected, then $\mathcal{G}$ is said to be *connected*. If this is not the case, then $\mathcal{G}$ is called *disconnected*, i.e. there exist two vertices in $\mathcal{G}$ such that no path in $\mathcal{G}$ has these vertices as endpoints. A *connected component* of $\mathcal{G}$ is a maximal connected subgraph.

Two graphs $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijection $f : V_1 \longrightarrow V_2$ such that

$$\{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2.$$

We define a *colouring* of $\mathcal{G}$ as an assignment of colours to the vertices of $\mathcal{G}$ such that no two adjacent vertices receive the same colour. The minimum number of colours necessary to colour $\mathcal{G}$, denoted by $\chi(G)$, is called the *chromatic number* of $\mathcal{G}$. We say also that $\mathcal{G}$ is k-*chromatic* if $\chi(\mathcal{G}) = k$. $\mathcal{G}$ is called *bipartite* if its set of vertices can be divided into two disjoint sets, say U and V, such that every edge $\{u, v\}$ either connects a vertex from U to V or a vertex from V to U. We can also say that no edge connects vertices of the same set. Note that a graph $\mathcal{G}$ is bipartite if and only if $\chi(\mathcal{G}) = 2$.

To study graphs in a given context, it is necessary to use the spectral theory of graphs, which consists of studying the properties of a graph's Laplacian and adjacency matrices, more specifically, its eigenvalues and eigenvectors. These are defined as follows. The *adjacency matrix* of $\mathcal{G} = (V, E)$ is the $n \times n$ matrix $A = (a_{ij})$ indexed by V, whose (i, j)-entry is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{if } \{v_i, v_j\} \notin E, \end{cases}$$

and the *Laplacian matrix* $L = (l_{ij})$ is defined as follows

$$l_{ij} = \begin{cases} -1 & \text{if } \{v_i, v_j\} \in E \\ \deg(v_i) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

These two matrices will later be used to determine a graph's number of connected components.

To conclude this section, we present relevant results on the symmetric group of degree n. For more details on notation and definitions of this particular subject, we refer the reader to [6] or [14, Chapter 6]. The *symmetric group on a finite set* Ω is the group whose elements are all permutations of Ω, that is, all bijective functions from Ω to Ω, and whose group operation is the composition. The *symmetric group of degree* n, denoted by $\mathrm{Sym}(n)$, is the symmetric group defined on the set $\Omega = \{1, 2, \dots, n\}$. The expression

$$\sigma = (a_1, a_2, \dots, a_k)$$

denotes the permutation σ that sends a_i to a_{i+1} for $i = 1, \dots, k-1$ and sends the last element a_k to the first element a_1. All other elements of Ω are held fixed. This permutation is called a k-*cycle* in $\mathrm{Sym}(n)$. A 2-cycle $\tau = (ab)$ is a transposition. The *support* of $\sigma \in \mathrm{Sym}(n)$, denoted by $\mathrm{supp}(\sigma)$, is defined as follows:

$$\mathrm{supp}(\sigma) := \{j \in \Omega \mid \sigma(j) \neq j\}.$$

Two permutations σ and π in $\mathrm{Sym}(n) =$ are *disjoint* if their supports are disjoint. In particular, two cycles

$$\sigma = (a_1 a_2 \dots a_k) \quad \text{and} \quad \pi = (b_1 b_2 \dots b_r)$$

are disjoint if the underlying sets $\{a_1, a_2, \dots a_k\}$ and $\{b_1, b_2, \dots b_r\}$, are disjoint. It is well known that disjoint permutations commute; that is, if σ and π are disjoint, then $\sigma\pi = \pi\sigma$.

In general, the group $\mathrm{Sym}(n)$ can be factorised in the form

$$\mathrm{Sym}(n) = \mathrm{Sym}(n-1)\langle(12\dots n)\rangle,$$

and, applying this property inductively, we have that $\mathrm{Sym}(n)$ is generated by the set

$$X := \{\sigma_j = (12\dots j) \mid j = 2, \dots, n\}.$$

That is, every element $\sigma \in \mathrm{Sym}(n)$ can be uniquely written as follows

$$\sigma = \prod_{j=2}^{n} \sigma_j^{\epsilon_j}, \tag{1}$$

where each ϵ_j is a natural number, with $0 \leqslant \epsilon_j < m_j$, and $2 \leqslant j \leqslant n$. The numbers m_j can be, for example, the orders of the corresponding elements in the finite case, but they may also differ from these orders.

Given $\sigma \in \mathrm{Sym}(n)$ we calculate the exponents ϵ_j with $\sigma = \sigma_2^{\epsilon_2} \dots \sigma_n^{\epsilon_n}$ as follows. First, consider $k = \sigma^{-1}(n)$, that is, $\sigma(k) = n$, then we set $\epsilon_n = n - k$. Now recursively, for $j \in \{2, \dots, n-1\}$ define $k_j = \sigma^{-1}(j)$ and then we put

$$\epsilon_j = j - \sigma_{j+1}^{\epsilon_{j+1}} \dots \sigma_n^{\epsilon_n}(k_j).$$

Since $\sigma_n = (1 \dots k \dots n)$ it follows that $\sigma_n^{\epsilon_n}(k) = \sigma_n^{n-k}(k) = n$. On the other hand, since σ_m fixes n for all $m < n$ we have that

$$\sigma_2^{\epsilon_2} \dots \sigma_n^{\epsilon_n}(k) = \sigma(k) = n.$$

Similar to the above, $\sigma_{n-1} = (1 \dots \sigma_n^{\epsilon_n}(k_{n-1}) \dots n-1)$ and it holds that σ_m fixes $n-1$ for all $m < n-1$. Then we get

$$\sigma_2^{\epsilon_2} \dots \sigma_n^{\epsilon_n}(k_{n-1}) = \sigma(k_{n-1}) = n-1.$$

Continuing in this form leads to the statement.

Example 1 Let $\sigma = (12)(34) \in \mathrm{Sym}(4)$. We find the exponents ϵ_j, $j = 2, 3, 4$, such that

$$\sigma = \sigma_2^{\epsilon_2}\sigma_3^{\epsilon_3}\sigma_4^{\epsilon_4} = (1,2)^{\epsilon_2}(1,2,3)^{\epsilon_3}(1,2,3,4)^{\epsilon_4}.$$

Following the above procedure, we first have $3 = \sigma^{-1}(4)$, then $\epsilon_4 = 1$. To find ϵ_3, note that $4 = \sigma^{-1}(3)$ and $\sigma_4(4) = 1$, and then

$$\epsilon_3 = 3 - 1 = 2.$$

Finally, we have $1 = \sigma^{-1}(2)$, and $\sigma_3^2\sigma_4(1) = 1$. Therefore $\epsilon_2 = 2 - 1 = 1$, and $\sigma = \sigma_2\sigma_3^2\sigma_4$.

3 The Minkowski t-graph of the symmetric group

Let $G = \langle g_1, \dots, g_n \rangle$ be a finite group and suppose that every element $g \in G$ can be uniquely written as follows

$$g = \prod_{i=1}^{n} g_i^{\epsilon_i}, \tag{2}$$

with $0 \leqslant \epsilon_i < m_i$, and $1 \leqslant i \leqslant n$. Further, we introduce the following distance map

$$d_1 : G \times G \longrightarrow \mathbb{N}_0$$

defined by

$$d_1(g,h) = d_1\Big(\prod_{i=1}^{n} g_i^{\epsilon_i}, \prod_{i=1}^{n} g_i^{\delta_i}\Big) = \sum_{i=1}^{n} |\epsilon_i - \delta_i|. \tag{3}$$

The distance map d_1 is a metric over G. Actually, it is the Minkowski l_p metric for $p = 1$ in the set $\{(\epsilon_1, \dots, \epsilon_n) \mid 0 \leqslant \epsilon_i < m_i\}$.

Definition 2 Let t be a natural number and G be a finite generated group. The *Minkowski* t*-graph* of G is defined as the pair $\mathcal{G} = (G, E)$, where the set $\{g, h\} \in E$ if and only if $d_1(g, h) = t$.

Theorem 3 *Let* t *be an even natural number and* $G = \langle g_1, \dots, g_n \rangle$ *be a finite group and suppose that every element* $g \in G$ *can be uniquely written as in* (2)*. Then the Minkowski* t*-graph* $\mathcal{G} = (G, E)$ *is non-connected.*

Proof — Let t be an even number. We define $V_1, V_2 \subseteq G$ as follows:

$$V_1 := \Big\{ \prod_{j=1}^{n} g_j^{\epsilon_j} \mid \sum_{j=1}^{n} \epsilon_j \equiv 0 \bmod 2 \Big\},$$

and

$$V_2 := \Big\{ \prod_{j=1}^{n} g_j^{\epsilon_j} \mid \sum_{j=1}^{n} \epsilon_j \equiv 1 \bmod 2 \Big\},$$

It is clear that

$$V_1 \cap V_2 = \emptyset \quad \text{and} \quad V_1 \cup V_2 = G.$$

Let suppose that $g, h \in V_1$, say

$$g = \prod_{j=1}^{n} g_j^{\epsilon_j} \quad \text{and} \quad h = \prod_{j=1}^{n} g_j^{\delta_j}.$$

Since $g \in V_1$, there exists a $t \in \mathbb{N}$ such that the number of odd exponents is 2t, similarly for h we have that the number of odd exponents is $2r$ for some natural number r. Let us define it now

$$P := \{i \mid \epsilon_i, \delta_i \equiv 1 \bmod 2\},$$

$$P_1 := \{j \mid \epsilon_j \equiv 1 \bmod 2 \wedge \delta_j \equiv 0 \bmod 2\}$$

and

$$P_2 := \{k \mid \epsilon_k \equiv 0 \bmod 2 \wedge \delta_k \equiv 1 \bmod 2\}.$$

Note that

$$|P_1| = 2t - |P| \quad \text{and} \quad |P_2| = 2r - |P|,$$

so $|P_1| + |P_2|$ is even, and hence $d_1(g,h) \equiv 0 \bmod 2$. Similarly, if $g, h \in V_2$, then $d_1(g,h) \equiv 0 \bmod 2$.

On the other hand, if $g \in V_1$, and $h \in V_2$, then $d_1(g,h) \equiv 1 \bmod 2$. This means that $\{g,h\}$ cannot belong to E. Now,

$$E_1 := \Big\{\{\prod_{j=1}^{n} g_j^{\epsilon_j}, \prod_{j=1}^{n} g_j^{\delta_j}\} \mid \sum_{i=1}^{n} \epsilon_i, \sum_{i=1}^{n} \delta_i \equiv 0 \bmod 2 \wedge \sum_{i=1}^{n} |\epsilon_i - \delta_i| = t\Big\},$$

and

$$E_2 := \Big\{\{\prod_{j=1}^{n} g_j^{\epsilon_j}, \prod_{j=1}^{n} g_j^{\delta_j}\} \mid \sum_{i=1}^{n} \epsilon_i, \sum_{i=1}^{n} \delta_i \equiv 1 \bmod 2 \wedge \sum_{i=1}^{n} |\epsilon_i - \delta_i| = t\Big\}.$$

This implies that

$$S_1 := (V_1, E_1) \quad \text{and} \quad S_2 := (V_2, E_2)$$

are subgraph of $\mathcal{G}$, and then $\mathcal{G}$ is a non-connected graph. □

Using (1) we can consider the Minkowski graph $\mathcal{G}$ with the group $\mathrm{Sym}(n)$ as the underlying vertices. That is, $\{\sigma, \sigma'\} \in E$ if and only if

$$d_1(\sigma, \sigma') = d_1\Big(\prod_{j=2}^{n} \sigma_j^{\epsilon_j}, \prod_{j=2}^{n} \sigma_j^{\delta_i}\Big) = \sum_{j=2}^{n} |\epsilon_j - \delta_j| = t.$$

Example 4 Following the above notation, we have

$$\mathrm{Sym}(4) = \langle \sigma_2, \sigma_3, \sigma_4 \rangle.$$

Then the table of distances would be given as shown in Table 1.

Table 1: Distances Table of Sym(4)

d	1	σ_2	σ_3	σ_3^2	σ_4	σ_4^2	σ_4^3	$\sigma_2\sigma_3$	$\sigma_2\sigma_3^2$	$\sigma_2\sigma_4$	$\sigma_2\sigma_4^2$	$\sigma_2\sigma_4^3$	$\sigma_3\sigma_4$	$\sigma_3\sigma_4^2$	$\sigma_3\sigma_4^3$	$\sigma_3^2\sigma_4$	$\sigma_3^2\sigma_4^2$	$\sigma_3^2\sigma_4^3$	$\sigma_2\sigma_3\sigma_4$	$\sigma_2\sigma_3\sigma_4^2$	$\sigma_2\sigma_3\sigma_4^3$	$\sigma_2\sigma_3^2\sigma_4$	$\sigma_2\sigma_3^2\sigma_4^2$	$\sigma_2\sigma_3^2\sigma_4^3$
1	0	1	1	2	1	2	3	2	3	2	3	4	2	3	4	3	4	5	3	4	5	4	5	6
σ_2	1	0	2	3	2	3	4	1	2	1	2	3	3	4	5	4	5	6	2	3	4	3	4	5
σ_3	1	2	0	1	2	3	4	1	2	3	4	5	1	2	3	2	3	4	2	3	4	3	4	5
σ_3^2	2	3	1	0	3	4	5	2	1	4	5	6	2	3	4	1	2	3	3	4	5	2	3	4
σ_4	1	2	2	3	0	1	2	3	4	1	2	3	1	2	3	2	3	4	2	3	4	3	4	5
σ_4^2	2	3	3	4	1	0	1	4	5	2	1	2	2	1	2	3	2	3	3	2	3	4	3	4
σ_4^3	3	4	4	5	2	1	0	5	6	3	2	1	3	2	1	4	3	2	4	3	2	5	4	3
$\sigma_2\sigma_3$	2	1	1	2	3	4	5	0	1	2	3	4	2	3	4	3	4	5	1	2	3	2	3	4
$\sigma_2\sigma_3^2$	3	2	2	1	4	5	6	1	0	3	4	5	3	4	5	2	3	4	2	3	4	1	2	3
$\sigma_2\sigma_4$	2	1	3	4	1	2	3	2	3	0	1	2	2	3	4	3	4	5	1	2	3	2	3	4
$\sigma_2\sigma_4^2$	3	2	4	5	2	1	2	3	4	1	0	1	3	2	3	4	3	4	2	1	2	3	2	3
$\sigma_2\sigma_4^3$	4	3	5	6	3	2	1	4	5	2	1	0	4	3	2	5	4	3	3	2	1	4	3	2
$\sigma_3\sigma_4$	2	3	1	2	1	2	3	2	3	2	3	4	0	1	2	1	2	3	1	2	3	2	3	4
$\sigma_3\sigma_4^2$	3	4	2	3	2	1	2	3	4	3	2	3	1	0	1	2	1	2	2	1	2	3	2	3
$\sigma_3\sigma_4^3$	4	5	3	4	3	2	1	4	5	4	3	2	2	1	0	3	2	1	3	2	1	4	3	2
$\sigma_3^2\sigma_4$	3	4	2	1	2	3	4	3	2	3	4	5	1	2	3	0	1	2	2	3	4	1	2	3
$\sigma_3^2\sigma_4^2$	4	5	3	2	3	2	3	4	3	4	3	4	2	1	2	1	0	1	3	2	3	2	1	2
$\sigma_3^2\sigma_4^3$	5	6	4	3	4	3	2	5	4	5	4	3	3	2	1	2	1	0	4	3	2	3	2	1
$\sigma_2\sigma_3\sigma_4$	3	2	2	3	2	3	4	1	2	1	2	3	1	2	3	2	3	4	0	1	2	1	2	3
$\sigma_2\sigma_3\sigma_4^2$	4	3	3	4	3	2	3	2	3	2	1	2	2	1	2	3	2	3	1	0	1	2	1	2
$\sigma_2\sigma_3\sigma_4^3$	5	4	4	5	4	3	2	3	4	3	2	1	3	2	1	4	3	2	2	1	0	3	2	1
$\sigma_2\sigma_3^2\sigma_4$	4	3	3	2	3	4	5	2	1	2	3	4	2	3	4	1	2	3	1	2	3	0	1	2
$\sigma_2\sigma_3^2\sigma_4^2$	5	4	4	3	4	3	4	3	2	3	2	3	3	2	3	2	1	2	2	1	2	1	0	1
$\sigma_2\sigma_3^2\sigma_4^3$	6	5	5	4	5	4	3	4	3	4	3	2	4	3	2	3	2	1	3	2	1	2	1	0

The t-graphs of Sym(4) for some t are presented in the next Figures.

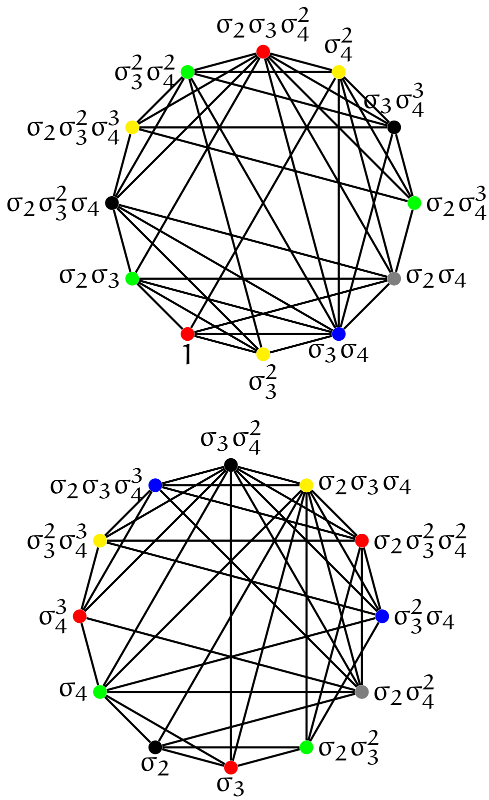
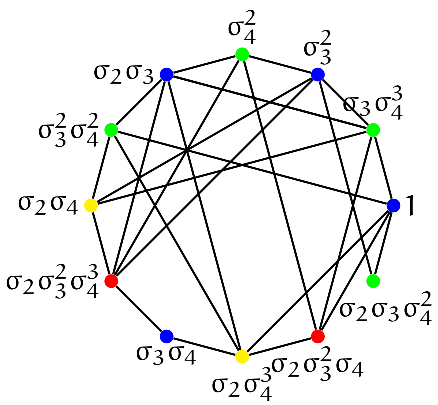


Figure 1: Minkowski 2-graph of Sym(4).



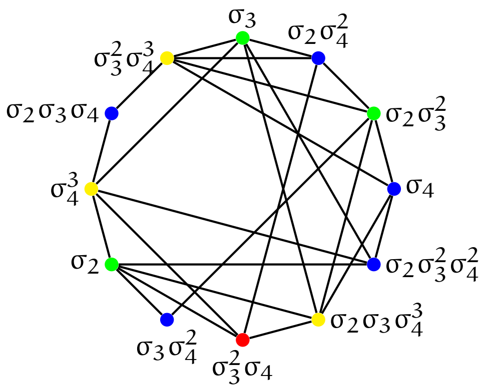


Figure 2: Minkowski 4-graph of Sym(4).

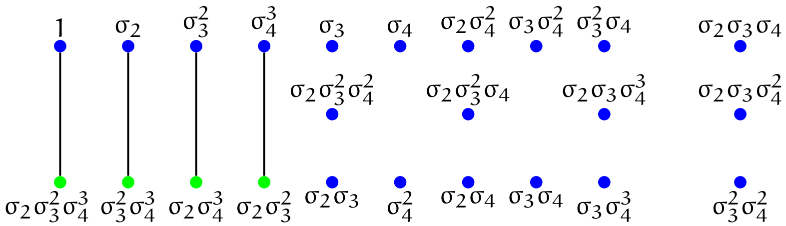


Figure 3: Minkowski 6-graph of Sym(4).

To analyse the behaviour of the number of connected components of the Kendall τ-graphs defined on Sym(n), we use the following theorem, which allows us to realize Tables 2 and 4. A proof of this theorem can be found in [12, Theorem 7.1].

Theorem 5 *A graph* $\mathcal{G}$ *has* k *connected components if and only if the algebraic multiplicity of zero as the Laplacian eigenvalue is* k*.*

Table 2: Number of connected components of the t-graphs of Sym(n)

n/t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	1	2	4	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4	1	2	1	2	6	20	-	-	-	-	-	-	-	-	-	-	-	-
5	1	2	1	2	1	2	5	26	68	112	-	-	-	-	-	-	-	-
6	1	2	1	2	1	2	1	2	1	10	57	186	386	584	704	-	-	-
7	1	2	1	2	1	2	1	2	1	2	1	2	9	74	313	882	1825	2978

4 The Kendall τ-graph of the symmetric group

Let n be a natural number, $[n] = \{1, \dots, n\}$. Throughout this section, we use vector notation for permutations. That is, if $\sigma \in \mathrm{Sym}(n)$ and $\sigma(k) = i_k$ for all $k \in \{1, \dots, n\}$, then we write $\sigma = [i_1, i_2, \dots, i_n]$. Permutation multiplication is defined as compositions of functions on two permutations from right to left.

An inversion is said to occur in a permutation $\sigma = [i_1, i_2, \dots, i_n]$ in $\mathrm{Sym}(n)$ whenever a larger integer precedes a smaller one. Formally, an *inversion* in $\sigma \in \mathrm{Sym}(n)$ is a pair $[\sigma(i), \sigma(j)]$ such that $i < j$ and $\sigma(i) > \sigma(j)$, where $\{i, j\} \in [n]$. With $\mathrm{Inv}(\sigma)$ we denote the set of inversions in σ, and $|\mathrm{Inv}(\sigma)|$ denotes the number of inversions in σ. For example, if $\sigma = [4, 2, 3, 1] \in \mathrm{Sym}(4)$, then $|\mathrm{Inv}(\sigma)| = 5$.

An *adjacent transposition* of a permutation $\sigma \in \mathrm{Sym}(n)$ is the local exchange of two adjacent elements in σ. That is,

$$\sigma = [i_1, \dots, i_{k-1}, i_k, i_{k+1}, \dots, i_n]$$

is changed to the permutation

$$\sigma' = [i_1, \dots, i_{k-1}, i_{k+1}, i_k, i_{k+2}, \dots, i_n].$$

If $\tau = (i, j) \in \mathrm{Sym}(n)$ is a transposition, then it can be expressed as a product of $2j - 2i - 1$ adjacent transpositions. Namely,

$$\tau = (i, i+1)(i+1, i+2) \dots (j-1, j)(j-1, j-2)(j-2, j-3) \dots (i+1, i).$$

Further $|\mathrm{Inv}(\tau)| = 2j - 2i - 1$.

It is well known that every element in $\mathrm{Sym}(n)$ can be factorised as a product of transpositions. It follows that the symmetric group of degree n is generated by the adjacent transpositions $\tau_j = (j, j+1)$, with $j = 1, \dots, n-1$. Every permutation can be written as a product of adjacent transpositions.

These adjacent transpositions satisfy the *Coxeter relations*:

$$\tau_j^2 = 1 \quad \text{for all} \quad j,$$

$$\tau_i \tau_j = \tau_j \tau_i \quad \text{for all} \quad |i - j| > 1, \tag{4}$$

$$\tau_j \tau_{j+1} \tau_j = \tau_{j+1} \tau_j \tau_{j+1} \quad \text{for all} \quad j \leqslant n-1. \tag{5}$$

The relations (4) and (5) are known as *commutations* and *braid rela-*

tions, respectively.

Let $\sigma \in \mathrm{Sym}(n)$, and $\tau = (i, j)$ be a transposition. Then the product $\sigma\tau$ is the same permutation as σ except that i and j are switched in the product. In a particular case that τ is an adjacent transposition, say $\tau = (j, j+1)$, it follows that $|\mathrm{Inv}(\sigma\tau)| = |\mathrm{Inv}(\sigma)| \pm 1$. Now, if we have n disjoint adjacent transpositions $\tau_1, \ldots, \tau_n$, then $\sigma\tau_1 \ldots \tau_n$ is the same permutation as σ except that each pair of numbers in each transposition has switched positions in the product. Further

$$|\mathrm{Inv}(\sigma\tau_1 \ldots \tau_n)| = \sum_{j=1}^{n} |\mathrm{Inv}(\sigma\tau_j)| + (1-n)|\mathrm{Inv}(\sigma)|.$$

In general, if $\tau_1, \ldots, \tau_n$ are adjacent transpositions there are some bounds for $|\mathrm{Inv}(\sigma\tau_1 \ldots \tau_n)|$, which are:

1. $|\mathrm{Inv}(\sigma)| - n \leqslant |\mathrm{Inv}(\sigma\tau_1 \ldots \tau_n)| \leqslant |\mathrm{Inv}(\sigma)| + n$,
2. There are just 4 possible values for $|\mathrm{Inv}(\sigma\tau_1 \ldots \tau_n)|$, and these are

   $|\mathrm{Inv}(\sigma\tau_1 \ldots \tau_n)| = |\mathrm{Inv}(\sigma)| + (n-2k)$, such hat $0 \leqslant k \leqslant \binom{n+1}{n} - 1$.

The proof of all these remarks can be found in [13].

Definition 6 For $\alpha, \beta \in \mathrm{Sym}(n)$ we denote with $d_K(\alpha, \beta)$ the minimal number of adjacent transpositions required to change α into β. The *Kendall* τ*-weight* of $\sigma \in \mathrm{Sym}(n)$, denoted by $\mathrm{wt}(\sigma)$, is defined as $d_K(\sigma, 1)$, where 1 is the identity permutation.

The distance d_K induces a metric over $\mathrm{Sym}(n)$ and is called the *Kendall* τ*-metric* (see [8]). It is well-known that the Kendall τ-metric is right invariant [3]. That is, for every $\alpha, \beta, \sigma \in \mathrm{Sym}(n)$ holds

$$d_K(\alpha\sigma, \beta\sigma) = d_K(\alpha, \beta).$$

By this observation, for any two permutations $\alpha, \beta \in \mathrm{Sym}(n)$ we have that

$$d_K(\alpha, \beta) = wt(\alpha\beta^{-1}).$$

A. Jiang et al. proved in [7] that d_K can be represented as follows:

$$d_K(\alpha, \beta) = |\{(i, j) \mid \alpha(i) < \alpha(j) \ \wedge \ \beta(i) > \beta(j)\}|.$$

Example 7 Let G be the symmetric group of degree 3. That is,

$$G = \{[1,2,3],[2,1,3],[3,2,1],[1,3,2],[2,3,1],[3,1,2]\}$$

in vector notation, the Kendall τ-distance table is given by

Table 3: The Kendall τ-distance of Sym(3)

d	[1,2,3]	[2,1,3]	[3,2,1]	[1,3,2]	[2,3,1]	[3,1,2]
[1,2,3]	0	1	3	1	2	2
[2,1,3]	1	0	2	2	1	3
[3,2,1]	3	2	0	2	1	1
[1,3,2]	1	2	2	0	3	1
[2,3,1]	2	1	1	3	0	2
[3,1,2]	2	3	1	1	2	0

Definition 8 Let $n, t \in \mathbb{N}$, with $1 \leqslant t \leqslant \binom{n}{2}$. The *Kendall* t-*graph* of $G = \mathrm{Sym}(n)$ is defined as the pair $\mathcal{G} = (G, E)$, where two permutations α and $\beta \in G$ are adjacent if and only if $d_K(\alpha, \beta) = t$.

Remark 9 The parameter t is restricted to those values since there is a unique permutation, namely, $\alpha = [n, n-1, \ldots, 2, 1]$ such that $wt(\alpha) = \binom{n}{2}$ (by Equation (2)), which is the largest value.

Example 10 We present the Kendall t-graph of Sym(3) for $t = 1, 2, 3$.

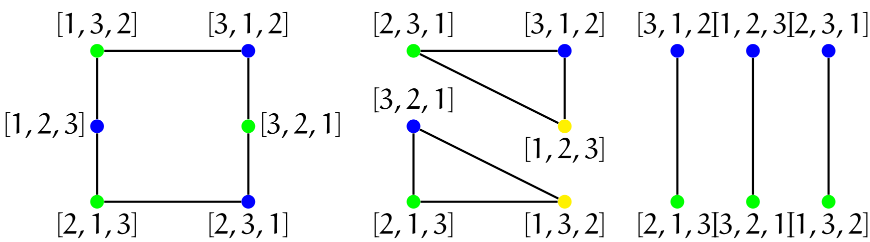


Figure 4: Some Kendall t-graphs of Sym(3)

Proposition 11 *Let* $n \in \mathbb{N}$, G *be the symmetric group of degree* n, *and* t *be a positive even number. Then the Kendall* t-*graph* $\mathcal{G}$ *is disconnected.*

Proof — Suppose $\mathcal{G}$ is connected, then there exists a finite path $(v_1, v_2, \dots, v_k)$ from $[1,2,3,\dots,n]$ to $[2,1,3,\dots,n]$, this implies that there are t adjacent permutations such that v_1 is the composition of these t permutations, similarly there exist other t permutations such that v_2 is the composition of v_1 and these t permutations, keeping these process, we see that $[2,1,3,\dots,n]$ is the composition of tk permutations, which is absurd since $[2,1,3,\dots,n]$ is an odd permutation, then $\mathcal{G}$ must be disconnected. □

Lemma 12 *Let* $\mathcal{G}$ *the Kendall* t*-graph, if* t *is odd then* $\mathcal{G}$ *is bipartite.*

Proof — Let A be the alternating subgroup of G and B be the complement of A, i.e., the set of odd permutations. If two vertex $u, v \in A$ were connected, then one of them is an odd permutation (since t is odd), and then there is no edge between elements in A, similarly for B. Therefore all edges must connect an element from A to one in B. This completes the proof. □

The converse is generally not true; if we take $G = \mathrm{Sym}(4)$ and $t = 6$, then $\mathcal{G}$ is bipartite.

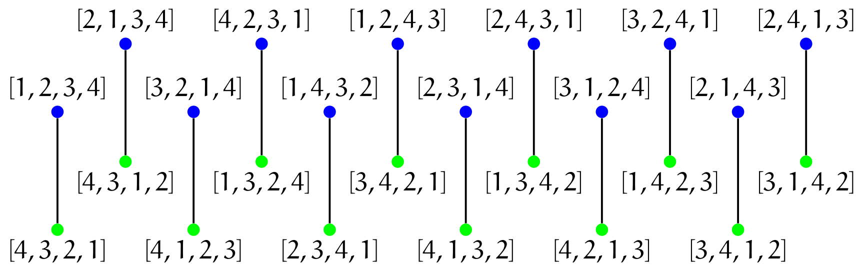


Figure 5: Kendall 6-graph of Sym(4)

The girth of a graph $\mathcal{G}$ denoted $g(\mathcal{G})$ is the length of the shortest cycle contained in the graph. If $\mathcal{G}$ is an acyclic graph, we say that $g(\mathcal{G}) = \infty$.

Corollary 13 *Let* $n \in \mathbb{N}$, G *be the symmetric group of degree* n*, and* t *be a positive odd number. Then the Kendall* t*-graph* $\mathcal{G}$ *has no triangles.*

Proof — It follows from the above lemma since any bipartite graph has no odd length cycle. □

The above proposition implies that for t odd, $g(\mathcal{G}) \geqslant 4$.

Theorem 14 *Let* $t \in \mathbb{N}$ *and* $n \geqslant 2t+2$, *then the Kendall* t*-graph* $\mathcal{G}$ *has a cycle of length* 4.

Proof — Consider the vertices

$$\alpha_1 = 1, \alpha_2 = \tau_1 \dots \tau_t, \alpha_3 = \tau_{t+2} \dots \tau_{2t+1}, \alpha_4 = \alpha_2 \alpha_3.$$

It is clear that $d_K(\alpha_1, \alpha_2) = d_K(\alpha_1, \alpha_3) = t$. On the other hand, since α_2 and α_3 are disjoint permutations then $\alpha_4 = \alpha_3\alpha_2$, therefore

$$d_K(\alpha_2, \alpha_4) = d_K(\alpha_2, \alpha_3\alpha_2) = d_K(\alpha_1, \alpha_3) = t.$$

Similarly we have that $d_K(\alpha_3, \alpha_4) = t$, then $\mathcal{G}$ has a length 4 cycle. □

This means that for those values of t and n, we have $g(\mathcal{G}) \leqslant 4$.

Corollary 15 *If* t *is an odd natural number and* $n \geqslant 2t+2$, *then* $g(\mathcal{G})=4$.

In Figure 5, we can see that the graph consists of 12 connected components, all isomorphic to the complete bipartite graph $K_{1,1}$. The following lemma characterises the Kendall $\binom{n}{2}$-graph for Sym(n).

Lemma 16 *Let* $n \in \mathbb{N}$. *The Kendall* $\binom{n}{2}$*-graph* $\mathcal{G}$ *of the symmetric group of degree* n *has* $n!/2$ *connected components, all isomorphic to* $K_{1,1}$.

Proof — It follows from the fact that there is a unique permutation α such that $wt(\alpha) = \binom{n}{2}$. □

Proposition 17 *The Kendall* t*-graph of* Sym(n) *is* k*-regular for some* $k \in \mathbb{N}$.

Proof — Let $\sigma \in$ Sym(n); we define $D(\sigma) := \{\alpha | d_K(\sigma, \alpha) = t\}$, clearly $deg(\sigma) = |D(\sigma)|$, since the Kendall metric is right invariant then it follows that $|D(\sigma)| = |D(1)|$, therefore $deg(\sigma) = |D(1)|$ for all $\sigma \in$ Sym(n). □

Theorem 18 *The Kendall* 1*-graph of* Sym(n) *is* $(n-1)$*-regular.*

Proof — Let $\sigma \in$ Sym(n). Every permutation of the form $\beta_j := \sigma\tau_j$ with $j = 1, \dots, n-1$ is adjacent to σ. Therefore $deg(\sigma) = n-1$, and the assertion is true. □

Theorem 19 *The Kendall* 2*-graph of* Sym(n) *is* $\left(\binom{n}{2} - 1\right)$*-regular.*

Proof — First notice that in $\mathrm{Sym}(n)$ there are $n-1$ adjacent transpositions, and that if $|i-j|>1$ then $\tau_i\tau_j=\tau_j\tau_i$ for all $i,j\in\{1,2,\dots,n-1\}$, let $\sigma\in\mathrm{Sym}(n)$ we want the number of $\beta_{i,j}$ such that $\beta_{i,j}=\sigma\tau_i\tau_j$. There are $n-1$ choices for i, and $n-2$ choices for j since j cannot be equal to i; from all of those choices, some duplicates correspond to commutativity, for $i=1,n-1$ there are $(n-3)$ adjacent transpositions which commute with τ_i, for $1<i<n-1$ there are $(n-4)$ of this transpositions. Therefore it leads to the following

$$\deg(\sigma)=(n-1)(n-2)-\frac{2(n-3)+(n-3)(n-4)}{2}=\binom{n}{2}-1.$$

Since the choice of σ was arbitrary, it follows that the Kendall 2-graph of $\mathrm{Sym}(n)$ is $\left(\binom{n}{2}-1\right)$-regular. □

Theorem 20 *The Kendall 3-graph of* $\mathrm{Sym}(n)$ *is* $\left(\binom{n+1}{3}-\binom{n}{1}\right)$*-regular.*

Proof — Given a permutation $\sigma=(a_1,\dots,a_n)$ there are 3 possibilities

1. We do (a_k,a_{k+1}), (a_m,a_{m+1}) and (a_l,a_{l+1}), that is, composition with 3 adjacent transpositions.

2. We do (a_k,a_{k+1}), (a_m,a_{m+2}), that is, composition with 1 adjacent transposition and a 2-cycle of weight 2.

3. We do (a_k,a_{k+3}), that is, composition with a 2-cycle of weight 3.

For the first there are $\binom{n-1}{3}$ ways of choosing, for the second there are $\binom{n-1}{1}\binom{n-2}{1}$ ways and for the latter $\binom{n-3}{1}$, and finally we exclude the $n-2$ braids. Therefore the degree of σ is

$$\deg(\sigma)=\binom{n-1}{3}+\binom{n-1}{1}\binom{n-2}{1}+\binom{n-3}{1}-(n+2)=\binom{n+1}{3}-\binom{n}{1}.$$

The statement is proved. □

Theorem 21 *The Kendall* 1*-graph* $\mathcal{G}$ *of* $\mathrm{Sym}(n)$ *is planar if and only if* $n\leqslant 4$.

Proof — The necessity can be checked straightforwardly. For the sufficiency, let's suppose that $n>4$ since $\mathcal{G}$ is planar and by Corollary 13 it has no triangles, then

$$|E|\leqslant 2|G|-4, \tag{6}$$

by Theorem 18 we have $|E| = \frac{n!(n-1)}{2}$, replacing in Equation 6 we have $(n-5)n! \leqslant -8$, which is absurd, then $n \leqslant 4$. □

Finally, we present a table with the number of connected components of the Kendall graph.

Table 4: Number of connected components of the τ-graphs of Sym(n)

n/τ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3	1	2	3	-	-	-	-	-	-	-	-	-	-	-	-
4	1	2	1	2	1	12	-	-	-	-	-	-	-	-	-
5	1	2	1	2	1	2	1	2	1	60	-	-	-	-	-
6	1	2	1	2	1	2	1	2	1	2	1	2	1	2	360

Conjecture 22 *Let $n \in \mathbb{N}$. Then the chromatic number of the Kendall 2-graph of* Sym(n) *is* n.

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