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On Generalized σ-Subnormal Subgroups of Finite Groups

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Abstract

Let $\sigma = \{\sigma_i \mid i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, G a finite group and $\sigma(G) = \{\sigma_i \mid \sigma_i \cap \pi(G) \neq \emptyset\}$. A subgroup A of G is said to be *generalized σ-subnormal* in G if $A = \langle L, T\rangle$, where L is a modular subgroup and T is a σ-subnormal subgroup of G. We study the structure of G being based on the assumptions that if all members of $\mathcal{H}$ and every maximal subgroup of any non-cyclic $H_i \in \mathcal{H}$ are generalized σ-subnormal in G, where $\mathcal{H}$ is a *complete Hall σ-set* of G.

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1 Introduction

Throughout this article, all groups are assumed to be finite and G always denotes a finite group. Moreover $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. A subgroup M of G is called *modular* (in the sense of Kurosh [12, p. 43]) if M is a *modular* element of the lattice $\mathcal{L}(G)$, that is,

(1) $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leqslant G, Z \leqslant G$ with $X \leqslant Z$, and

(2) $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leqslant G, Z \leqslant G$ such that $M \leqslant Z$.

If n is an integer, the symbol $\pi(n)$ denotes the set of all prime dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i \mid i \in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We write

$$\sigma(n) = \{\sigma_i \mid \sigma_i \cap \pi(n) \neq \emptyset\}$$

and $\sigma(G) = \sigma(|G|)$.

Following [4],[5],[9],[10],[14],[13], a set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma(G)$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup for every $\sigma_i \in \sigma(G)$. The group G is said to be: *σ-full* if it possesses a complete Hall σ-set; *σ-primary* if $|\sigma(G)| \leqslant 1$; *σ-nilpotent* if it has a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $G = H_1 \times \ldots \times H_t$. A subgroup H of G is said to be *σ-subnormal* in G if there exists a subgroup chain $H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, \ldots, n$.

Definition 1.1 (see [6]) A subgroup A of G is said to be *generalized σ-subnormal* in G if $A = \langle L, T\rangle$, where L is a modular subgroup and T is a σ-subnormal subgroup of G.

Some properties of generalized σ-subnormal were analyzed in the paper [6]. In this paper, we also study some other new properties of generalized σ-subnormal subgroups. In fact, our main result are as follows.

Theorem 1.2 *Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a complete Hall σ-set of G. Suppose that every subgroup $H_i \in \mathcal{H}$ is generalized σ-subnormal in G, then the derived subgroup G' of G is σ-nilpotent.*

Remark 1.3 It is clear from Example 1.2 of [6] that every modular, σ-permutable and every σ-subnormal subgroups of G are generalized σ-subnormal subgroup but the converse is not necessarily true. Moreover in the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ every subnormal subgroup is σ-subnormal subgroup of G.

Corollary 1.4 (see [6],Theorem 1.3) *If all Schmidt subgroups of G are generalized σ-subnormal in G, then $G/F_\sigma(G)$ is abelian.*

Corollary 1.5 (see [17], Theorem A(ii)) *If G possesses a complete Hall σ-set $\mathcal{H}$ all of whose members are $\mathfrak{m}$-σ-permutable in G, then the derived subgroup G' of G is σ-nilpotent.*

In the case, when $\sigma = \{\{2\},\{3\},\dots\}$, by using Theorem 1.2 we have the following corollary.

Corollary 1.6 *If every Sylow subgroup of* G *is either modular or subnormal in* G*, then* G' *is nilpotent.*

Remark 1.7 Also note from [6, Example 1.2] that the modular subgroup L of G is not σ-subnormal in G and the σ-subnormal subgroup T of G is not modular in G. By this observation we have the following corollaries.

Corollary 1.8 *If every Sylow subgroup of* G *is modular in* G*, then* G' *is nilpotent.*

Corollary 1.9 (see [11]) *If every Schmidt subgroup of* G *is subnormal in* G*, then* $G/F(G)$ *is abelian.*

Theorem 1.10 *Let* G *be a σ-full group of Sylow type and* $\mathcal{H}=\{H_1,\dots,H_t\}$ *be a complete Hall σ-set of* G *such that every subgroup* $H_i \in \mathcal{H}$ *is generalized σ-subnormal in* G*, then* G *is σ-soluble.*

Theorem 1.11 *Let* G *be a σ-full group of Sylow type and* $\mathcal{H}=\{H_1,\dots,H_t\}$ *be a complete Hall σ-set of* G *such that every subgroup* $H_i \in \mathcal{H}$ *is a nilpotent* σ_i*-subgroup. If every maximal subgroup of any non-cyclic* H_i *is generalized σ-subnormal in* G*, then* G *is supersoluble.*

In the case when $\sigma = \{\{2\},\{3\},\dots\}$ every normal subgroup, every *permutable subgroup* (recall that a subgroup H of G is said to be permutable in G if $HS = SH$ for any subgroup S of G) and every *s-permutable subgroup* (recall that a subgroup H of G is said to be s-permutable in G if $HP = PH$ for any Sylow subgroup P of G) of G are generalized σ-subnormal in G. The following corollaries are true for Theorem 1.11.

Corollary 1.12 (see [16], Theorem 1) *If all maximal subgroups of every Sylow subgroup of* G *are normal in* G*, then* G *is supersoluble.*

Corollary 1.13 (see [16], Theorem 2) *If all maximal subgroups of every Sylow subgroup of* G *are* s*-permutable in* G*, then* G *is supersoluble.*

Corollary 1.14 (see [7], VI, Theorem 10.3) *If every Sylow subgroup of* G *is cyclic, then* G *is supersoluble.*

All unexplained terminologies and notations are standard. The reader is referred to [1],[2],[3],[7] if necessary.

2 Preliminaries

We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups and $\mathfrak{S}_\sigma$ to denote the class of all σ-soluble groups. Also we use $\mathfrak{U}$ to denote the class of all supersoluble groups.

Lemma 2.1 (see [15], Theorems A and B) *If G is σ-soluble, then G is a σ-full group.*

Lemma 2.2 (see [13], Corollary 2.4 and Lemma 2.5) *The following statements hold.*

(1) *The class $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroup, homomorphic images and subgroups.*

(2) *If G/N and G/M are σ-nilpotent, then $G/M \cap N$ is σ-nilpotent.*

(3) *If R is a normal subgroup of G and $R/R \cap \Phi(G)$ is σ-nilpotent, then R is σ-nilpotent.*

Lemma 2.3 (see [13], Lemma 2.1, or [8], Lemma 2.3) *The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the σ-soluble group by a σ-soluble group is a σ-soluble group as well.*

Lemma 2.4 (see [6], Lemma 2.5) *Let A, B and N be subgroups of G, where A is generalized σ-subnormal in G and N is normal in G, then AN/N is generalized σ-subnormal in G/N.*

Lemma 2.5 (see [12], Lemma 2.5) *If H is modular in the finite group G, then every chief factor of G between H^G and H_G is cyclic.*

Lemma 2.6 (see [13], Lemma 2.6) *Let A, K and N be subgroups of G. Suppose that A is σ-subnormal in G and N is normal in G. If A is a σ-Hall subgroup of G, then A is normal in G.*

A normal subgroup H of G is said to be *hypercyclically embedded* in G (see [12], p.217) if either $H = 1$ or $H \neq 1$ and every chief factor of G below H is cyclic. We use $Z_{\mathfrak{U}}(G)$ to denote the product of all normal hypercyclically embedded subgroup of G. It is clear that a normal subgroup H of G is hypercyclically embedded in G if and only if $H \leqslant Z_{\mathfrak{U}}(G)$.

Lemma 2.7 (see [12], Theorem 5.2.5) *If M is modular in the finite group G, then $M^G/M_G \leqslant Z_{\mathfrak{U}}(G/M_G)$.*

Recall that $G^{\mathfrak{N}_\sigma}$ denotes the *σ-nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

3 Proofs of the results

Proof of Theorem 1.2 — Let $1 \neq D = G^{\mathfrak{N}_\sigma}$ be the σ-nilpotent residual of G. Assume that the theorem is false and let G be a counter example with $|G|$ minimal, then G is not σ-nilpotent. So $|\sigma(G)| > 1$. Furthermore, G/D is σ-nilpotent by Lemma 2.2(2), and therefore by Lemma 2.3 G is σ-soluble. By Lemma 2.1, $\mathcal{H} = \{H_1, \dots, H_t\}$ be a complete Hall σ-set of G.

Let N be a minimal normal subgroup of G. Then N is a σ_i-group for some i, therefore the hypothesis holds for G/N by Lemma 2.4. Hence $(G/N)' = G'N/N = G'/G' \cap N$ is σ-nilpotent by the choice of G. Thus by Lemma 2.2(3) $N \leqslant G'$ and $N \nleqslant \Phi(G)$. Furthermore, if G has a minimal normal subgroup $R \neq N$ of G, then $R \leqslant G'$ and $G'/N \cap R = G'/1 \simeq G'$ is σ-nilpotent, a contradiction. This implies that N is the unique minimal normal subgroup of G and $C_G(N) \leqslant N$ by [2], Chapter A, 15.2. We can assume without loss of generality that $i = 1$ and $N \leqslant H_1$.

Let M be a maximal subgroup of G such that $N \nleqslant M$. So $M_G = 1$ and $|G : M|$ is a σ_i-number. This shows that for some $x \in G$ we have $A = H_2^x \leqslant M$. Then $A = \langle L, T\rangle$ for some modular subgroup L and σ-subnormal subgroup T of G. Moreover, $L_G \leqslant M_G = 1$, so L^G is hypercyclically embedded in G by Lemma 2.5. If $L \neq 1$, then $N \leqslant L^G$ and thus $|N| = p$ for some prime p. But then $C_G(N) = N$ and $G/N = G/C_G(N)$ is cyclic. Hence G' is σ-nilpotent. This contradiction shows that $L = 1$, thus $A = T$ is σ-subnormal in G. This implies that A is normal in G by Lemma 2.6 since A is a Hall σ_i-subgroup of G for some i. So $1 < A \leqslant M_G$, a contradiction. Thus theorem holds. □

Proof of Theorem 1.10 — Suppose that this is false, and let G be a counterexample of minimal order, then $|\sigma(G)| > 1$.

(1) G/R *is σ-soluble for every non-identity normal subgroup* R *of* G.

Assume that R is a non-trivial normal subgroup of G and H/R is any Hall σ_i-subgroup of G/R, where $\sigma_i \cap \pi(G/R) \neq \emptyset$. Then we have that $H/R = H_iR/R$ for some Hall σ_i-subgroup H_i of G. By the hypothesis, $H_i = \langle L, T\rangle$ for some modular subgroup L and σ-subnormal subgroup T of G. By Lemma 2.4 $H_iR/R = \langle LR/R, TR/R\rangle$ is genralized σ-subnormal in G/R. This shows that G/R satisfies the hypothesis. The minimal choice of G implies that G/R is σ-soluble.

(2) G *is not a simple group.*

Assume that G is non-abelian simple group. Then 1 is the only proper σ-subnormal of G. Let H_i be any non-identity Hall σ_i-subgroup of G, where $\sigma_i \in \sigma(G)$. By the hypothesis and $|\sigma(G)| > 1$, we have $H_i = \langle L, T\rangle$, where L is modular subgroup and T is σ-subnormal subgroup of G. If $T = G$, then $H_i = \langle L, G\rangle = G$, a contradiction of $|\sigma(G)| > 1$. If $T = 1$, then $H_i = L$ is a modular subgroup of G. By Lemma 2.7, $H_i^G \leqslant Z_{\mathfrak{U}}(G)$. But since G is simple group either $H_i^G = 1$ or $H_i^G = Z_{\mathfrak{U}}(G) = G$, a contradiction of $H_i \leqslant H_i^G = 1$ and G is σ-soluble. Hence we have (2).

(3) *If* N *is minimal normal subgroup of* G*, then* N *is σ-soluble.*

Since $|\sigma(G)| > 1$, therefore let H_i be any Hall σ_i-subgroup of G. By the hypothesis $H_i = \langle L, T\rangle$, where L is modular subgroup and T is σ-subnormal subgroup of G. If $L_G \neq 1$, then $N \leqslant L_G$. Since L is σ-nilpotent, so N is σ-soluble. On the other hand if $L_G = 1$, then L^G is supersoluble by Lemma 2.7 and therefore minimal normal subgroup N of G contained in L^G is σ-soluble. Now suppose that if $L = 1$, then $H_i = T$ is σ-subnormal subgroup of G. This implies that H_i is normal in G by Lemma 2.6. Thus $N \leqslant H_i$. Since H_i is σ-nilpotent, therefore N is σ-soluble. If $T = 1$, then $H_i = L$ is modular subgroup of G. Then every chief factor of G between H_i^G and H_{iG} is cyclic by Lemma 2.5. This follows that N is cyclic and therefore σ-soluble. Hence (3) holds.

(4) *Final Contradiction.*

In view of Claims (1), (2) and (3), we have G is σ-soluble by Lemma 2.3. The final contradiction completes the proof of the theorem. □

Proof of Theorem 1.11 — Assume that the assertion is false, and let G be a counterexample of minimal order, then $|\sigma(G)| > 1$.

Let q be the smallest prime dividing |G|. Without loss of generality, we can assume that $q \in \sigma_1$. We now continue the proof via the following steps.

(1) *The hypothesis holds on* G/N *for every σ-primary minimal normal subgroup* $1 \neq N$ *of* G*. Consequently,* G/N *is supersoluble and* G *is soluble. Hence* N *is non-cyclic.*

It is clear that $\overline{\mathcal{H}} = \{H_1N/N, \ldots, H_tN/N\}$ is a complete Hall σ-set of G/N and $H_iN/N \simeq H_i/H_i \cap N$ is nilpotent. Now assume that W/N is a maximal subgroup of H_iN/N, thus $|(H_iN/N) : (W/N)| = q$ is a prime. Then $W = N(W \cap H_i)$. Hence

$$\begin{aligned} q &= |(H_iN/N) : (W/N)| = |(H_iN/N) : (N(W \cap H_i)/N)| \\ &= |H_iN : N(W \cap H_i)| = |H_i||N||N \cap (W \cap H_i)|/|W \cap H_i||N||H_i \cap N| \\ &= |H_i|/|W \cap H_i| = |H_i : (W \cap H_i)|, \end{aligned}$$

therefore $W \cap H_i$ is a maximal subgroup of H_i. If H_iN/N is non-cyclic, then H_i is non-cyclic, thus by the hypothesis $W \cap H_i$ is generalized σ-subnormal in G. Then $(W \cap H_i)N/N$ is generalized σ-subnormal in G/N by Lemma 2.4. Thus hypothesis holds for G/N, so the choice of G implies that G/N is supersoluble. Hence G is σ-soluble. Finally, since $N \leqslant H_i$ for some i, N is soluble and so G is soluble. Hence we have (1).

(2) G *is σ-soluble. Hence* G *is soluble.*

Assume that this is false, then $(H_1)_G = 1$ and $O_j(G) = 1$ for all σ_j in $\sigma(G)$ by Claim (1). Furthermore, H_1 is non-cyclic. In fact, if H_1 is cyclic, then G is q-nilpotent by [7], IV, 2.8. Thus by the Feit-Thompson theorem G is soluble since q is the smallest prime dividing the order of G.

Since H_1 is nilpotent non-cyclic by the hypothesis and $q \in \pi(H_1)$. We have $H_1 = W_1W_2$ for some maximal subgroups W_1, W_2 of H_1 and W_1, W_2 are generalized σ-subnormal in G. Then $W_k = \langle L_k, T_k \rangle$ for some modular subgroup L_k and σ-subnormal subgroup T_k of G with $\langle L_k, T_k \rangle_G = (L_k)_G = (T_i)_G = 1$, where $k = 1, 2$. Since G has no cyclic minimal normal subgroup by Claim (1). Therefore it follows by Lemma 2.5 that $L_k = 1$, and so $V_k = T_k$. If $T_k > 1$, then

$$1 > T_k = O_{\sigma_1}(T_k) \leqslant O_{\sigma_1}(G)$$

by [10, Lemma 2.3(1)], a contradiction. Therefore $V_k = T_k = 1$, and it follows that $H_1 = 1$, again a contradiction. Hence (2) holds.

(3) N *is the unique minimal normal subgroup of* G, $\Phi(G) = 1$, $N = C_G(N)$ $= F(G) = O_q(G)$ *and* $|N| > q$ *for some prime* q.

It follow from Claim (2), [7, VI, 8.6] and [2, A, 15.2].

(4) *The final contradiction.*

Let $p \in \sigma_i$, then $N \leqslant H_i$. Thus for some maximal subgroup M of G

we have $G = N \rtimes M$ by Claim (3). Then we have $H_i = N \rtimes (H_i \cap M)$. By the hypothesis H_i is nilpotent, so some maximal subgroup E of N is normal in H_i. Then $W = E(H_i \cap M)$ is a maximal subgroup of H_i, and by the hypothesis $W = \langle L, T \rangle$ for some modular subgroup L and σ-subnormal subgroup T of G. Claim (3) and Lemma 2.5 implies that $L_G = 1$ and $L^G \leqslant Z_{\mathfrak{U}}(G)$. Therefore $N \leqslant Z_{\mathfrak{U}}(G)$. It follows from Claim (1) that $G \in \mathfrak{U}$. This contradiction completes the proof of the theorem. □

REFERENCES

[1] A. Ballester-Bolinches – R. Esteban-Romero – M. Asaad: "Products of Finite Groups", *de Gruyter*, Berlin (2010).

[2] K. Doerk – T. Hawkes: "Finite Soluble Groups", *de Gruyter*, Berlin (1992).

[3] W. Guo: "Structure Theory for Canonical Classes of Finite Groups", *Springer*, New York (2015).

[4] W. Guo – C. Cao – A.N. Skiba – D.A. Sinitsa: "Finite groups with $\mathcal{H}$-permutable subgroups", *Comm. Math. Stat.* 5 (2017), 83–92.

[5] W. Guo – A.N. Skiba: "On Π-quasinormal subgroups of finite groups", *Monatsh. Math.* 185 (2018), 443–453.

[6] B. Hu – J. Huang: "On finite groups with generalized σ-subnormal Schmidt subgroups", *Comm. Algebra* 46 (2018), 3127–3134.

[7] B. Huppert: "Endliche Gruppen I", *Springer*, Berlin (1967).

[8] M.T. Hussain: "Finite groups with given σ-conditionally permutable subgroups", *Iran. J. Math. Sci. Inform.*, to appear.

[9] M.T. Husssain – C. Cao – C. Zhang: "The influence of weakly σ-permutably embedded subgroups on the structure of finite groups", *Bull. Iran. Math. Soc.* 46 (2020), 1341–1356.

[10] M.T. Hussain – C. Cao – L. Zhang: "Finite groups with given weakly τ_σ-quasinormal subgroups", *Hacet. J. Math. Stat.* 49 (2020), 1706–1717.

[11] V.N. Knyagina – V.S. Monakhov: "On finite groups with some subnormal Schmidt subgroups", *Siberian Math. J.* 45 (2004), 1075–1079.

[12] R. Schmidt: "Subgroup Lattices of Groups", *de Gruyter*, Berlin (1994).

[13] A.N. Skiba: "On σ-subnormal and σ-permutable subgroups of finite groups", *J. Algebra* 436 (2015), 1–16.

[14] A.N. Skiba: "On some results in the theory of finite partially soluble groups", *Comm. Math. Stat.* 4 (2016), 281–309.

[15] A.N. Skiba: "A generalization of a Hall theorem", *J. Algebra Appl.* 15 (2016), 1650085, 13pp.

[16] S. Srinivasan: "Two sufficient conditions for supersolvability of finite groups", *Israel J. Math.* 35 (1980), 210–214.

[17] X. Wei: "On weakly $\mathfrak{m}$-σ-permutable subgroups of finite groups", *Comm. Algebra* 47 (2019), 945–956.

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