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Two Generator p-Groups of Class Two of Central Type

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Abstract

We investigate the 2-generator p-groups of class two of central type.

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1 Introduction

Groups of central type were first introduced by DeMeyer and Janusz in [3]. A finite group Γ is called of *central type* if there is an irreducible character χ of Γ such that $\chi(1)^2 = |\Gamma : Z(\Gamma)|$. Theorem 2 in [3] establishes that Γ is of central type if and only if for any Sylow p-subgroup P of Γ, P is of central type and $Z(P) = Z(\Gamma) \cap P$. Thus, it is important to study the p-groups of central type in order to understand groups of central type. Using the classification of finite simple groups, Isaacs and Howlett proved in [7] that a group of central type is solvable.

In [12], Shariari called a finite group G a *central type factor group* if there exists a central type group Γ such that $G \simeq \Gamma/Z(\Gamma)$. By abuse of language, we will call such a group G a group of central type as well.

Definition 1.1 A finite group G is *of central type* if $G \simeq \Gamma/Z(\Gamma)$, where Γ is a finite group such that $|\Gamma : Z(\Gamma)| = \chi(1)^2$ for some irreducible character χ of Γ.

The classification of abelian groups of central type is known. Shariari proved in Corollary 3.10 of [12] that G is an abelian group of central type if and only if $G = H \times \phi(H)$, where $\phi : H \to G$ is an embedding of abelian groups. Other results about central type factor groups are known. For instance, Ginosar and Schnabel classified all the groups of central type of order n^2 where n is a square-free number (see [5]). The groups of central type of order p^4 have been classified in [6] and [11].

In this paper we classify the 2-generator p-groups of class two of central type. We will use the classification of 2-generator p-groups of class two proved in [1]. The following is Theorem 1.1 in [1].

Theorem 1.2 *Let p be a prime and $n > 2$ a positive integer. Every 2-generated p-group of class exactly 2 corresponds to an ordered 5-tuple of integers $(\alpha, \beta.\gamma; \rho, \sigma)$ such that:*

(i) $\alpha \geqslant \beta \geqslant \gamma \geqslant 1$,

(ii) $\alpha + \beta + \gamma = n$,

(iii) $0 \leqslant \rho \leqslant \gamma$ *and* $0 \leqslant \sigma \leqslant \gamma$,

where $(\alpha, \beta, \gamma; \rho, \sigma)$ corresponds to the group presented by

$$G_p(\alpha, \beta, \gamma; \rho, \sigma) = \big\langle a, b \mid [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1,$$
$$a^{p^\alpha} = [a, b]^{p^\rho}, b^{p^\beta} = [a, b]^{p^\sigma} \big\rangle.$$

Our main result is the following one.

Theorem 1.3 *Let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be a 2-generator p-group of class 2. Then G is of central type if and only if one of the following conditions holds:*

(1) $G = G_2(2, 1, 1; 1, 1)$.

(2) $p > 2$, $G = G_p(\beta + \gamma, \beta, \gamma; \gamma, 0)$.

(3) $G = G_2(\beta + \gamma, \beta, \gamma; \gamma, 0)$, $\gamma > 1$.

2 2-Generators p-groups of class two

The 2-generator p-groups of class two have been classified in [1]. In this section we introduce some notations and definitions in order to restate the main result from [1]. In what follows, let p be a prime and let $n > 2$ be an integer.

Definition 2.1 We define $\mathcal{T}_n$ to be the set of ordered 5-tuples of integers satisfying the following conditions:

$$\mathcal{T}_n = \left\{ (\alpha, \beta, \gamma; \rho, \sigma) \ : \ \begin{array}{l} \alpha \geqslant \beta \geqslant \gamma \geqslant 1 \\ \alpha + \beta + \gamma = n \\ 0 \leqslant \rho, \sigma \leqslant \gamma \end{array} \right\}.$$

Definition 2.2 Let $(\alpha, \beta, \gamma; \rho, \sigma) \in \mathcal{T}_n$. We define $G_p(\alpha, \beta, \gamma; \rho, \sigma)$ to be the group presented by

$$G_p(\alpha, \beta, \gamma; \rho, \sigma) = \Big\langle a, b \ \mid \ [a, b]^{p^\gamma} = [a, b, a] = [a, b, b] = 1,$$
$$a^{p^\alpha} = [a, b]^{p^\rho}, b^{p^\beta} = [a, b]^{p^\sigma} \Big\rangle$$

where $[a, b] = a^{-1}b^{-1}ab$.

Remark 2.3 Let G be a group as in Definition 2.2. In [1] it is proved that G is a 2-generator p-group of class two, of order p^n, and $|G'| = p^\gamma$.

Definition 2.4 We define the following subsets of $\mathcal{T}_n$*:*

$$\mathcal{S}_{1.a} = \{(\alpha, \beta, \gamma; \rho, \gamma) \in \mathcal{T}_n \ : \ \alpha > \beta \geqslant \gamma \geqslant \rho \geqslant 0\},$$

$$\mathcal{S}_{1.b} = \{(\alpha, \beta, \gamma; \gamma, \sigma) \in \mathcal{T}_n \ : \ \alpha > \beta \geqslant \gamma > \sigma \geqslant 0\},$$

$$\mathcal{S}_{1.c} = \{(\alpha, \beta, \gamma; \rho, \sigma) \in \mathcal{T}_n \ : \ \alpha > \beta \geqslant \gamma \text{ and } \min(\gamma, \sigma + \alpha - \beta) > \rho > \sigma \geqslant 0\},$$

$$\mathcal{S}_2 = \{(\alpha, \alpha, \gamma; \rho, \gamma) \in \mathcal{T}_n \ : \ \alpha > \gamma \geqslant \rho \geqslant 0\},$$

$$\mathcal{S}_{3.a} = \{(\gamma, \gamma, \gamma; \rho, \gamma) \in \mathcal{T}_n \ : \ 0 \leqslant \rho \leqslant \gamma\},$$

$$\mathcal{S}_{3.b} = \{(\gamma, \gamma, \gamma; \rho, \gamma) \in \mathcal{T}_n : 0 \leqslant \rho \leqslant \gamma, \rho \neq \gamma - 1\} \cup \{(\gamma, \gamma, \gamma; \gamma - 1, \gamma - 1) \in \mathcal{T}_n\}.$$

Now we define

$$\mathcal{S}_{n,p} = \begin{cases} \mathcal{S}_{1.a} \cup \mathcal{S}_{1.b} \cup \mathcal{S}_{1.c} \cup \mathcal{S}_2 \cup \mathcal{S}_{3.a} & \text{if } p \text{ is odd} \\ \mathcal{S}_{1.a} \cup \mathcal{S}_{1.b} \cup \mathcal{S}_{1.c} \cup \mathcal{S}_2 \cup \mathcal{S}_{3.b} & \text{if } p = 2 \end{cases}$$

Definition 2.5 A 5-tuple $(\alpha, \beta, \gamma; \rho, \sigma) \in \mathcal{T}_n$ is a p-good 5-tuple if $(\alpha, \beta, \gamma; \rho, \sigma) \in \mathcal{S}_{n,p}$.

Theorem 2.6 *Let G be a 2-generator p-group of class two and order p^n. There exists a unique p-good 5-tuple $(\alpha, \beta, \gamma; \rho, \sigma) \in \mathcal{S}_{n,p}$ such that G is isomorphic to $G_p(\alpha, \beta, \gamma; \rho, \sigma)$.*

Proof — This is a restatement of Theorem 1.1 in [1]. □

In the remainder of the paper we will assume that every 2-generator p-group of class two and order p^n corresponds to a unique p-good 5-tuple $(\alpha, \beta, \gamma; \rho, \sigma) \in \mathcal{S}_{n,p}$ and is generated by a, b as in Definition 2.2.

Remark 2.7 Let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be 2-generator p-group of class two. Then

(1) $Z(G) \simeq \begin{cases} C_{p^{\alpha-\rho}} \times C_{p^{\beta-\gamma}} \times C_{p^{\rho}} & \text{if } \rho \leqslant \sigma \\ C_{p^{\alpha-\gamma}} \times C_{p^{\beta-\sigma}} \times C_{p^{\sigma}} & \text{if } \sigma < \rho \end{cases}$

(2) $G/G' \simeq C_{p^{\alpha}} \times C_{p^{\beta}}$

where C_n denotes the cyclic group of order n.

Definition 2.8 A group G is *capable* if there exists a group Γ such that $G \simeq \Gamma/Z(\Gamma)$.

The capable 2-generator p-groups of class two have been classified in [10]. For completeness we state the main results from [10].

Theorem 2.9 (see Theorem 63 in [10]) *Let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be any 2-generator p-group of class 2 with presentation as in* Definition 2.2 *and $p > 2$. Then G is capable if and only if*

$$\alpha - \beta = \rho - \sigma \qquad \textit{and} \qquad \gamma = \rho.$$

For $p = 2$, we have thee following result.

Theorem 2.10 (see Theorem 67 in [10]) *Let $G = G_2(\alpha, \beta, \gamma; \rho, \sigma)$ be any 2-generator 2-group of class 2 with presentation as in* Definition 2.2. *Then G is capable if and only if it meets one of the following conditions*

(i) $\rho \leqslant \sigma$, $\alpha = \beta$, *and* $\rho = \gamma$;

(ii) $\rho > \sigma$, $\alpha - \beta = \rho - \sigma > \delta_{\beta\gamma}$, *and* $\rho = \gamma$;

(iii) $\rho = \sigma = \gamma = \beta$, *and* $\alpha = \gamma + 1$;

where δ_{ij} is Kronecker's delta.

3 Groups of central type

It is straightforward from Definition 1.1 that any group of central type is capable.

Suppose we have $G \simeq \Gamma/Z(\Gamma)$. Then we have an exact sequence:

$$1 \to Z(\Gamma) \to \Gamma \to G \to 1.$$

In this case, an irreducible representation of Γ induces a projective representation $\mathfrak{X}$ of G.

Given $\mathfrak{X}$ a projective representation of G, we have a 2-cocycle:

$$f : G \times G \to \mathbb{C}^*.$$

Then we can define the twisted group algebra $\mathbb{C}^f G$ as follows:

Definition 3.1 Let G be a finite group and let $f \in Z^2(G, \mathbb{C}^*)$ be a 2-cocycle. The *twisted group algebra* is the $\mathbb{C}$-algebra with basis $\{u_g\}_{g \in G}$ and multiplication

$$u_g u_h = f(g, h) u_{gh}$$

extended by distributivity.

The twisted algebra depends up to $\mathbb{C}$-isomorphism only on the cohomology class of f. We denote the twisted group algebra by $\mathbb{C}^f G$.

The twisted algebra $\mathbb{C}^f G$ is always semisimple (see Theorem 3.2.10 in [8]). By Maschke's Theorem, the group algebra $\mathbb{C}G$ is also semisimple. However, $\mathbb{C}G$ is not simple unless G is trivial. By contrast $\mathbb{C}^f G$ can be simple, as illustrated in the following theorem:

Theorem 3.2 (Theorem 1 in [3]) *A group G is a group of central type if and only if there exists $f \in Z^2(G, \mathbb{C}^*)$ such that $\mathbb{C}^f G$ is simple.*

Definition 3.3 If $\mathbb{C}^f G$ is a central simple $\mathbb{C}$-algebra, then f is called *non-degenerate*.

If $\mathbb{C}^f G$ is not a central simple $\mathbb{C}$-algebra, then $\dim Z(\mathbb{C}^f G) > 1$. Hence, there exists $1 \neq x \in G$ such that

$$f(x, g) = f(g, x)$$

for all $g \in C_G(x)$.

Definition 3.4 Let $f \in Z^2(G, \mathbb{C}^*)$. A non-identity element $x \in G$ is called f-*regular* if

$$f(x, g) = f(g, x)$$

for all $g \in C_G(x)$.

Remark 3.5 (1) x is f-regular if and only if

$$[u_x, u_g] = 1$$

for all $g \in C_G(x)$, where $[u_x, u_g] = u_x^{-1} u_b^{-1} u_x u_g$.

(2) G is of central type if there exists $f \in Z^2(G, \mathbb{C}^*)$ such that $x = 1$ is the only f-regular element. So in order to show that G is not of central type it is enough to show that there exists $1 \neq x \in G$ such that for all $g \in C_G(x)$ we have

$$[u_x, u_g] = 1$$

for all $f \in Z^2(G, \mathbb{C}^*)$.

The following lemma is Lemma 2.5 in [11].

Lemma 3.6 *Let G be a finite group and let $f \in Z^2(G, \mathbb{C}^*)$ and let $a, b \in G$ be commuting elements such that in $\mathbb{C}^f G$ we have $[u_a, u_b] = \lambda$. Then λ is a root of unity of order dividing the greatest common divisor of $o(a)$ and $o(b)$.*

Theorem 3.7 (see Theorem C in [4]) *Let G be a group of central type. Then the center of G embeds into the group $\widehat{G}$ of 1-dimensional characters of G.*

Summarizing we have the following: if G is a group of central type then

- By Definition 1.1, $|G| = |\Gamma : Z(\Gamma)| = \chi(1)^2$, so the order of G must be a perfect square.
- $Z(G)$ is isomorphic to a subgroup of G/G'.
- Since $\mathbb{C}$ is algebraically closed, if $o(g) = m$ then we can always assume that for any $f \in Z^2(G, \mathbb{C}^*)$, we have $u_g^m = 1$ in the corresponding twisted algebra $\mathbb{C}^f G$ (see page 4 in [11]).
- If $a, b \in G$ are commuting elements such that $[u_a, u_b] = \lambda$. Then λ is a root of unity of order dividing the $\gcd\big(o(a), o(b)\big)$.

4 Main Results

In this section let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be any 2-generator p-group of class 2 with presentation as in Definition 2.2. Let $f \in Z^2(G, \mathbb{C}^*)$ be a 2-cocycle. Then $\mathbb{C}^f G$ is generated by u_a, u_b, and u_c and the following hold:

$$u_a^{p^{\alpha+\gamma-\rho}} = u_b^{p^{\beta+\gamma-\sigma}} = u_c^{p^\gamma} = 1,$$
$$[u_a, u_b] = \zeta u_c, \; [u_a, u_c] = \lambda, \; \text{ and } \; [u_c, u_b] = \mu,$$

where ζ, λ, and μ are complex roots of unity.

The following lemma gives the commutator relations in $\mathbb{C}^f G$.

Lemma 4.1 *With the above notations for all integers $r, s \geqslant 1$ we have:*

(1) $[u_a^r, u_c] = [u_a, u_c^r] = \lambda^r$.

(2) $[u_c^r, u_b] = [u_c, u_b^r] = \mu^r$.

(3) $[u_a^r, u_b^s] = \zeta^{rs} \lambda^{-\frac{r(r-1)}{2}s} \mu^{\frac{s(s-1)}{2}r} u_c^{rs}$.

In particular, $\lambda^{p^\rho} = \mu^{p^\sigma} = 1$.

Proof — It follows by induction on r and s. In particular, since $a^{p^\alpha} = c^{p^\rho} = 1$ we have $\lambda^{p^\rho} = [u_a, u_c^{p^\rho}] = [u_a, u_a^{p^\rho}] = 1$. Similarly, for $\mu^{p^\sigma} = 1$. □

Theorem 4.2 *If G is of central type, then $\rho = \gamma$.*

Proof — It follows immediately from Theorems 2.9 and 2.10, and the fact that every group of central type is capable. □

Theorem 4.3 *Let G be a group of central type. Then $\sigma = 0$ or $\sigma = \gamma$.*

Proof — Since G is of central type, we have $\rho = \gamma$ and $Z(G) \hookrightarrow G/G'$. Also, $Z(G) \simeq C_{p^{\alpha-\gamma}} \times C_{p^{\beta-\sigma}} \times C_{p^\sigma}$ and $G/G' \simeq C_{p^\alpha} \times C_{p^\beta}$. We obtain $\alpha = \beta = \gamma$. If p is odd then we cannot have $\alpha - \beta = \rho - \sigma$, so the condition of Theorem 2.9 is not satisfied. If $p = 2$ then again the condition (ii) of Theorem 2.10 is not satisfied. Thus, G is not of central type. □

Theorem 4.4 *If $p > 2$ and $G \in \mathcal{S}_{1,a}$, then G is not of central type.*

Proof — Let $G = G_p(\alpha, \beta, \gamma; \rho, \gamma) \in \mathcal{S}_{1,a}$ be of central type, so $\alpha > \beta$ and from Theorem 2.9 we must have

$$\alpha - \beta = \rho - \sigma.$$

This is impossible beacuse $\rho = \sigma = \gamma$, but $\alpha > \beta$. Thus, G is not of central type. $\square$

Theorem 4.5 *If* $p = 2$ *and* $G \in \mathcal{S}_{1,a}$, *then* G *is of central type if and only if* $G = G_2(2, 1, 1; 1, 1)$.

Proof — It has been shown in [11] that $G_2(2, 1, 1; 1, 1)$ is of central type (see Theorem 3.9 in [11]). Assume

$$G = G_2(\alpha, \beta, \gamma; \rho, \gamma) \in \mathcal{S}_{1,a}$$

is of central type. We claim that $G = G_2(2, 1, 1; 1, 1)$. First, we must have $\alpha > \beta$ and $\rho = \gamma$. Hence, by Theorem 2.10 we must have $\gamma = \beta$ and $\alpha = \gamma + 1$.

We will show that if $\gamma > 1$, then $G = G_2(\gamma + 1, \gamma, \gamma; \gamma, \gamma)$ is not of central type. Note that since $|G| = 2^{3\gamma+1}$ and $|G|$ must be a perfect square we must have γ is odd. We have that $\mathbb{C}^f G$ is generated by u_a, u_b, and u_c where

$$u_a^{2^{\gamma+1}} = u_b^{2^\gamma} = u_c^{2^\gamma} = 1,$$

$$[u_a, u_b] = \zeta u_c,\ [u_a, u_c] = \lambda,\ [u_c, u_b] = \mu,$$

and ζ, λ, μ are complex numbers such that $\zeta^{2^{\gamma+1}} = \lambda^{2^\gamma} = \mu^{2^\gamma} = 1$. We also have $1 = [u_a, u_b^{2^\gamma}] = \zeta^{2^\gamma} \mu^{-2^{\gamma-1}}$. Thus, $\zeta^{2^\gamma} = \mu^{2^{\gamma-1}}$.

Case 1 Suppose that $o(\lambda) = o(\mu) = 2^\gamma$. Then $\zeta^{2^\gamma} = \mu^{2^{\gamma-1}} = \lambda^{2^{\gamma-1}} = -1$. We have $1 \neq a^{2^\gamma} \in Z(G)$ and for any $y = a^r b^s c^t \in G$ we obtain

$$\begin{aligned} [u_{a^{2^\gamma}}, u_y] &= [u_a^{2^\gamma}, u_a^r u_b^s u_c^t] = \zeta^{2^\gamma s} \lambda^{\frac{-2^\gamma(2^\gamma-1)}{2} s} \mu^{\frac{s(s-1)}{2} 2^\gamma} \lambda^{2^\gamma t} \\ &= \zeta^{2^\gamma s} \lambda^{2^{\gamma-1} s} = (-1)^s (-1)^s = 1. \end{aligned}$$

This shows that a^{2^γ} is an f-regular element. Thus, G is not of central type.

Case 2 Suppose that $o(\lambda), o(\mu) < 2^\gamma$, and let $2^\delta = \max\big(o(\lambda), o(\mu)\big)$.

Then $x = c^{2^\delta}$ is an f-regular element since

$$[u_c^{2^\delta}, u_a^r u_b^s u_c^t] = \lambda^{-2^\delta r} \mu^{2^\delta s} = 1$$

for any $y = a^r b^s c^t \in G$. Thus, G is not of central type.

Case 3 Suppose that $o(\lambda) = 2^\gamma$ and $o(\mu) = 2^{\gamma-\delta}$, where $1 \leqslant \delta \leqslant \gamma$. Then $\zeta^{2^\gamma} = \mu^{2^{\gamma-1}} = 1$. Without loss of generality we can assume that $\mu = \lambda^{2^\delta m}$ for some odd integer m.

We will show that $x = a^{2^{\delta+1} m} b^2 c$ is f-regular. We have:

$$C_G(x) = \left\{ a^r b^s c^t \;:\; 2^{\delta+1} ms - 2r \equiv 0 \bmod 2^\gamma \right\}.$$

Let $y = a^r b^s c^t \in C_G(x)$ and assume that $2^{\delta+1} ms - 2r = u \cdot 2^\gamma$ for some integer u. Then using properties of commutators we have:

$$[u_x, u_y] = [u_a^{2^{\delta+1} m} u_b^2 u_c, u_a^r u_b^s u_c^t]$$

$$= \left([u_a^{2^{\delta+1} m}, u_c^t][u_a^{2^{\delta+1} m}, u_a^r u_b^s] \right)^{u_b^2} [u_b^2, u_c^t][u_b^2, u_a^r u_b^s][u_c, u_b^s][u_c, u_a^r]$$

$$= \zeta^{2^{\delta+1} ms - 2r} \lambda^{2^{\delta+1} mt - \frac{2^{\delta+1} m(2^{\delta+1} m - 1)}{2} s + \frac{2r(r-1)}{2} - r} w\, u_c^{2^{\delta+1} ms - 2rs},$$

where

$$w = \mu^{\frac{s(s-1)}{2} 2^{\delta+1} m - r + s - 2t + 2 \cdot 2^{\delta+1} ms - 2rs}.$$

Since $2^{\delta+1} ms - 2r = u \cdot 2^\gamma$, $\zeta^{2^\gamma} = 1$, $u_c^{2^\gamma} = 1$, and $\mu = \lambda^{2^\delta m}$, we obtain $[u_x, u_y] = \lambda^E$, where

$$E = \frac{2^{\delta+2} mt - 2^{2\delta+2} m^2 s + 2^{\delta+1} ms + 2r^2 - 2r - 2r + 2^{2\delta+1} m^2 s^2 - 2^{2\delta+1} m^2 s - 2^{\delta+1} mr}{2}$$

$$+ \frac{2^{\delta+1} ms - 2^{\delta+2} mt + 2 \cdot 2^{2\delta+2} m^2 s - 2 \cdot 2^{\delta+1} mrs}{2}.$$

Using again the fact that $2^{\delta+1} ms - 2r = u \cdot 2^\gamma$, we can simplify E. We obtain:

$$E = \frac{u \cdot 2^\gamma + u^2 \cdot 2^{2\gamma-1} + u \cdot 2^\gamma - 2^{\delta+1} m(2^{\delta+1} ms + 2^\delta ms + r - 2^{\delta+2} ms)}{2}.$$

Now, $2^{\delta+2}ms = 2^{\delta+1}ms + 2^{\delta+1}ms = 2^{\delta+1}ms + 2^{\delta}ms + 2^{\delta}ms$. Hence,

$$\begin{aligned} E &= \frac{2u2^{\gamma} + u^2 2^{\gamma+\gamma-1} - 2^{\delta+1}m(-2^{\delta}ms + r)}{2} \\ &= u \cdot 2^{\gamma} + u^2 \cdot 2^{\gamma+\gamma-2} - 2^{\delta}m \cdot u \cdot 2^{\gamma-1}. \end{aligned}$$

Since $\gamma \geqslant 3$ and $\delta \geqslant 1$, we obtain $E \equiv 0 \bmod 2^{\gamma}$. Hence,

$$[u_x, u_y] = \lambda^E = 1$$

and x is an f-regular element.

Case 4 Assume that $o(\mu) = 2^{\gamma}$ and $o(\lambda) = 2^{\gamma-\delta}$ for some $1 \leqslant \delta < \gamma$. As in the previous case we can assume that $\lambda = \mu^{2^{\delta}m}$ for some odd integer m.

We will show that $x = ab^{2^{\delta}m}c$ is f-regular. We have:

$$C_G(x) = \left\{a^r b^s c^t \ : \ s - 2^{\delta}mr \equiv 0 \bmod 2^{\gamma}\right\}.$$

Let $y = a^r b^s c^t \in C_G(x)$ and assume that $s - 2^{\delta}mr = u \cdot 2^{\gamma}$ for some integer u. Then as in the previous case,

$$[u_x, u_y] = [u_a u_b^{2^{\delta}m} u_c, u_a^r u_b^s u_c^t]$$

$$= \zeta^{s-2^{\delta}mr} \lambda^{\frac{r(r-1)}{2}2^{\delta}m - r + t} \mu^{\frac{s(s-1)}{2} - \frac{2^{\delta}m(2^{\delta}m-1)}{2}r - 2^{\delta}mt + s + 2^{\delta}ms - 2^{\delta}mrs}.$$

Since $\zeta^{s-2^{\delta}mr} = \zeta^{u\cdot 2^{\gamma}} = \mu^{u\cdot 2^{\gamma-1}}$ and $\lambda = \mu^{2^{\delta}m}$, we have $[u_x, u_y] = \mu^E$, where

$$\begin{aligned} E &= \frac{u \cdot 2^{\gamma} + s^2 - s + 2 \cdot 2^{\delta}ms + 2^{\delta+1}mt + 2^{2\delta}m^2r^2 - 2^{2\delta}m^2r - 2^{2\delta}m^2r}{2} \\ &+ \frac{2^{\delta}mr - 2 \cdot 2^{\delta}mrs - 2^{\delta+1}mt - 2^{\delta+1}mr + 2s}{2} \\ &= \frac{u \cdot 2^{\gamma} + u^2 \cdot 2^{2\gamma} - u \cdot 2^{\gamma} + 2 \cdot u \cdot 2^{\gamma} + 2 \cdot 2^{\delta}m(s - 2^{\delta}mr)}{2} \\ &= u^2 \cdot 2^{\gamma+\gamma-1} + u \cdot 2^{\gamma} + u \cdot 2^{\gamma+\delta}m. \end{aligned}$$

Since $\gamma \geqslant 3$ and $\delta \geqslant 1$, we obtain $E \equiv 0 \bmod 2^{\gamma}$. Hence,

$$[u_x, u_y] = \mu^E = 1$$

and x is an f-regular element.

Thus, $G = G_2(\gamma+1, \gamma, \gamma; \gamma, \gamma)$ is not of central type if $\gamma > 1$. $\square$

Theorem 4.6 *If $p > 2$, then $G \in \mathcal{S}_{1,b}$ is of central type if and only if $\sigma = 0$ and $\alpha = \beta + \gamma$.*

Proof — Let $G = G_p(\alpha, \beta, \gamma; \gamma, \sigma) \in \mathcal{S}_{1,b}$ be of central type. Then from Theorem 4.3 we must have $\sigma = 0$. From Theorem 2.9 we must have $\alpha - \beta = \rho - \sigma$. This implies $\alpha = \beta + \gamma$.

Conversely, we will show that $G = G_p(\beta+\gamma, \beta, \gamma; \gamma, 0)$ is of central type. We have:

$$G = \langle a, b \ :\ a^{p^{\beta+\gamma}} = b^{p^{\beta+\gamma}} = 1,\ [a,b] = b^{p^{\beta}},\ [a,b,a] = [a,b,b] = 1\rangle.$$

Let $R_1 = \mathbb{C}\langle b\rangle$ be the group algebra corresponding to the group generated by b. Consider the $C_{p^{\beta+\gamma}} = \langle a\rangle$ action on R_1 given by

$$\Psi_a(u_b) = \zeta u_b^{p^{\beta}+1},$$

where ζ is a root of unity of order $p^{\beta+\gamma}$. By setting $\Psi_a(u_b^i) = \Psi_a(u_b)^i$ we ensure that Ψ_a is a homomorphism of R_1.

We will show that Ψ_a has order $p^{\beta+\gamma}$. We have that

$$\Psi_a^k(u_b) = \zeta^{1+r+\ldots+r^{k-1}} \cdot u_b^{r^k},$$

for all integers $k \geqslant 1$, where $r = p^{\beta}+1$. It is easy to see that

$$\Psi_a^{p^{\beta+\gamma}}(u_b) = u_b.$$

Since $o(u_b) = p^{\beta+\gamma}$, if $\Psi_a^k(u_b) = u_b$ then we must have $p^{\beta+\gamma}$ divides $(p^{\beta}+1)^k - 1$. So, there exists an integer s such that

$$(p^{\beta}+1)^k - 1 = sp^{\beta+\gamma}.$$

In this case the exponent of ζ is

$$\frac{r^k-1}{r-1} = \frac{(p^{\beta}+1)^k-1}{p^{\beta}} = \frac{sp^{\beta+\gamma}}{p^{\beta}} = p^{\gamma}s.$$

Since the order of ζ is $p^{\beta+\gamma}$ we must have $p^{\beta} \mid s$. We obtain

$$(p^{\beta}+1)^{k}-1=p^{\beta+\beta+\gamma}t,$$

for some integer t. But

$$(p^{\beta}+1)^{k}-1=p^{\beta}(p^{\beta(k-1)}+kp^{\beta(k-2)}+\ldots+\frac{k(k-1)}{2}p^{\beta}+k).$$

It follows that

$$p^{\beta+\gamma} \mid p^{\beta(k-1)}+kp^{\beta(k-2)}+\ldots+\frac{k(k-1)}{2}p^{\beta}+k.$$

We must have that p^{β} divides k. Hence, $k=p^{\beta}l$, for some integer l and

$$\begin{aligned} & p^{\beta(k-1)}+kp^{\beta(k-2)}+\ldots+\frac{k(k-1)}{2}p^{\beta}+k \\ =\; & p^{\beta}(p^{\beta(k-2)}+lp^{\beta(k-2)}+\ldots+\frac{k(k-1)}{2}+l)=p^{\beta+\gamma}\nu. \end{aligned}$$

Note that since $p>2$ we must have $k \geqslant 3$. By the same argument as before, we must have $p^{\gamma} \mid l$. Thus, $p^{\beta+\gamma} \mid k$. Hence, Ψ_a has order $p^{\beta+\gamma}$.

Therefore, the ring R generated by u_a, u_b is a crossed product of $\langle u_a \rangle$ over R_1. Hence, R is an associative complex algebra generated by u_a, u_b with relations

$$u_a^{p^{\beta+\gamma}}=u_b^{p^{\beta+\gamma}}=1,\ [u_a,u_b]=\zeta u_b^{p^{\beta}}$$

where ζ is a root of unity of order $p^{\beta+\gamma}$. Thus, there exists a cocycle $f \in Z^2(G,\mathbb{C}^*)$ which satisfies the above relations.

It remains to show that the only f-regular element is 1. Let $x=a^i b^j$, with $0 \leqslant i,j < p^{\beta+\gamma}$, be an f-regular element of our group G. We show that $i=j=0$.

Case 1 Suppose that $p \nmid i$. Then $b^{p^{\gamma}} \in C_G(x)$ and $[u_x,u_{b^{p^{\gamma}}}]=\zeta^{ip^{\gamma}} \neq 1$, since $p \nmid i$.

Case 2a Suppose that $i \equiv 0 \bmod p^{\gamma}$. In this case, we have $b \in C_G(x)$ and $[u_x,u_b]=\zeta^i \cdot u_b^{ip^{\beta}}=\zeta^i \neq 1$.

Case 2b Suppose that $p \mid i$ but $i \not\equiv 0 \bmod p^{\gamma}$. Then there is an integer r such that $ir \equiv 0 \bmod p^{\gamma}$ but $ir \not\equiv 0 \bmod p^{\beta+\gamma}$. Then $b^{r} \in C_G(x)$ and $[u_x, u_b] = \zeta^{ir}(u_b^{ir})^{p^{\beta}} = \zeta^{ir} \neq 1$.

Hence, we can assume that $i = 0$, so $x = b^{j}$, $0 \leqslant j < p^{\beta+\gamma}$.

Case 3 If $p \nmid j$ then $a^{p^{\gamma}} \in C_G(x)$ and $[u_x, u_{a^{p^{\gamma}}}] = \zeta^{-jp^{\gamma}} \neq 1$.

Case 4a If $j \equiv 0 \bmod p^{\gamma}$ then $a \in C_G(x)$ and $[u_x, u_a] = \zeta^{p^{j}} \neq 1$.

Case 4b Suppose that $p \mid j$ but $j \not\equiv 0 \bmod p^{\gamma}$. Then there is an integer s such that $js \equiv 0 \bmod p^{\gamma}$ and $js \not\equiv 0 \bmod p^{\beta+\gamma}$. We have $a \in C_G(x)$ and $[u_x, u_{a^{s}}] = \zeta^{js} \neq 1$.

Thus, $j = 0$ and $x = 1$ is the only f-regular element. Therefore G is of central type. □

Theorem 4.7 *If* $p = 2$ *and* $G \in \mathcal{S}_{1,b}$, *then* G *is of central type if and only if* $\sigma = 0$, $\alpha = \beta + \gamma$, *and* $\gamma > 1$.

Proof — As in the proof of the previous theorem we have $\sigma = 0$ and $\alpha = \beta + \gamma$. We will show that if $\gamma = 1$ then

$$G = G_2(\beta + 1, \beta, 1; 1, 0)$$

is not of central type. In this case $\mathbb{C}^{f}G$ is generated by u_a, u_b with relations $u_a^{2^{\beta+1}} = u_b^{2^{\beta+1}} = 1$ and $[u_a, u_b] = \zeta$, where ζ is a root of unity of order dividing $2^{\beta+1}$. Let $x = b^{2^{\beta}} \neq 1$. Then

$$a^{2} \in C_G(x) \quad \text{and} \quad [u_x, u_{a^2}] = \zeta^{-2^{\beta+1}} = 1.$$

Thus, x is f-regular and G is not of central type.

Conversely, assuming that $\gamma > 1$, we will show that the group $G = G_2(\beta + \gamma, \beta, \gamma; \gamma, 0)$ is of central type. The proof is identical to the proof we presented in the previous theorem. We need the fact that $\beta \geqslant \gamma \geqslant 2$ to show that Ψ_a has order $2^{\beta+\gamma}$. If $\gamma = 1$, then Ψ_a does not have order $2^{\beta+\gamma}$. □

Theorem 4.8 *If* $G \in \mathcal{S}_{1,c}$ *then* G *is not of central type.*

Proof — Since $G \in \mathcal{S}_{1,c}$ we have $\rho < \gamma$. Hence, by Theorem 4.2 G is not of central type. □

Theorem 4.9 *If* $G \in \mathcal{S}_2$ *then* G *is not of central type.*

Proof — Let $G = G_p(\alpha, \alpha, \gamma; \rho, \gamma) \in \mathcal{S}_2$ and assume that G is of central type. By Theorem 4.2, we have $\rho = \gamma$. Also, $Z(G) \hookrightarrow G/G'$. In this case $Z(G) = C_{p^{\alpha-\gamma}} \times C_{p^{\alpha-\gamma}} \times C_{p^\gamma}$ and $G/G' = C_{p^\alpha} \times C_{p^\alpha}$. It is easy to see that $Z(G)$ does not embed into G/G'. Thus, G is not of central type. □

Theorem 4.10 *The group* $G = G_p(\gamma, \gamma, \gamma; \gamma, \gamma)$ *is not of central type.*

Proof — The twisted algebra $\mathbb{C}^f G$ is generated by u_a, u_b, u_c with relations

$$u_a^{p^\gamma} = u_b^{p^\gamma} = u_c^{p^\gamma} = 1,\ [u_a, u_b] = \zeta u_c,\ [u_a, u_c] = \lambda,\ [u_c, u_b] = \mu,$$

where $c = [a, b]$ and $\zeta, \lambda, \mu \in \mathbb{C}$. We have that $\lambda^{p^\gamma} = \mu^{p^\gamma} = 1$.

If there exists and integer $0 \leqslant \delta < \gamma$ such that $\lambda^{p^\delta} = \mu^{p^\delta} = 1$, then it easy to see that $1 \neq x = c^{p^\delta}$ is an f-regular element. Hence, we must have that either λ or μ have order p^γ. Without loss of generality assume that $o(\lambda) = p^\gamma$ and $\mu = \lambda^m$, for some integer $0 \leqslant m \leqslant p^\gamma$.

We will show that $x = a^m b$ is an f-regular element. We have

$$C_G(x) = \left\{a^r b^s c^t \ :\ ms - r \equiv 0 \bmod p^\gamma\right\}.$$

Let $y = a^r b^s c^t \in C_G(x)$ and assume that $ms - r = up^\gamma$ for some integer u. Using the fact that $\mu = \lambda^m$ we obtain

$$\begin{aligned} [u_x, u_y] &= [u_a^m u_b, u_a^r u_b^s u_c^t] \\ &= \zeta^{ms-r} \lambda^{\frac{-m(m-1)}{2}s + \frac{r(r-1)}{2} + mt} \mu^{\frac{s(s-1)}{2}m + ms - rs - t} \\ &= \zeta^{ms-r} \lambda^E \end{aligned}$$

where

$$E = \frac{-m^2 s + ms + r^2 - r + 2mt + m^2 s^2 - m^2 s + 2m^2 s - 2mrs - 2mt}{2}.$$

Using again the fact that $ms - r = up^\gamma$ we obtain

$$E = \frac{up^\gamma (up^\gamma + 1)}{2}.$$

Thus,
$$[u_x, u_y] = \zeta^{u \cdot p^\gamma} \lambda^{\frac{up^\gamma(up^\gamma+1)}{2}}.$$

Case 1 Suppose that p is odd. Then we have

$$1 = [u_{a^{p^\gamma}}, u_b] = \zeta^{p^\gamma} \lambda^{\frac{-p^\gamma(p^\gamma-1)}{2}} = \zeta^{p^\gamma}.$$

Using this we obtain

$$[u_x, u_y] = \lambda^{\frac{up^\gamma(up^\gamma+1)}{2}}.$$

Regardless of whether u is even or odd we have p^γ divides the exponent of λ. This implies that $[u_x, u_y] = 1$ and x is an f-regular element. Thus, G is not of central type.

Case 2 Suppose that $p = 2$. Then

$$1 = [u_{a^{2^\gamma}}, u_b] = \zeta^{2^\gamma} \lambda^{\frac{-2^\gamma(2^\gamma-1)}{2}} = \zeta^{2^\gamma} \lambda^{2^{\gamma-1}}.$$

Hence, $\zeta^{2^\gamma} = \lambda^{-2^{\gamma-1}}$. We obtain

$$\begin{aligned}[u_x, u_y] &= \zeta^{u2^\gamma} \lambda^{\frac{u2^\gamma(u2^\gamma+1)}{2}} = \lambda^{-u2^{\gamma-1}} \lambda^{u2^{\gamma-1}(u2^\gamma+1)} \\ &= \lambda^{u2^{\gamma-1}(-1+u2^\gamma+1)} = 1.\end{aligned}$$

Hence, x is f-regular element. Thus, G is not of central type. □

Theorem 4.11 *If* $G \in \mathcal{S}_{3,a}$*, then* G *is not of central type.*

Proof — Let $G = G_p(\gamma, \gamma, \gamma; \rho, \gamma) \in \mathcal{S}_{3,a}$. Then by Theorem 4.2 we must have $\rho = \gamma$. From Theorem 4.10 we have that $G = G_p(\gamma, \gamma, \gamma; \gamma, \gamma)$ is not of central type. □

Theorem 4.12 *If* $G \in \mathcal{S}_{3,b}$*, then* G *is not of central type.*

Proof — If $G = G_2(\gamma, \gamma, \gamma; \rho, \gamma) \in \mathcal{S}_{3,b}$, then by Theorem 4.2 we must have $\rho = \gamma$. From Theorem 4.10 we have that $G = G_p(\gamma, \gamma, \gamma; \gamma, \gamma)$ is not of central type. If $G = G_2(\gamma, \gamma, \gamma; \gamma - 1, \gamma - 1)$ then G cannot be of central type by Theorem 4.2. □

Summarizing, we have the following result.

Theorem 4.13 *Let* $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ *be a 2-generator* p*-group of class* 2*. Then* G *is of central type if and only if one of the following conditions holds:*

(1) $G = G_2(2,1,1;1,1)$.

(2) $p > 2$, $G = G_p(\beta+\gamma,\beta,\gamma;\gamma,0)$.

(3) $G = G_2(\beta+\gamma,\beta,\gamma;\gamma,0)$, $\gamma > 1$.

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