Advances in Group Theory and Applications

16 (2023), pp. 23–38
ISSN: 2499-1287
DOI: 10.32037/agta-2023-006

On z-Reachable Subgroups of Finite Groups

Victor S. Monakhov — Irina L. Sokhor

(*Received Jan. 5, 2022; Accepted Feb. 28, 2022 — Communicated by A. Skiba*)

Abstract

A finite group G is a z-*group* if every Sylow subgroup of G is cyclic. A subgroup H of a group G is called c-*reachable* (z-*reachable*) in G if there is a subgroup chain

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_i \leqslant H_{i+1} \leqslant \ldots \leqslant H_{n-1} \leqslant H_n = G$$

such that $H_{i+1} = H_i K_i$, where K_i is a cyclic subgroup (z-subgroup) for every i. The aim of the paper is to study c-reachable and z-reachable subgroups. In particular, we prove that in soluble groups, a z-reachable subgroup is c-reachable, and we establish sufficient conditions under which all indices $|H_{i+1} : H_i|$ are primes. We obtain the structure of a group in which all Sylow subgroups are z-reachable. Besides, we prove that in Baer's Theorem on supersolubility of a group $G = AB$ with the nilpotent derived subgroup and supersoluble normal subgroups A and B, the requirement for the subgroups A and B to be normal can be weakened to z-reachability.

Mathematics Subject Classification (*2020*): 20D10, 20D20, 20E26

Keywords: finite group; z-subgroup; supersoluble group

1 Introduction

All groups in this paper are finite. We use the standard notation and terminology. If A and B are subgroups of a group G and $G = AB$, then B is said to be a supplement for A in G.

According to a theorem of Huppert (see [6, VI.9.5]), a group is supersoluble if every its maximal subgroup is of prime index. In this case, every maximal subgroup has a cyclic supplement of prime-power order. In 1965, O.H. Kegel proved that a group G is soluble if every maximal subgroup of G admits a supplement which is cyclic and of prime-power order [7, Proposition 1]. The symmetric group S_4 of degree 4 possesses this property but S_4 is not supersoluble.

J.C. Beidleman and D.J.S. Robinson [4] investigated a group G in which for every proper subgroup H there is an element $g \in G \setminus H$ such that $H\langle g\rangle = \langle g\rangle H$. In this case, maximal subgroups of G have cyclic supplements.

A. Ballester-Bolinches, J. Cossey and S. Qiao [2, Theorem 4] gave a detailed description of groups with cyclic supplements for maximal subgroups.

A group G is a z-*group* if every Sylow subgroup of G is cyclic. In view of Zassenhaus's Theorem [6, IV.2.11], the derived subgroup of a z-group is a Hall cyclic subgroup and the quotient group of a z-group with respect to the derived subgroup is also cyclic.

Based on the above results, we propose the following definitions.

A subgroup H of a group G is c-*supplemental* (z-*supplemental*) in G if there is a subgroup K such that $G = HK$ and K is a cyclic subgroup (z-subgroup, respectively), and we say K is a c-*supplement* (z-*supplement*, respectively) for H in G. It is clear that every c-supplemental subgroup is z-supplemental.

A subgroup H of a group G is c-*reachable* (z-*reachable*) in G if there is a subgroup chain

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_i \leqslant H_{i+1} \ldots \leqslant H_{n-1} \leqslant H_n = G \qquad (\star)$$

such that H_i is c-supplemental (z-supplemental, respectively) in H_{i+1} for every i. It is clear that every c-reachable subgroup is z-reachable.

Let $\mathbb{P}$ be the set of all primes and let H be a subgroup of a group G. If $H = G$ or there is a subgroup chain $(\star)$ such that $|H_{i+1} : H_i| \in \mathbb{P}$ for every i, then H is $\mathbb{P}$-*reachable* in G. In [14], the concept "$\mathbb{P}$-subnormal subgroup" was used instead of "$\mathbb{P}$-reachable subgroup". Every $\mathbb{P}$-reachable subgroup is c-reachable, see Lemma 2.1 (1), and so it is z-reachable.

From Huppert's Theorem [6, VI.9.5], it follows that every subgroup of a supersoluble group is $\mathbb{P}$-reachable.

Example 1.1 In the symmetric group S_4 of degree 4, every subgroup is c-reachable. In fact, all subgroups of S_4 is $\mathbb{P}$-reachable, except for a Sylow 3-subgroup C_3 and S_3. Since S_3 has a c-supplement $\langle(1\,2\,3\,4)\rangle$, S_3 is c-supplemental in S_4. From $|S_3 : C_3| = 2$, it follows that C_3 is c-reachable in S_4.

Example 1.2 In the symmetric group S_5 of degree 5, every subgroup is z-reachable. In fact, in S_5 [5, SmallGroup(120,34)], maximal subgroups are isomorphic to F_{20}, A_5, S_4 and D_{12}. Here, $F_{20} = C_5 \rtimes C_4$ is the Frobenius group of order 20, D_{12} is the dihedral group of order 12. Since

$$S_5 = F_{20}\langle(1\,2)(3\,4\,5)\rangle = A_5\langle(1\,2)\rangle = S_4\langle(1\,2\,3\,4\,5)\rangle = D_{12}F_{20},$$

in S_5, F_{20} is c-reachable, but F_{20} is not $\mathbb{P}$-reachable, A_5, S_4 are $\mathbb{P}$-reachable, D_{12} is z-reachable, but D_{12} is not c-reachable. Since F_{20} and D_{12} are supersoluble, all their subgroups are z-reachable in S_5. All subgroups of S_4 are c-reachable in S_4, so all subgroups of S_4 are c-reachable in S_5. Since every 2-maximal subgroup of S_5 is conjugate with a subgroup of F_{20}, S_4 or D_{12}, it implies that all subgroups of S_5 are z-reachable in S_5.

By Lemma 2.1 (3), every subnormal subgroup of a soluble group is $\mathbb{P}$-reachable, so it is also c-reachable and z-reachable.

We establish that in a soluble group, every z-reachable subgroup is c-reachable, and if H is a z-reachable subgroup of a soluble group G, then H is $\mathbb{P}$-reachable in G when one of the following conditions holds: G is S_4-free; 4 does not divide $|G : H|$; $(|H|, 6) = 1$.

We also study soluble groups G with all Sylow subgroups z-reachable. In particular, we prove that in G, all Sylow subgroups are $\mathbb{P}$-reachable, except for, maybe, a Sylow 3-subgroup, that a Hall $\{2,3\}'$-subgroup $G_{\{2,3\}'}$ is normal in G and has a Sylow tower of supersoluble type. It implies that G with z-reachable Sylow normalizers is either supersoluble or contains a normal subgroup N and $G/N \simeq S_4$. Besides, we prove that in Baer's Theorem on supersolubility of a group $G = AB$ with the nilpotent derived subgroup and supersoluble normal subgroups A and B, the requirement for subgroups A and B to be normal can be weakened to z-reachability.

2 General properties of z-reachable subgroups

Lemma 2.1 *Let* H *be a subgroup of a group* G. *The following statements hold.*

(1) *If* H *is* $\mathbb{P}$*-reachable in* G*, then* H *is* c*-reachable in* G*.*

(2) *If* H *is* c*-reachable in* G*, then* H *is* z*-reachable in* G*.*

(3) *If* H *is subnormal in* G *and* G *is soluble, then* H *is* $\mathbb{P}$*-reachable in* G*.*

Proof — (1) Let H be a $\mathbb{P}$-reachable subgroup of G. Then there is a subgroup chain ($\star$) such that $|H_{i+1} : H_i| = p_i \in \mathbb{P}$, $i = 0, \dots, n-1$. If P_i is a Sylow p_i-subgroup of H_{i+1} and $x_i \in P_i \setminus H_i$, then

$$|H_i\langle x_i\rangle| = |H_i||\langle x_i\rangle : H_i \cap \langle x_i\rangle| \geqslant |H_i|p_i = |H_{i+1}|.$$

Hence $H_{i+1} = H_i\langle x_i\rangle$. Since this is true for every i, we deduce that H is c-reachable in G.

(2) This follows from the definitions of c-reachable and z-reachable subgroups.

(3) Let H be a subnormal subgroup of a soluble group G. Then in G, there is a composition series

$$1 = H_0 \leqslant H_1 \leqslant \dots \leqslant H_j = H \leqslant H_{j+1} \leqslant \dots \leqslant H_t = G.$$

Since G is soluble, we get $|H_{i+1} : H_i| \in \mathbb{P}$ for every i. Therefore H is $\mathbb{P}$-reachable in G. □

Lemma 2.2 *Let* H *be a* z*-reachable subgroup of a group* G *and let* $N \lhd G$*. The following statements hold.*

(1) H^g *is* z*-reachable in* G *for every* $g \in G$*.*

(2) HN *is* z*-reachable in* G*.*

(3) HN/N *is* z*-reachable in* G/N*.*

(4) *Let* $A \leqslant B \leqslant G$*. If* A *is* z*-reachable in* B *and* B *is* z*-reachable in* G*, then* A *is* z*-reachable in* G*.*

Proof — Let H be a z-reachable subgroup of a group G. Then there is a subgroup chain ($\star$) such that $H_{i+1} = H_iK_i$, where K_i is a z-subgroup for every $i = 0, \dots, m-1$.

(1) Since $H_{i+1}^g = (H_i)^g(K_i)^g$ and $(K_i)^g$ is a z-subgroup, in ($\star$), we can replace H_i by $(H_i)^g$ and conclude that H^g is z-reachable in G.

(2) Since N is normal in G, we have $A_i = H_iN$ is a subgroup of G for every $i = 0, \dots, m-1$. Consider a subgroup chain

$$A = HN = A_0 \leqslant A_1 \leqslant \dots \leqslant A_i \leqslant A_{i+1} \leqslant \dots \leqslant A_m = G.$$

Since $A_{i+1} = H_{i+1}N = (H_iK_i)N = (H_iN)K_i = A_iK_i$, we deduce A_i is z-supplemental in A_{i+1} by K_i and $A = HN$ is z-reachable in G.

(3) Since $K_iN/N \simeq K_i/K_i \cap N$ is a z-group and

$$A_{i+1}/N = A_iK_i/N = (A_i/N)(K_iN/N),$$

we get A_i/N is z-supplemental in A_{i+1}/N by K_iN/N, and in view of

$$A/N = HN/N = A_0/N \leqslant \dots \leqslant A_m/N = G/N,$$

we conclude that $HN/N = A/N$ is z-reachable in G/N.

(4) It is obviously. □

Similarly, we can proof the following lemma.

Lemma 2.3 *Let H be a c-reachable subgroup of a group G and let $N \lhd G$. The following statements hold.*

(1) H^g *is c-reachable in G for every $g \in G$.*

(2) HN *is c-reachable in G.*

(3) HN/N *is c-reachable in G/N.*

(4) *Let $A \leqslant B \leqslant G$. If A is c-reachable in B and B is c-reachable in G, then A is c-reachable in G.*

In S_4, a Sylow 3-subgroup C_3 is c-reachable, but $C_3 \leqslant A_4 \leqslant S_4$ and C_3 is not z-reachable in A_4. Hence a subgroup H can be c-reachable (z-reachable) in a group and non-z-reachable in a subgroup containing H.

Lemma 2.4 *Let H and L be subgroups of a group G and let N be a normal subgroup of G.*

(1) *If H is $\mathbb{P}$-reachable in G, then $H \cap N$ is $\mathbb{P}$-reachable in N and HN/N is $\mathbb{P}$-reachable in G/N* — [15, Lemma 3.1 (1)].

(2) *If* $N \leqslant H$ *and* H/N *is* $\mathbb{P}$*-reachable in* G/N*, then* H *is* $\mathbb{P}$*-reachable in* G — [15, Lemma 3.1 (2)].

(3) *If* H *is* $\mathbb{P}$*-reachable in a soluble group* G *and* $U \leqslant G$*, then* $H \cap U$ *is* $\mathbb{P}$*-reachable in* U — [15, Lemma 3.4].

(4) *If* $H \leqslant L$*,* H *is* $\mathbb{P}$*-reachable in* L *and* L *is* $\mathbb{P}$*-reachable in* G*, then* H *is* $\mathbb{P}$*-reachable in* G — [15, Lemma 3.1 (4)].

Lemma 2.5 *Every subgroup of a supersoluble group is* $\mathbb{P}$*-reachable.*

Proof — Let H be a subgroup of a supersoluble group G. Consider a subgroup chain

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_{n-1} \leqslant H_n = G$$

such that $H_i \lessdot H_{i+1}$, $i = 0, 1, \ldots, n-1$. Here we write $H_i \lessdot H_{i+1}$ to denote that H_i is a maximal subgroup of H_{i+1}. Since $|H_{i+1} : H_i| \in \mathbb{P}$ by [6, IV.2.11], we conclude H is $\mathbb{P}$-reachable in G. □

3 z-Reachability in soluble groups

Lemma 3.1 (see Lemma 2.1 of [13]) *Let* M *be a maximal subgroup of a soluble group* G*, and assume that* $G = MC$ *for a cyclic subgroup* C*. Then* $|G : M|$ *is a prime or* 4*. Also, if* $|G : M| = 4$*, then* $G/M_G \simeq S_4$*.*

Theorem 3.2 *In a soluble group* G*, a subgroup* H *is* z*-reachable if and only if there is a subgroup chain*

$$H = M_0 \leqslant M_1 \leqslant \ldots \leqslant M_i \leqslant M_{i+1} \leqslant \ldots \leqslant M_{n-1} \leqslant M_n = G \quad (\star\star)$$

such that, for every $i = 0, 1, \ldots, n-1$*, either* $M_{i+1}/(M_i)_{M_{i+1}} \simeq S_4$ *and* $|M_{i+1} : M_i| = 4$*, or* $|M_{i+1} : M_i| \in \mathbb{P}$*.*

Proof — Let G be a soluble group and let H be a z-reachable subgroup of G. Compact chain $(\star)$ to a chain of maximal subgroups. Assume that U_i is a maximal subgroup of H_{i+1}, $H_i \leqslant U_i$ and K_i is a z-supplement for H_i in H_{i+1}. Then

$$H_{i+1} = H_i K_i, \; U_i = H_i (U_i \cap K_i), \; H_{i+1} = U_i K_i.$$

Since G is soluble, we have $|H_{i+1} : U_i| = p_i^{\alpha_i}$ for a prime $p_i \in \pi(H_{i+1})$ and $\alpha_i \in \mathbb{N}$. Let P_i be a Sylow p_i-subgroup of K_i. Then $H_{i+1} = U_i P_i$, where P_i is a cyclic p_i-subgroup. Since all Sylow subgroups of $U_i \cap K_i$ are cyclic, we deduce that in the chain $H_i \leqslant U_i \leqslant H_{i+1}$, H_i is z-reachable in U_i. Repeating this compaction several times, we obtain a chain ($\star\star$) such that M_i is maximal in M_{i+1} and $M_{i+1} = M_i P_i$, where P_i is a cyclic p_i-subgroup for every $i = 0, 1, \dots, n-1$. In particular, every z-reachable subgroup of a soluble group is c-reachable. By Lemma 3.1, for every $i = 0, 1 \dots, n-1$, either $M_{i+1}/(M_i)_{M_{i+1}} \simeq S_4$ and $|M_{i+1} : M_i| = 4$, or $|M_{i+1} : M_i| \in \mathbb{P}$.

Conversely, assume that there is a chain ($\star\star$) such that either the index $|M_{i+1} : M_i|$ is a prime, or $|M_{i+1} : M_i| = 4$ and $M_{i+1}/A_i \simeq S_4$, where $A_i = (M_i)_{M_{i+1}}$, for every $i = 0, 1, \dots, n-1$. If $M_{i+1}/A_i \simeq S_4$, then

$$M_i/A_i \simeq S_3,\ M_{i+1}/A_i = (M_i/A_i)(B_i/A_i),\ B_i/A_i \simeq C_4.$$

Let $B_i/A_i = \langle b_i A_i \rangle$. Then $M_{i+1} = M_i \langle b_i \rangle$, i.e. M_i is z-reachable in M_{i+1}.

Suppose that $|M_{i+1} : M_i| = p_i \in \mathbb{P}$, P_i is a Sylow p_i-subgroup of M_{i+1} and $x_i \in P_i \setminus M_i$. In that case,

$$|M_i\langle x_i \rangle| = |M_i||\langle x_i \rangle : M_i \cap \langle x_i \rangle| \geqslant |M_i| p_i = |M_{i+1}|.$$

Therefore $M_{i+1} = M_i \langle x_i \rangle$, i.e. M_i is z-supplemental in M_{i+1}. Since this is true for any i, we conclude H is z-reachable in G. □

Corollary 3.3 *Every* z*-reachable subgroup of a soluble group is* c*-reachable.*

If G is a group, $A \leqslant B \leqslant G$ and $A \lhd B$, then the quotient group B/A is called a section of G. If G has no sections that are isomorphic to S_4, then G is said to be S_4-free.

Corollary 3.4 *Let* H *be a* z*-reachable subgroup of a soluble group* G*. Then* H *is a* $\mathbb{P}$*-reachable subgroup of* G *when one of the following conditions holds:*

(1) G *is* S_4*-free.*

(2) 4 *does not divide* $|G : H|$.

(3) $(|H|, 6) = 1$.

Proof — By Theorem 3.2, there is a subgroup chain ($\star\star$) such that either $|M_{i+1} : M_i| = 4$ and $M_{i+1}/(M_i)_{M_{i+1}} \simeq S_4$ or $|M_{i+1} : M_i| \in \mathbb{P}$ for every $i = 0, 1, \ldots, n-1$.

(1) By the hypotheses, G is S_4-free, therefore $|M_{i+1} : M_i| \in \mathbb{P}$ for every i, and H is $\mathbb{P}$-reachable in G.

(2) By the hypotheses, 4 does not divide $|G : H|$. Hence in ($\star\star$), all indices are primes, and H is $\mathbb{P}$-reachable in G.

(3) We proceed by induction on $|G|$. Of course, it is possible to assume that $H \leqslant M_{n-1} \lessdot M_n = G$. Since H is z-reachable in M_{n-1}, we conclude H is $\mathbb{P}$-reachable in M_{n-1} by induction. If $|G : M_{n-1}|$ is prime, then H is $\mathbb{P}$-reachable in G by Lemma 2.4 (4). If $|G : M_{n-1}| \notin \mathbb{P}$, then

$$|G : M_{n-1}| = 4, \ G/(M_{n-1})_G \simeq S_4.$$

By hypotheses, $|H|$ is not divided by 2 and by 3, so $H \leqslant (M_{n-1})_G$ and H is $\mathbb{P}$-reachable in $(M_{n-1})_G$ by Lemma 2.4 (1). Since $(M_{n-1})_G$ is $\mathbb{P}$-reachable in G by Lemma 2.1 (3), H is $\mathbb{P}$-reachable in G by Lemma 2.4 (4). □

Corollary 3.5 *Let* G *be a soluble group,* $H \leqslant G$, $N \lhd G$, $N \leqslant H$. *If* H/N *is z-reachable in* G/N, *then* H *is z-reachable in* G.

Proof — Since H/N is z-reachable in G/N and G is soluble, then by Theorem 3.2, there is a subgroup chain

$$H/N = M_0/N \leqslant \ldots \leqslant M_i/N \leqslant M_{i+1}/N \leqslant \ldots \leqslant M_n/N = G/N$$

with either $|M_{i+1}/N : M_i/N| = 4$ and $(M_{i+1}/N)/(M_i/N)_{M_{i+1}/N} \simeq S_4$ or $|M_{i+1}/N : M_i/N| \in \mathbb{P}$ for every $i = 0, 1, \ldots, n-1$. Consider a subgroup chain

$$H = M_0 \leqslant \ldots \leqslant M_i \leqslant M_{i+1} \ldots \leqslant M_{n-1} \leqslant M_n = G.$$

If $|M_{i+1}/N : M_i/N| \in \mathbb{P}$, then $|M_{i+1} : M_i|$ is prime. Assume that $|M_{i+1}/N : M_i/N| = 4$ and $(M_{i+1}/N)/(M_i/N)_{M_{i+1}/N} \simeq S_4$. Then $|M_{i+1} : M_i| = 4$ and

$$M_{i+1}/(M_i)_{M_{i+1}} \simeq (M_{i+1}/N)/(M_i/N)_{M_{i+1}/N} \simeq S_4.$$

Consequently, H is z-reachable in G by Theorem 3.2. □

Corollary 3.6 *Let* G *be a soluble group, let* $\pi \subseteq \pi(G)$*, and let* K *be a z-reachable* π*-subgroup of* G*. Then* K *is z-reachable in a Hall* π*-subgroup* G_π *of* G*.*

Proof — Since K is a z-reachable subgroup of a soluble group G, in view of Theorem 3.2, there is a subgroup chain

$$K = M_0 \leqslant M_1 \leqslant \ldots \leqslant M_i \leqslant M_{i+1} \leqslant \ldots \leqslant M_{n-1} = M \leqslant M_n = G$$

such that, for every $i = 0, 1, \ldots, n-1$, either $M_{i+1}/(M_i)_{M_{i+1}} \simeq S_4$ and $|M_{i+1} : M_i| = 4$, or $|M_{i+1} : M_i| \in \mathbb{P}$. Since K is a z-reachable subgroup of a soluble group M, then by induction, K is z-reachable in a Hall π-subgroup M_π of M.

If $\pi\big(|G : M|\big) \cap \pi = \emptyset$, then M_π is a Hall π-subgroup of G and the statement is true.

Assume that $\pi\big(|G : M|\big) \cap \pi \neq \emptyset$ and G_π is a Hall π-subgroup of G that contains M_π. In that case, $G = MG_\pi$ and

$$|G : M| = |G_\pi : G_\pi \cap M| = |G_\pi : M_\pi|.$$

If M_π is $\mathbb{P}$-reachable in G_π, then K is z-reachable in G_π. Suppose that M_π is not $\mathbb{P}$-reachable in G_π. Then

$$|G_\pi : M_\pi| = 4, \; M_\pi \lessdot G_\pi, \; N_{G_\pi}(M_\pi) = M_\pi, \; G_\pi/(M_\pi)_{G_\pi} \simeq S_4.$$

Since all subgroups of S_4 are z-reachable in S_4, we get $M_\pi/(M_\pi)_{G_\pi}$ is z-reachable in $G_\pi/(M_\pi)_{G_\pi}$. By Corollary 3.5, M_π is z-reachable in G_π and K is z-reachable in G_π by Lemma 2.2 (4). □

Let G be a soluble group with all subgroups z-reachable. In view of Corollary 3.3, every subgroup of G is c-reachable in G. Hence for every proper subgroup H of G, there is an element $g \in G \setminus H$ such that $H < H\langle g\rangle = \langle g\rangle H$. The description of these groups was obtained in [4].

Let G be a soluble group with all maximal subgroups z-reachable. According to Corollary 3.3, every maximal subgroup of G is c-reachable in G. The description of these groups was obtained in [2].

Later, $\mathfrak{U}$ is the formation of all supersoluble groups, $A^{\mathfrak{U}}$ denotes the $\mathfrak{U}$-residual of a group A.

Corollary 3.7 *A group* G *is supersoluble if and only if* A *is z-reachable in* B *for any subgroups* A *and* B *such that* $A \leqslant B$*.*

Proof — If G is supersoluble, then every subgroup of G is supersoluble. Therefore by Lemma 2.5 and Lemma 2.1 (1), for any subgroups A and B such that $A \leqslant B$, A is z-reachable in B, and the necessity of the condition is proved.

To prove the sufficiency we proceed by induction on $|G|$. Suppose that G is a nonsupersoluble group of least order in which A is z-reachable in B for any subgroups A and B such that $A \leqslant B$. Let H be a proper subgroup of G. By induction, H is supersoluble and G is a minimal nonsupersoluble group.

Suppose $\Phi(G) \neq 1$. For subgroups $A/\Phi(G) \leqslant B/\Phi(G)$ of $G/\Phi(G)$, A is z-reachable in B by the hypothesis. By Lemma 2.2 (3), $A/\Phi(G)$ is z-reachable in $B/\Phi(G)$. Consequently, by induction, $G/\Phi(G) \in \mathfrak{U}$, and $G \in \mathfrak{U}$, a contradiction. Therefore $\Phi(G) = 1$, and in view of [8, Lemma 2.1], $G = G^{\mathfrak{U}} \rtimes H$, where $G^{\mathfrak{U}}$ is a Sylow p-subgroup for a prime $p \in \pi(G)$, $G^{\mathfrak{U}}$ is a minimal normal subgroup of G, $|G^{\mathfrak{U}}| > p$, H is a maximal subgroup of G. By the choice of G, H is z-reachable in G. In view of Theorem 3.2, either $|G : H| \in \mathbb{P}$ or $|G : H| = 4$ and $G/H_G \simeq S_4$. Since $|G : H| = |G^{\mathfrak{U}}| > p$ and $H_G = 1$, we get $G \simeq S_4$. But S_4 contains the nonsupersoluble subgroup A_4, a contradiction. Therefore G is supersoluble. □

4 Soluble groups with z-reachable Sylow subgroups

From Huppert's Theorem [6, VI.9.5], it follows that the formation $\mathfrak{U}$ of all supersoluble groups can be defined as a class of all groups in which all subgroups are $\mathbb{P}$-reachable. Let $\pi \subseteq \mathbb{P}$; $w_\pi\mathfrak{U}$ is the class of all groups with $\mathbb{P}$-reachable Sylow r-subgroups for every $r \in \pi \cap \pi(G)$ (see [16]). If $\pi = \mathbb{P}$ we write $w\mathfrak{U}$ instead of $w_{\mathbb{P}}\mathfrak{U}$. The class $w\mathfrak{U}$ is fairly well studied [8, 10, 14]. In particular, $w\mathfrak{U}$ is a subgroup-closed saturated formation. According to [16, Theorem 3.1], $w_\pi\mathfrak{U}$ is a subgroup-closed formation.

Lemma 4.1 *Let* $r = \max \pi(G)$ *and let* R *be a Sylow* r*-subgroup of a soluble group* G*. If* R *is z-reachable in* G *and* $r > 3$*, then* R *is normal in* G*.*

Proof — We proceed by induction on $|G|$. By Theorem 3.2, there is a subgroup chain

$$R = R_0 \leqslant R_1 \leqslant \ldots \leqslant R_i \leqslant R_{i+1} \leqslant \ldots \leqslant R_{n-1} \leqslant R_n = G \qquad (\dagger)$$

such that either $|R_{i+1} : R_i| = 4$ and $R_{i+1}/(R_i)_{R_{i+1}} \simeq S_4$ or $|R_{i+1} : R_i| \in \mathbb{P}$ for $i = 0, 1, \ldots, n-1$. Since R is z-reachable in R_{n-1}, R is normal in R_{n-1} by induction. If R is not normal in G, then $R_{n-1} = N_G(R)$. By the Sylow Theorem, $|G : R_{n-1}| \equiv 1 \pmod r$. Since $r = \max \pi(G)$, we get $|G : R_{n-1}| \notin \mathbb{P}$. If $|G : R_{n-1}| = 4$, then $r = 3$, a contradiction. Thus R is normal in G. □

Theorem 4.2 *If every Sylow subgroup of a soluble group* G *is z-reachable, then the following statements hold.*

(1) $G \in w_{3'}\mathfrak{U}$.

(2) *A Hall* $\{2,3\}'$*-subgroup* $G_{\{2,3\}'}$ *of* G *is normal in* G.

(3) $G_{2'} \in w\mathfrak{U}$, $G_{3'} \in w\mathfrak{U}$, $G_{\{2,3\}'} \in w\mathfrak{U}$.

(4) *If* G *is a* S_4*-free group, then* $G \in w\mathfrak{U}$.

Proof — Assume that every Sylow subgroup of a soluble group G is z-reachable in G.

(1) By Corollary 3.4 (2,3), every Sylow r-subgroup R, with $r \neq 3$, is $\mathbb{P}$-reachable in G, i. e. $G \in w_{3'}\mathfrak{U}$.

(2) We proceed by induction on $|G|$. Let R be a Sylow r-subgroup of $H = G_{\{2,3\}'}$ for $r = \max \pi(G)$. It is clear that if $r \leqslant 3$, then $H = 1$, and the statement is true. Let $r > 3$. By the hypotheses, R is z-reachable in G, and by Lemma 4.1, R is normal in G. In view of Lemma 2.2 (3), every Sylow subgroup of G/R is z-reachable in G/R. By induction, H/R is normal in G/R. Hence H is normal in G.

(3) All Sylow subgroups of $G_{3'}$ and of $G_{\{2,3\}'}$ are $\mathbb{P}$-reachable in G in view of Statement (1) and $\mathbb{P}$-reachable in $G_{3'}$ and, respectively, in $G_{\{2,3\}'}$ by Lemma 2.4 (3). Therefore $G_{3'} \in w\mathfrak{U}$ and $G_{\{2,3\}'} \in w\mathfrak{U}$.

Since $R = G_3$ is z-reachable in G, R is z-reachable in a Hall $2'$-subgroup $G_{2'}$ of G by Corollary 3.6. In view of Theorem 3.2, there is a subgroup chain (†) (in this chain, we assume that $G = G_{2'}$) such that either $|R_{i+1} : R_i| = 4$ or $|R_{i+1} : R_i| \in \mathbb{P}$ for $i = 0, 1, \ldots, n-1$. Since $G_{2'}$ is a group of odd order, we deduce that R is $\mathbb{P}$-reachable in $G_{2'}$. From Statement (1), it follows that a Sylow p-subgroup of $G_{2'}$ is $\mathbb{P}$-reachable in G for every $p \in \pi(G_{2'}) \setminus \{3\}$. By Lemma 2.4 (3), all Sylow subgroups of $G_{2'}$ are $\mathbb{P}$-reachable in $G_{2'}$. Thus, $G_{2'} \in w\mathfrak{U}$.

(4) If G is S_4-free, then $G \in w\mathfrak{U}$ in view of Corollary 3.4 (1). □

Later, the Sylow normalizer is the normalizer of a Sylow subgroup of a group. If every Sylow normalizer of a group G is $\mathbb{P}$-reachable, then G is supersoluble (see [11]). For a group with all Sylow normalizer z-reachable, the following statement is true.

Corollary 4.3 *If every Sylow normalizer of a soluble group* G *is z-reachable, then either* G *is supersoluble or* G *contains a normal subgroup* N *such that* $G/N \simeq S_4$.

PROOF — Note that in view of Lemma 2.2 and Lemma 2.5, every Sylow subgroup of G is z-reachable in G.

We proceed by induction on $|G|$. Assume that N is a normal subgroup of G, $N \neq 1$, and $\overline{R}$ is a Sylow r-subgroup of $\overline{G} = G/N$ for a prime $r \in \pi(G/N)$. Then in G, there is a Sylow r-subgroup R such that $\overline{R} = RN/N$. By the hypotheses, $N_G(R)$ is z-reachable in G. Since

$$N_{\overline{G}}(\overline{R}) = N_{G/N}(RN/N) = N_G(R)N/N,$$

according to Lemma 2.2 (3), $N_{\overline{G}}(\overline{R})$ is z-reachable in $\overline{G}$. Consequently, the hypotheses is true for every quotient subgroup of G. By induction, either G/N is supersoluble or G/N has a normal subgroup K/N such that $(G/N)/(K/N) \simeq S_4$. In the latter case, $G/K \simeq S_4$ and the statement is true. Therefore we consider that G/N is supersoluble for every non-identity normal subgroup N of G. By [11, Lemma 2.2], G is primitive, $F = F(G)$ is a unique minimal normal subgroup, $G = F \rtimes H$, H is a maximal subgroup of G, $H_G = 1$ and H is supersoluble. Let $q = \max \pi(H)$ and let Q be a Sylow q-subgroup of H. Since H is supersoluble, Q is normal in H and $N_H(Q) = H$. Hence Q is a Sylow q-subgroup of G, and by the hypotheses, H is z-reachable in G. In view of Theorem 3.2, either $|G : H| = |F| \in \mathbb{P}$ or $|G : H| = |F| = 4$. If $|G : H| = |F| \in \mathbb{P}$, then G is supersoluble. If $|G : H| = |F| = 4$, then $G \simeq G/H_G \simeq S_4$ by Theorem 3.2. □

Example 4.4 The group $G = C_2^4 \rtimes (S_3 \times S_3)$ [5, SmallGroup(576,8654)] is soluble and contains the following classes of non-conjugate maximal subgroups:

$$M \simeq S_3 \times S_3, M_1 \simeq C_2^4 \rtimes (C_3 \times S_3), M_2 \simeq C_2^4 \rtimes (C_3 \times S_3),$$
$$M_3 \simeq (A_4 \times A_4) \rtimes C_2, M_4 \simeq C_2^4 \rtimes D_{12}, M_5 \simeq C_2^4 \rtimes D_{12},$$
$$|G : M| = 16, |G : M_1| = |G : M_2| = |G : M_3| = 2,$$
$$|G : M_4| = |G : M_5| = 3.$$

Since G contains the maximal subgroup M of index 16, $G \notin \mathfrak{U}$, and in view of $|\pi(G)| = 2$, $G \notin w\mathfrak{U}$. As G has no maximal subgroups of index 4, G does not contain a normal subgroup N such that $G/N \simeq S_4$.

In G, the Sylow 2-subgroup $P \simeq C_2^4 \rtimes C_2^2 = F(G) \rtimes P_1$. Here $P_1 \simeq C_2^2$ is a Sylow 2-subgroup of M. Since M is supersoluble, P_1 is $\mathbb{P}$-reachable in M by Lemma 2.5, and P is $\mathbb{P}$-reachable in G. By Lemma 2.1, P is z-reachable in G. For the Sylow 3-subgroup Q of G, there is a subgroup chain

$$Q \simeq C_3 \times C_3 \lessdot C_3 \rtimes S_3 \lessdot C_3 \rtimes S_4 \lessdot M_3 \lessdot G.$$

It is clear that Q is z-reachable in $C_3 \rtimes S_3$, $C_3 \rtimes S_3$ is z-reachable in $C_3 \rtimes S_4$ (see Example 1.1). In $M_3 \simeq (A_4 \times A_4) \rtimes C_2$ [5, SmallGroup(288,1026)], $C_3 \rtimes S_4$ is z-reachable in view of Theorem 3.2, since $|M_3 : C_3 \rtimes S_4| = 4$ and $M_3/(C_3 \rtimes S_4)_{M_3} \simeq S_4$. Therefore Q is z-reachable in G.

This example shows that for a group with z-reachable Sylow subgroups, the analog of Corollary 4.3 is not true.

5 To Baer's theorem

If $\mathfrak{X}$ is a formation and A is a group, then $A^{\mathfrak{X}}$ is an $\mathfrak{X}$-residual of A. Recall $\mathfrak{A}$, $\mathfrak{N}$ and $\mathfrak{U}$ denote the formations of all abelian, nilpotent and supersoluble groups, respectively, $[A, B] = \langle [a, b] \mid a \in A,\ b \in B \rangle$ denotes the commutator of subgroups A and B.

The following is a well known result due to Baer.

Theorem 5.1 (see [1, p.186]) *Let A and B be supersoluble normal subgroups of a group G and let $G = AB$. If the derived subgroup of G is nilpotent, then G is supersoluble.*

Since nilpotency of the derived subgroup of a group G is equivalent to $(G')^{\mathfrak{N}} = 1$, Theorem 5.1 arises from the following theorem.

Theorem 5.2 *Let A and B be supersoluble subgroups of a group G and let $G = AB$.*

(1) *If A and B are subnormal subgroups of G, then* $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = [A, B]^{\mathfrak{N}}$ — [9, Theorem 2].

(2) *If A and B are $\mathbb{P}$-reachable subgroups of G, then* $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$ — [12, Theorem 3.3].

We prove a more general statement.

Theorem 5.3 *Let* A *and* B *be supersoluble* z*-reachable subgroups of a group* G *and let* $G = AB$*. Then* $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} \leqslant [A, B]$*. In particular, if the derived subgroup of* G *is nilpotent, then* G *is supersoluble.*

Proof — Since the derived subgroup of a supersoluble group is nilpotent [6, VI.9.1], we have $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$ and $(G')^{\mathfrak{N}} \leqslant G^{\mathfrak{U}}$. Since A and B are supersoluble subgroups of G, then $G^{\mathfrak{U}} \leqslant [A, B]$ by [9, Lemma 11]. Thus, $(G')^{\mathfrak{N}} \leqslant G^{\mathfrak{U}} \leqslant [A, B]$. Now we prove that $G^{\mathfrak{U}} \leqslant (G')^{\mathfrak{N}}$.

Consider separately the case when $(G')^{\mathfrak{N}} = 1$. In that case, G' is nilpotent and G is soluble. Since A is z-reachable, by Theorem 3.2, there is a subgroup chain

$$A = M_0 \leqslant M_1 \leqslant \ldots \leqslant M_i \leqslant M_{i+1} \leqslant \ldots \leqslant M_{n-1} \leqslant M_n = G$$

such that, for every $i = 0, 1, \ldots, n-1$, either $M_{i+1}/(M_i)_{M_{i+1}} \simeq S_4$ and $|M_{i+1} : M_i| = 4$, or $|M_{i+1} : M_i| \in \mathbb{P}$. Since the derived subgroup of S_4 is not nilpotent, case $M_{i+1}/(M_i)_{M_{i+1}} \simeq S_4$ is impossible. Hence $|M_{i+1} : M_i| \in \mathbb{P}$ for every $i = 0, 1, \ldots, n-1$, and A is $\mathbb{P}$-reachable in G. Similarly, B is $\mathbb{P}$-reachable in G. Thus, in view of Theorem 5.2 (2), $G^{\mathfrak{U}} = 1$, i. e. G is supersoluble and $1 = (G')^{\mathfrak{N}} = G^{\mathfrak{U}}$.

Now, assume that $(G')^{\mathfrak{N}} \neq 1$. Consider

$$G/(G')^{\mathfrak{N}} = A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} \cdot B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}.$$

Since A and B are supersoluble z-reachable subgroups of G, we have that also the subgroups $A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$ and $B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$ are supersoluble and z-reachable in $G/(G')^{\mathfrak{N}}$ in view of Lemma 2.2 (3). Moreover,

$$\left(G/(G')^{\mathfrak{N}}\right)' = G'(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} = G'/(G')^{\mathfrak{N}} \in \mathfrak{N}, \ \left(\left(G/(G')^{\mathfrak{N}}\right)'\right)^{\mathfrak{N}} = 1.$$

By the above, $G/(G')^{\mathfrak{N}} \in \mathfrak{U}$ and $G^{\mathfrak{U}} \leqslant (G')^{\mathfrak{N}}$. □

Corollary 5.4 *Let* A *and* B *be abelian subgroups of a group* $G = AB$*. If* A *and* B *are* z*-reachable in* G*, then* G *is supersoluble.*

Proof — Since $(G')' = 1$ (see for instance [6, VI.4.4]), by Theorem 5.3, we have that $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} \leqslant (G')' = 1$ and G is supersoluble. □

Example 5.5 In S_4, every subgroup is z-reachable (see Example 1.1). Furthermore,

$$S_4 = AB, \ A \simeq C_3, \ B \simeq D_8,$$
$$(S_4)' = [A, B] \simeq A_4, \ \left((S_4)'\right)^{\mathfrak{N}} = (S_4)^{\mathfrak{U}} \simeq C_2^2 < A_4.$$

Therefore in Theorem 5.3, we can not replace inclusion $(G')^{\mathfrak{N}} \leqslant [A, B]$ by equality.

REFERENCES

[1] R. Baer: "Classes of finite groups and their properties", *Illinois J. Math.* 1 (1957), 115–187.

[2] A. Ballester-Bolinches – J. Cossey – S. Qiao: "A note on finite groups with the maximal permutiser condition", *RACSAM* 110 (2016), 247–250.

[3] A. Ballester-Bolinches – R. Esteban-Romero – M. Asaad: "Products of Finite Groups", *de Gruyter*, Berlin (2010).

[4] J.C. Beidleman – D.J.S. Robinson: "On finite groups satisfying the permutizer condition", *J. Algebra* 191 (1997), 686–703.

[5] "The GAP Group: GAP — Groups, Algorithms, and Programming", Ver. 4.11.1 (02-03-2021).

[6] B. Huppert: "Endliche Gruppen I", *Springer*, Berlin (1967).

[7] O.H. Kegel: "On Huppert's characterization of finite supersoluble groups", in: Proc. Internat. Conf. Theory of Groups (Canberra, 1965), *Gordon and Breach*, New York (1967), 209–215.

[8] V.S. Monakhov: "Finite groups with abnormal and $\mathfrak{U}$-subnormal subgroups", *Siberian Math. J.* 57 (2016), 352–363.

[9] V.S. Monakhov – I.K. Chirik: "On the supersoluble residual of a product of subnormal supersoluble subgroups", *Siberian Math. J.* 58 (2017), 271–280.

[10] V.S. Monakhov – V.N. Kniahina: "Finite groups with $\mathbb{P}$-subnormal subgroups", *Ric. Mat.* 62 (2013), 307–323.

[11] V.S. Monakhov – V.N. Kniahina: "On supersolvability of finite groups with $\mathbb{P}$-subnormal subgroups", *Int. J. Group Theory* 2:4 (2013), 21–29.

[12] V.S. Monakhov – A.A. Trofimuk: "On the supersoluble residual of a product of supersoluble subgroups", *Adv. Group Theory Appl.* 9 (2020), 51–70.

[13] S. Qiao – G. Qian – Y. Wang: "Finite groups with the maximal permutizer condition", *J. Algebra Appl.* 12:5 (2013), 1250217, 5pp.

[14] A.F. Vasil'ev – T.I. Vasil'eva – V.N. Tyutyanov: "On the finite groups of supersoluble type", *Siberian Math. J.* 51 (2010), 1004–1012.

[15] A.F. Vasil'ev – T.I. Vasil'eva – V.N. Tyutyanov: "On the products of $\mathbb{P}$-subnormal subgroups of finite groups", *Siberian Math. J.* 53 (2012), 47–54.

[16] A.F. Vasil'ev – T.I. Vasil'eva – A.S. Vegera: "Finite groups with generalized subnormal embedding of Sylow subgroups", *Siberian Math. J.* 57 (2016), 200–212.

Victor S. Monakhov, Irina L. Sokhor
Department of Mathematics and Technologies of Programming
Francisk Skorina Gomel State University
Kirova Str. 119, 246019 Gomel (Belarus)
e-mail: victor.monakhov@gmail.com; irina.sokhor@gmail.com