Advances in Group Theory and Applications
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16 (2023), pp. 91–108 ISSN: 2499-1287
DOI: 10.32037/agta-2023-010

Gelfand Pairs for Affine Weyl Groups *

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(*Received Dec. 17, 2021; Accepted Feb. 22, 2023 — Communicated by J. Zhang*)

Abstract

This paper is motivated by several combinatorial actions of the affine Weyl group of type $\widetilde{C}_n$. Addressing a question of David Vogan, it was shown in an earlier paper that these permutation representations are essentially multiplicity-free [2]. However, the Gelfand trick, which was indispensable in [2] to prove this property for types $\widetilde{C}_n$ and $\widetilde{B}_n$, is not applicable for other classical types. Here we present a unified approach to fully answer the analogous question for all irreducible affine Weyl groups. Given a finite Weyl group W with maximal parabolic subgroup $P \leqslant W$, there corresponds to it a reflection subgroup H of the affine Weyl group $\widetilde{W}$. It turns out that while the Gelfand property of $P \leqslant W$ does not imply that of $H \leqslant \widetilde{W}$, however the pair $Q = N_W(P) \leqslant W$ has the Gelfand property if and only if $K = QH \leqslant \widetilde{W}$ has. Finally, for each irreducible type we describe when (W, Q) is a Gelfand pair.

Mathematics Subject Classification (2020): 20C15, 20F55

Keywords: affine Weyl group; multiplicity-freeness; Gelfand subgroup

1 Introduction

Our previous paper [2] was motivated by several natural actions of the affine Weyl group G of type $\widetilde{C}_n$ on finite sets of combinatorial objects, see Section 2 there. We concluded that these are "essentially" multiplicity-free, that is, the actions are not multiplicity-free,

* This research was partially supported by Hungarian National Research, Development and Innovation Office (NKFIH) Grant No. K138596. The project leading to this application has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 741420).

but there exists a G-equivariant coupling of the underlying set such that the G-action on the couples is multiplicity-free.

Our approach included an inflated action of G on an infinite set $\mathbb{Z}_3^n \times \mathbb{Z}$ with point stabilizer H. Dealing with an infinite action led to the definition of a *proto-Gelfand pair* [2, Definition 3.3]: if $A \leqslant B$ for a not necessarily finite group B, we call (B, A) a proto-Gelfand pair if for every homomorphism $\varphi : B \mapsto B_1$ with a finite image, $\big(\varphi(B), \varphi(A)\big)$ is a Gelfand pair. We also say that A is a *proto-Gelfand subgroup* of B.

Theorem 1.1 (see [2], Theorem 1.7) *Let* G *be an affine Weyl group of type* $\widetilde{C}_n$ *or* $\widetilde{B}_n$*. For every* $\omega \in \mathbb{Z}_3^n \times \mathbb{Z}$ *whose stabilizer is not all of* G (*namely,* ω *is not* G*-invariant*), *there exists a double cover of the stabilizer which is a proto-Gelfand subgroup of* G*.*

The action of G on $\mathbb{Z}_3^n \times \mathbb{Z}$ can be understood in terms of affine permutations and also in terms of the fundamental domain. Here we do not explore the possible actions but concentrate only on the Gelfand property of the stabilizer. We prove the following, analogous theorem for all affine Weyl groups. As a reference for Coxeter groups in general, and affine Weyl groups in particular, we use [7].

Let Φ be an irreducible root system with fundamental roots Δ. Let W be its Weyl group, $X = L(\Phi^\vee)$ the translation group with its coroot lattice and $G = \widetilde{W} = W \ltimes X$ the corresponding affine Weyl group.

Delete a node from Δ, let Φ_0 be the (typically reducible) root system generated by the remaining fundamental roots, and let H be the reflection subgroup of G generated by the affine reflections corresponding to Φ_0. Let $P = H \cap W$ denote the corresponding parabolic subgroup of W and $Q = N_W(P)$ its normaliser. The main result of the paper is the following one.

Theorem 1.2 *The index* $|Q : P|$ *is at most* 2*.* $K = QH$ *is a proto-Gelfand subgroup of* G *if and only if* Q *is a Gelfand subgroup of* W*. In particular, this property holds for the following types:*

- A_n, B_n, C_n, G_2 *for every removed node;*
- D_n *for every removed node if* n *is odd, and for* $k \leqslant 2[n/4] + 1$ *or* $k \geqslant n - 1$ *if* n *is even* (*for the numbering see Figure 2 below*)*;*
- E_6, E_7 *if one of the three degree one nodes is removed;*

- E_8 *if one of the two farthest endnodes is removed;*
- F_4 *if one of the two endnodes is removed.*

One part of the theorem is easy. Let φ denote the natural homomorphism $G \to G/X \simeq W$. If K is a proto-Gelfand subgroup of G then $Q \simeq QX/X = KX/X = \varphi(K)$ is a Gelfand subgroup of $W \simeq G/X = \varphi(G)$.

If P is not a Gelfand subgroup of W then, by a similar argument, H cannot be a proto-Gelfand subgroup of G. However, even if P is Gelfand in W but $P < Q$ then H is still not a proto-Gelfand subgroup of G; that is why K is needed in the theorem instead of the reflection subgroup H (see Remark 2.4 below). The fact that $|Q : P| = |K : H| \leqslant 2$, which appears as property (c) in Hypothesis 2.1 below, is a simple consequence of [7, Theorem 1.12 (d)]; see in the proof of Theorem 1.2.

Our proof in [2] was based on the so called "Gelfand's trick": if G has an involutive anti-automorphism (typically the group inversion) that fixes every $H - H$ double coset, then (G, H) is a (proto-)Gelfand pair. In the present paper we use, instead, character theory for two reasons. First, it enables a unified approach for all affine Weyl groups. Second, in some cases Gelfand's trick does not suffice. One such case is type $\widetilde{A}_n$. The following is a special case of our main theorem, removing the k-th node from the Dynkin diagram of A_n.

Corollary 1.3 *Let G be a Weyl group of type $\widetilde{A}_n$. It contains naturally a reflection subgroup H_k of type $\widetilde{A}_{k-1} \times \widetilde{A}_{n-k}$. If $2k \neq n+1$ then H_k is a proto-Gelfand subgroup of G. If $2k = n+1$ then H_k has a double cover K_k which is a proto-Gelfand subgroup of G.*

It is possible to use direct calculations to show that for every homomorphism $\varphi : G \to G_1$ with finite image, the Hecke(double-coset) algebra $H\big(\varphi(G), \varphi(H_k)\big)$ (or $H\big(\varphi(G), \varphi(K_k)\big)$ for $2k = n+1$) is commutative. Still, the simplest Gelfand's trick does not work, not even for $2k \neq n+1$: the double coset $H_k x H_k = xH_k \neq H_k$ (with $x \in X$) does not contain involutions. Albeit, H_k is not a parabolic subgroup, this phenomenon also indicates why in the theorem of Curtis, Iwahori and Kilmoyer (see Theorem 1.4, below) finiteness is required.

The study of the Gelfand property for subgroups of Coxeter groups goes back to at least half a century. To our knowledge, [3] and [10] are among the first general results. Saxl [10] gave a list of potential candidates of Gelfand subgroups $G \leqslant S_n$ for $n > 18$. His list was later made exact by Godsil and Meagher [6] who also dealt with $n \leqslant 18$.

Curtis, Iwahori and Kilmoyer considered parabolic subgroups of finite Coxeter groups. The following is abridged from [3, Theorem 3.1].

Theorem 1.4 *Let* (W, R) *be a Coxeter system with* W *finite,* $J \subseteq R$ *and* $W_J = \langle s_j \mid j \in J \rangle$ *the corresponding parabolic subgroup. The Hecke algebra (double coset algebra)* $H(W, W_J)$ *over an algebraically closed field of characteristic* 0 *is commutative if and only if the shortest element of each double coset* $W_J w W_J \subseteq W$ *is an involution.*

A complete classification of commutative Hecke algebras of Coxeter groups over parabolic subgroups was given by Abramenko, Parkinson and Van Maldeghen [1]. Let $X_{n,i}$ denote the case of the Coxeter system (W, S) of type X_n with $I = S \setminus \{i\}$, removing node i from S according to the standard numbering of the nodes. Similarly $\widetilde{X}_{n,i}$ denotes the case of the affine Weyl group of type $\widetilde{X}_n$ with $I = S \setminus \{i\}$.

Theorem 1.5 (see [1], Theorem 2.1) *Let* (W, S) *be an irreducible Coxeter system,* $I \subseteq S$ *be spherical (that is,* W_I *is finite), and let* $\mathbf{fi} = (\tau_s)$ *be a specialization with* $\tau_s \geqslant 1$ *for each* $s \in S$*. The* I*-parabolic Hecke algebra* $\mathcal{H}^I$ *and its specialization* $\mathcal{H}^I_\tau$ *are noncommutative if* $|S \setminus I| > 1$*. If* $I = S \setminus \{i\}$ *then* $\mathcal{H}^I$ *and* $\mathcal{H}^I_\tau$ *are commutative in the cases*

- $A_{n,i}\,(1 \leqslant i \leqslant n)$, $B_{n,i}\,(1 \leqslant i \leqslant n)$, $D_{n,i}\,(1 \leqslant i \leqslant n/2$ *or* $i = n-1, n)$, $E_{6,1}$, $E_{6,2}$, $E_{6,6}$, $E_{7,1}$, $E_{7,2}$, $E_{7,7}$, $E_{8,1}$, $E_{8,8}$, $F_{4,1}$, $F_{4,4}$, $H_{3,1}$, $H_{3,3}$, $H_{4,1}$, $I_2(p)_i (i = 1, 2)$, *and*
- *all affine cases* $\widetilde{X}_{n,i}$ *with* i *a special node (that is, the removed node* i *and the extra node of the affine Coxeter diagram are in the same orbit under graph automorphisms),*

and noncommutative otherwise.

We will use the list of this theorem for exceptional Weyl groups in the following short form. The parabolic subgroup is a Gelfand subgroup if and only if the deleted node is an endnode ("leaf") of the Dynkin diagram, but not the middle leaf of E_8. See Figure 2 for the labelling of the diagrams.

We finish the Introduction by a natural question. Beyond parabolic subgroups one may consider all reflection subgroups. Dyer and Lehrer gave a complete description of reflection subgroups of finite Coxeter groups and affine Weyl groups [4]. Our subgroup H is always a reflection subgroup. The following is a slight modification of Question 6.2 of [2].

Question 1.6 *Which reflection subgroups (or their finite covers) of affine Weyl groups are proto-Gelfand?*

2 Gelfand pairs for affine Weyl groups

We prove first a general result, and then show that its assumptions hold for all affine Weyl groups. These assumptions are collected in Hypothesis 2.1 about a group G. Note that the definition of Q in (e) is different from the definition of Q preceding Theorem 1.2. That they are equivalent will become clear in the proof of the theorem: when we establish property (c) we also prove that $N_W(Y) = N_W(P)$. For the proof of Proposition 2.3 the version in Hypothesis 2.1 is more convenient.

Hypothesis 2.1 *Let* $G = W \ltimes X$ *be a semidirect product of a finite group* W *and a free Abelian group* X*. Further, let* $H \leqslant G$ *be a subgroup for which the following* (a), (b), (c), (d), (e) *or* (a), (b), (c′), (d′), (e′) *hold:*

(a) $X \cap H = Y$ *and* $X = Y \times \langle x \rangle$ *for some* $x \in X$*, which we fix;*

(b) $H \lhd L = H\langle x \rangle$*;* (c) $|N_G(Y) : L| \leqslant 2$*;*

(d) *if* $g \in G \setminus N_G(Y)$*, then* $y^g = x^m z$ *for some* $y, z \in Y$ *and* $m \in \{\pm 1, \pm 2\}$*;*

(e) $Q = N_W(Y)$ *is a Gelfand subgroup of* W*;* (c′) $|N_G(Y) : L| = 2$*;*

(d′) *if* $g \in G \setminus N_G(Y)$*, then* $y^g = x^m z$ *for some* $y, z \in Y$ *and* $m \in \{\pm 1, \pm 2, \pm 3\}$*;*

(e′) $P = H \cap W$ *is a Gelfand subgroup of* W*.*

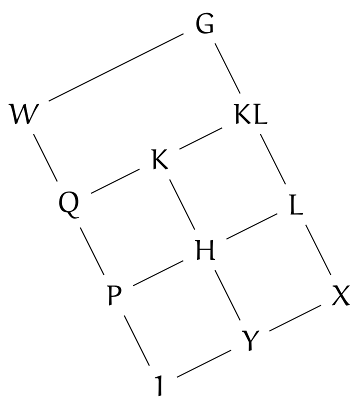


Figure 1: Part of the subgroup lattice of G

Remark 2.2 The (c′), (d′), (e′) version is needed in the sole case when Φ is a root system of type G_2 and we remove the short fundamental root. Here (d) does not hold, so we somewhat relax it and strengthen (c) and (e). After the proof of the main case of Proposition 2.3 we briefly cover the necessary changes for this special case.

Proposition 2.3 *With the assumptions of Hypothesis 2.1,* $K = N_W(Y)Y$ *is a proto-Gelfand subgroup of* G*.*

Proof — We use all the notations of Hypothesis 2.1.

First we prove that $H \triangleleft K$ of index at most 2. Suppose that $gv \in H$ with $g \in W$ of order n and $v \in x^iY$ for some integer i. By (b), H trivially acts on X/Y, so $x^g \in xY$. Hence

$$(gv)^n = g^n v^{g^{n-1}} v^{g^{n-2}} \dots v^g v \in x^{in}Y,$$

but also $(gv)^n \in H$. By (a), $in = 0$ and $i = 0$. It follows that

$$H = (H \cap W)Y = PY \leqslant QY = K.$$

Observe that $L = HX = PX$ and $N_G(Y) = QX = KX = KHX = KL$. Also $Q \cap L = P$ and $K \cap L = H$ so, by (c), $P \triangleleft Q$ and $H \triangleleft K$ are of index at most 2, as claimed (see Figure 1).

We also have $L/H \triangleleft KL/H$ infinite cyclic. If K/H is also normal in KL/H then all the assumptions hold for K in place of H. (If (e′) holds for H then it also holds for K because overgroups of Gelfand subgroups are also Gelfand subgroups.) So, in the following we assume that either $K = H$, or $K > H$ and $K \ntriangleleft KL$. In other words, KL/H is either cyclic or infinite dihedral. Similarly, the action of KL on $X/Y = Y\langle x\rangle/Y$ is the same as its action on $L/H = H\langle x\rangle/H$. That is,

$$x^gY = \begin{cases} xY, & \text{if } g \in L; \\ x^{-1}Y, & \text{if } g \in KL \setminus L. \end{cases} \tag{1}$$

We need show that the permutation character of $\phi(G)$ on the $\phi(K)$-cosets is multiplicity-free in every finite quotient $\phi(G)$ of G. Let $N \triangleleft G$ be any normal subgroup of finite index. It contains $N \cap X$, of finite index in X. Given this finite index we can pick h sufficiently large so that $N \cap X \supseteq X^{(h)} = \{y^h \mid y \in X\}$, which is still a normal subgroup of finite index in G. If $ZX^{(h)}/X^{(h)}$ is a Gelfand subgroup of $G/X^{(h)}$ then all the more so is ZN/N a Gelfand subgroup of G/N, because $ZN/N \leqslant G/N$ are the homomorphic images of $ZX^{(n)}/X^{(h)} \leqslant G/X^{(h)}$ under the natural homomorphism with kernel $N/X^{(h)}$. So it is enough to consider factors by the normal subgroups $X^{(h)}$. For ease of notation we use the same symbol for subgroups of the factor group as for subgroups of G. In other words, instead of being a free Abelian group, X is assumed to be isomorphic to the homogeneous Abelian group Z_h^n. The rest of Hypothesis 2.1 remains intact.

In particular,

$$L/H \simeq Z_h \text{ and if } K > H, \text{ then } KL/H \simeq D_{2h} \text{ is dihedral of order } 2h. \tag{2}$$

As $1_K^G = \left(1_K^{KL}\right)^G$, first we treat the irreducible constituents of 1_K^{KL} separately.

Claim 1: *Suppose* $\varphi \in \mathrm{Irr}(KL)$ *is a linear constituent of* 1_K^{KL}*. Then* φ^G *is multiplicity free.*

Note that for a linear character φ of KL Frobenius reciprocity implies $0 \leqslant \left(\varphi, 1_K^{KL}\right) = \left(\varphi_K, 1_K\right) \leqslant 1$. Hence $\varphi_K = 1_K$ is equivalent to φ being a constituent of 1_K^{KL}. As $G = WKL$, we may use a special case of Mackey's theorem [8, Problem (5.2)] for the restriction of the induced character $\varphi^G{\downarrow_W} = (\varphi_{\downarrow Q})^W = 1_Q^W$ which is multiplicity free by (e), since $KL \cap W = K \cap W = Q$. Hence φ^G itself is also multiplicity free.

Let ε be a primitive h-th root of 1 and η a linear character of L with kernel H and $\eta(x) = \varepsilon$. We consider $1_H^L = \sum_{s=1}^h \eta^s$, a sum of linear characters, each a power of η. For any $1 \leqslant t \leqslant h$, an integer i and $g \in H$ we have $\eta^t(gx^i) = \varepsilon^{ti}$. Let $\lambda_t = \eta^t_{\downarrow X} \in \mathrm{Irr}(X)$.

Recall that if $Z \lhd G$, ζ a character of Z and $g \in G$ then ζ^g is the conjugate character of Z defined by $\zeta^g(z^g) = \zeta(z)$ for every $z \in Z$.

Claim 2: *If for some* $g \in G$, $\lambda_t = \lambda_s^g$*, where* $t \neq h, h/2$ *then* $g \in KL$*, and either* $s = t$*, or* $s = h - t$ *and* $K > H$.

Indeed, if $g \notin KL = N_G(Y)$ then, by assumption (d), there exist $y, z \in Y$ such that $y^g = x^m z$ with $m \in \{\pm 1, \pm 2\}$. So

$$1 = \lambda_s(y) = \lambda_s^g(x^m z) = \lambda_t(x^m) = \varepsilon^{mt},$$

impossible. If $g \in L$ and $v = x^i y \in X$ ($y \in Y$) then $v^g = x^i z$ ($z \in Y$), so, by (1),

$$\lambda_t(x^i y) = \lambda_s^g(x^i y) = \lambda_s^g(x^i z) = \lambda_s^g(\nu^g) = \lambda_s(x^i y).$$

Hence $s = t$.

On the other hand, if $g \in KL \setminus L$ and $\nu = x^i y \in X$ ($y \in Y$) then $\nu^g = x^{-i} z$ ($z \in Y$), so, by (1),

$$\lambda_t(x^{-i} y) = \lambda_s^g(x^{-i} y) = \lambda_s^g(x^{-i} z) = \lambda_s^g(\nu^g) = \lambda_s(x^i y) = \lambda_{h-t}(x^i y).$$

Hence $s = h - t$, but this is possible only if $KL > L$, that is, if $K > H$.

Claim 3: *If $(\eta^t)^G \neq (\eta^s)^G$ then they share no common irreducible constituent. Of course, if $K > H$ and $s = h - t$ then $(\eta^s)^{KL} = (\eta^t)^{KL}$ so $(\eta^t)^G = (\eta^s)^G$.*

Note that the irreducible constituents of $(\eta^t)^G$ lie above λ_t. So, by Clifford's theorem (see [8], (6.2)), it is enough to prove that λ_t and λ_s are not G-conjugate, unless $s = t$ or $s = h - t$ and $K > H$. By Claim 2, we have to check only $s, t \in \{h, h/2\}$. But then $s = t$ as $\mathrm{Ker}(\lambda_h) = X \neq \mathrm{Ker}(\lambda_{h/2})$.

If $K = H$ then $KL = L$ and, by Claim 1, all $(\eta^s)^G$ are multiplicity free and, by Claim 3, the $(\eta^s)^G$ ($s = 1, \dots, h$) share no common irreducible constituent. Hence $1_H^G = \sum_{s=1}^h (\eta^s)^G$ is multiplicity free, indeed.

So let $K > H$. If $t = h, h/2$ (the second only for even h) then there are unique linear extensions 1_{KL} of $\eta^h = 1_L$ and μ of $\eta^{h/2}$ (for h even) to KL that are trivial on K. So, using Claim 1 again, we conclude that 1_{KL}^G and μ^G are multiplicity free with all constituents lying above $\lambda_h = 1_X$ and $\lambda_{h/2}$, respectively.

Assume now $t \neq h, h/2$. Let $I_t = I_G(\lambda_t) = \{g \in G \mid \lambda_t^g = \lambda_t\} \geqslant L$ denote the inertia subgroup of λ_t. By Claim2, $I_t \subseteq KL$. If $g \in K \setminus H$ then, by (1), $\lambda_t^g = \lambda_{h-t} \neq \lambda_t$ so $I_t = L$ and $(\eta^s)^G$ is irreducible as it is induced from the inertia subgroup [8], (6.11).

Now 1_K^{KL} is a sum of irreducible, degree 2 characters $(\eta^s)^{KL}$, $1 \leqslant s < h/2$, of 1_{KL}, and if h is even then of an additional linear character μ extending $\eta^{h/2}$. For $1 \leqslant s < h/2$ $(\eta^s)^G$ are distinct irreducible while μ^G (for h even) and 1_{KL}^G are multiplicity free. By Claim 3, there are no shared constituents among these, so 1_K^G is multiplicity free.

If together with (a) and (b), the alternatives (c'), (d'), (e') hold instead of (c), (d) and (e), then we make the following modifications. For Claim 1: if $\varphi \in \mathrm{Irr}(L)$ is a linear constituent of 1_H^L then φ^G is multiplicity free. The proof is the same, using (e') and that $G = WL$

and $P = W \cap L$. For Claim 2, the above proof still works, using (d′), unless $m = \pm 3$ and $t = h/3, 2h/3$. But even in that case either $s = t$ or $s = h - t$. So, if for some $g \in G$, $\lambda_t = \lambda_s^g$ then $s = t$ or $s = h - t$, even if $g \notin KL$.

Claim 3 holds, that is $(\eta^t)^G, (\eta^s)^G$ share no common irreducible summand unless $s = t, s = h - t$ but in these cases they are the same, as $K > H$ by (c′). The conclusion of the proof is the same: we use Claim 1 (and (c′)) to show that for $t \neq h, h/2$

$$(\eta^t)^G = \left((\eta^t)^{KL}\right)^G$$

are all multiplicity free. Hence, by Claim 3 as above, 1_K^G is multiplicity free. □

Remark 2.4 If $H < K$ then (G, H) is not a proto-Gelfand pair. Indeed, let

$$\varphi : G \to W \ltimes Z_3^n$$

be the natural homomorphism with kernel $X^{(3)}$. Then the image

$$\varphi(KL)/\varphi(H) \simeq D_6,$$

the dihedral group of order 6, by (2). However, in D_6 the regular character is not multiplicity free, so $\big(\varphi(KL), \varphi(H)\big)$ is not a Gelfand pair, nor is then $\big(\varphi(G), \varphi(H)\big)$. Proposition 2.3 claims that even if H itself is not a proto-Gelfand subgroup, a double cover of H is.

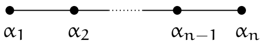
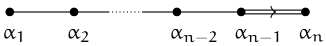
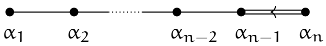
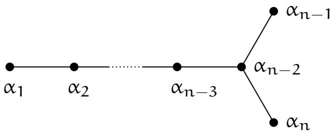
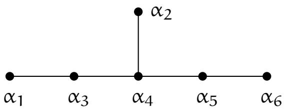
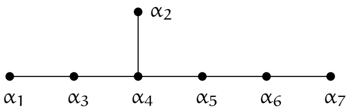
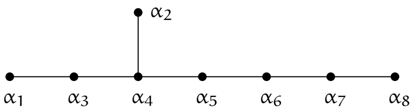
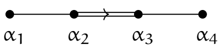
Proof of Theorem 1.2 — To finish the proof we have to confirm the assumptions of Hypothesis 2.1 for the subgroup H defined before the statement of Theorem 1.2 and described below in more detail.

Let Φ be an irreducible root system of rank n, W be its Weyl group and G be the corresponding affine Weyl group. Let $\Delta = \{\alpha_i\}_{i=1,\dots,n}$ denote the fundamental roots and

$$\Phi^\vee = \left\{\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} \mid \alpha \in \Phi\right\}$$

the corresponding coroot system. Then the normal subgroup of translations X is naturally identified with $L(\Phi^\vee)$, the coroot lattice. We record some properties of the root systems in Figure 2, see [7, Sections 2.9, 2.10, 4.9].

Figure 2: Coefficients of the highest root in irreducible root sytems

	Dynkin diagram	coefficients of $\widetilde{\alpha}$
A_n		$1, \ldots, 1$
B_n		$1, 2, \ldots, 2$
C_n		$2, \ldots, 2, 1$
D_n		$2, \ldots, 2, 1, 1$
E_6		$1, 2, 2, 3, 2, 1$
E_7		$2, 2, 3, 4, 3, 2, 1$
E_8		$2, 3, 4, 6, 5, 4, 3, 2$
F_4		$2, 3, 4, 2$
G_2		$3, 2$

Pick α from among the fundamental roots. Let Φ_0 be the (reducible) root system generated by the remaining fundamental roots $\Delta \setminus \{\alpha\}$. Finally, let H denote the reflection subgroup of G generated by the affine reflections corresponding to Φ_0. Note that $P = W \cap H$ is the parabolic subgroup $W_{\Delta\setminus\{\alpha\}} \leqslant W$ generated by the fundamental reflections corresponding to $\Delta \setminus \{\alpha\}$.

Of course, H is the direct product of the affine reflection groups on the (usually two) connected components of $\Delta \setminus \{\alpha\}$. Namely, if $\Gamma = \{\alpha_i \mid i \in I\}$ is one component then $H_\Gamma = \langle g, s_i \mid i \in I\rangle$ is one direct factor of H, where $g = s_{\beta,1}$ is an affine reflection flipping β, the highest root in Φ_Γ, the root system generated by Γ. Let $Y = H \cap X$, naturally isomorphic to $L(\Phi_0^\vee) \leqslant L(\Phi^\vee)$, and let $x = t(\alpha^\vee)$, the translation by the coroot in the direction of the missing fundamental root, α. Of course, $X = Y \times \langle x\rangle$. So Property (a) follows.

If $s = s_{\alpha_i} \in H$ is among the ordinary reflections generating H then

$$\begin{aligned} x^{-1}sxs &= t(-\alpha^\vee)s_{\alpha_i}t(\alpha^\vee)s_{\alpha_i} = t(-\alpha^\vee)t(s_{\alpha_i}\alpha^\vee) \\ &= t(-\alpha^\vee)t(\alpha^\vee - (\alpha^\vee, \alpha_i)\alpha_i^\vee) \in Y. \end{aligned}$$

As all of the generators of H/Y commute with xY, we get that H/Y and X/Y commute. In other words, $H \lhd L = H\langle x\rangle$. So Property (b) follows.

Recall, that $P = W \cap H$. Let U denote the intersection of the reflecting hyperplanes corresponding to the $n-1$ fundamental roots in $\Delta \setminus \{\alpha\}$. So

$$U = \langle \Phi_0 \rangle^\perp = \bigcap_{g \in P} \mathrm{Ker}(g-1)$$

is a line. Let

$$M = \{w \in W \mid wU = U\} = \{w \in W \mid w\Phi_0 \subseteq \langle \Phi_0 \rangle\} = N_W(Y).$$

So the elements of M either reflect U through the origin or fix U pointwise. In this latter case w is in the isotropy group of U, which is P, by [7, Theorem 1.12 (d)]. Hence $|M : P| \leqslant 2$, in particular $M \leqslant N_W(P)$. If $w \in W$ then

$$wU = \bigcap_{g \in P} \mathrm{Ker}(wgw^{-1} - 1)$$

therefore U is $N_W(P)$-invariant. That is, $N_W(P) \leqslant M$, so

$$N_W(Y) = M = N_W(P)$$

and Property (c) follows.

To prove Property (d) we will use that the coefficient of the deleted fundamental root α in the highest root $\widetilde{\alpha}$ is 1 or 2. For types A, B, C and D the highest root has coefficients $1, 2$ only, while for the exceptional types this holds for the endnodes, the "leaves" (see our remark after Theorem 1.5 in the introduction). There are two exceptions: the middle leaf of E_8 (for which the parabolic subgroup itself is not a Gelfand subgroup) and the short fundamental root of G_2 (whose coefficient in the highest root is 3). For the middle leaf of E_8 the parabolic subgroup is self-normalizing so (e) does not hold. For the short root of G_2, (c′), (d′) and (e′) do hold, see below.

Suppose that $w \in W$ does not normalize $L(\Phi_0^\vee)$, hence there is a root $\beta \in \Phi_0$ such that in the decomposition of the root $w\beta \in \Phi$ the coefficient of α is non-zero. As both $\widetilde{\alpha} - w\beta$ and $w\beta + \widetilde{\alpha}$ are non-negative combinations of the fundamental roots, the coefficient of α in $w\beta$ is between -2 and 2. As it is also non-zero, Property (d) follows.

For classical types Property (e) follows from Theorem 1.5 save the case of $D_{n,i}$ $n/2 < i < n-1$, which requires special attention. To obtain Q in each of the exceptional cases E_6, E_7, E_8, F_4 we used GAP [5]. We conclude that if the node is not a leaf then Q is not a Gelfand subgroup.

A_n, B_n, C_n By Theorem 1.5, P is a Gelfand subgroup, so (e) holds.

D_n By Theorem 1.5, if we remove α_k then P is a Gelfand subgroup, so (e) holds, unless $n/2 < k < n-1$. However, $Q = N_W(P)$ is a Gelfand subgroup of W if and only if n is odd or n is even and $k \leqslant 2[n/4] + 1$ or $k \geqslant n-1$ (see Theorem 3.1 below for a proof). Hence (e) holds unless n is even and $2[n/4]+1<k<n-1$.

E_6 If α is one of the three endnodes then P is a Gelfand subgroup, by Theorem 1.5, so (e) holds. For the other three nodes even Q is not a Gelfand subgroup (among these, $Q > P$ holds only for the middle node).

E_7 If α is an endnode then P is a Gelfand subgroup, by Theorem 1.5, so (e) holds. For the other four nodes even Q is not a Gelfand subgroup (even though $Q > P$ always).

E_8 If α is one of the two farthest endnodes then P is a Gelfand subgroup, by Theorem 1.5, so (e) holds. For the other six nodes even Q is not a Gelfand subgroup (even though $Q > P$ always).

F_4 If α is an endnode then P is a Gelfand subgroup, by Theorem 1.5, so (e) holds. For the two middle nodes even Q is not a Gelfand subgroup (even though $Q > P$ always).

G_2 The natural action of the dihedral group on the vertices of a hexagon is multiplicity-free, so (e′) holds. As $Q = N_W(P)$ is of order 4, a double cover, (c′) holds. The coefficients of $\widetilde{\alpha}$ are 2 and 3, so (d′) holds.

The statement is proved. □

3 Type D

This section is devoted to the statement and proof of Theorem 3.1, which completes the proof of Theorem 1.2 for the classical type D_n. It seems that Lehrer [9] was the first to determine which parabolic subgroups of D_n are Gelfand subgroups. His list was incomplete, later Abramenko, Parkinson and Van Maldeghem [1] provided the complete answer.

We need (and prove) the claim for the double cover of the reflection subgroup P but for completeness we also state the claim for the parabolic subgroup itself. The proof resembles the proof of Theorem 1.2.

Theorem 3.1 *Let $n > 3$ and W a Weyl group of type D_n. Let $1 \leqslant k \leqslant n$ and $P \leqslant W$ the maximal parabolic subgroup corresponding to removing the k-th node.*

(1) *P is a Gelfand subgroup of W if and only if either $1 \leqslant k \leqslant n/2$ or $n-1 \leqslant k$.*

(2) *Suppose $n/2 < k < n-1$. The double cover $Q = N_W(P)$ is a Gelfand subgroup of W if and only if either n is odd, or n is divisible by 4 and $k = n/2+1$.*

Proof — We use the isomorphism $W \simeq V \rtimes S_n$, where $V = \mathbb{F}_2^{n-1}$. This V is the 1-codimensional submodule $\{\sum a_i e_i \mid \sum a_i = 0\}$ of the natural $\mathbb{F}_2 S_n$ module $\langle e_1, \dots, e_n \rangle$. Let a basis of V consist of

$$\{v_i = e_i + e_n \mid i = 1, \dots, n-1\}$$

on which $\sigma \in S_n$ acts by

$$v_i\sigma = \begin{cases} v_{i\sigma}, & \text{if } n\sigma = n, \\ v_{n\sigma}, & \text{if } i\sigma = n, \text{ while} \\ v_{i\sigma} + v_{n\sigma}, & \text{if } i\sigma,\ n\sigma < n. \end{cases} \tag{3}$$

The claim about the Gelfand property of P is covered by Theorem 1.5. From now on we assume $n/2 < k < n-1$. Then P is the Weyl group of the decomposable root system of type $A_{k-1} \oplus D_{n-k}$ (with minor notational changes due to degeneration if $k = n-3, n-2$) and

$$P \simeq S_k \times (V_0 \rtimes S_{n-k}),$$

where $V_0 = \langle v_{k+1}, \dots v_{n-1} \rangle$. Let $Q = N_W(P)$. As $k > n/2$,

$$Q = N_V(P)P = \langle v_1 + v_2 + \dots + v_k \rangle P.$$

So let

$$V_1 = Q \cap V = \langle v_1 + v_2 + \dots + v_k, v_{k+1}, \dots, v_{n-1} \rangle.$$

To determine 1_Q^W we first decompose 1_Q^{VQ}. Using a special case of Mackey's theorem (see [8, Problem (5.2)]), $1_Q^{VQ} \downarrow_V = 1_{V_1}^V$, whose constituents correspond to $E \subseteq \{1, \dots, k\}$ of cardinality $|E| \equiv k \pmod 2$. Namely, for each such E let

$$\eta_E(v_f) = \begin{cases} 1, & \text{if } f \in E \cup \{k+1, \dots, n-1\}; \\ -1, & \text{if } f \notin E \cup \{k+1, \dots, n-1\}. \end{cases} \tag{4}$$

Then

$$\eta_E \in \mathrm{Irr}(V) \quad \text{and} \quad 1_{V_1}^V = \sum_E \eta_E.$$

By (3), $v_i\sigma = v_{i\sigma}$ for σ in $S_k \times S_{n-k}$, hence the orbit of a subset $E \subseteq \{1, \dots, k\}$ under the action of $S_k \times S_{n-k}$ consists of the subsets of the same cardinality. So

$$1_Q^{VQ} = \sum_{i=0}^{[k/2]} \chi_i,$$

where $\chi_{iV} = \sum_{|E|=k-2i} \eta_E$. Fix $E = E_i = \{2i+1, \ldots, k\}$ and the inertia subgroup of η_E in VQ is

$$M = V(S_{2i} \times S_{k-2i} \times S_{n-k}) = I_{VQ}(\eta_E).$$

If $\nu_E \in \mathrm{Irr}(M)$ denotes the extension of η_E to M which is trivial on $S_{2i} \times S_{k-2i} \times S_{n-k}$, then $\chi_i = \nu_E^{VQ}$ is irreducible by [8, (6.11)].

Claim: *For every* $0 \leqslant i \leqslant [k/2]$, χ_i^W *is multiplicity free and* $\left(\chi_i^W, \chi_j^W\right) > 0$ *if and only if* $i = j$ *or* $2i + 2j = n$.

By the claim, 1_Q^W is multiplicity free unless n can be written a sum of two distinct even integers between 0 and k. This latter possibility can occur only for even n. But if n is divisible by 4 then $k = n/2 + 1$ is odd and the largest distinct $2i, 2j$ are $k-1, k-3$, whose sum is $2k - 4 = n - 2 < n$. Let now n be even. If $k > n/2$ is also even then for $i = k/2 > (n-k)/2 = j$ we get $2i + 2j = n$, while if $k > 2[n/4] + 1$ is odd then $i = (k-1)/2 > (n-k+1)/2 = j$ implies $2i + 2j = n$. These confirm the theorem.

We now prove the claim. To show multiplicity freeness, let $i \leqslant k/2$ be fixed, $E = E_i = \{2i+1, \ldots, k\}$ and $N = S_{2i} \times VS_{n-2i} = I_W(\eta_E)$. Then η_E extends to $\mu_E \in \mathrm{Irr}(N)$ such that $S_{2i} \times S_{n-2i} \subseteq \mathrm{Ker}(\mu_E)$. Of course, μ_E also extends ν_E. The other constituents of η_E^N are $\mu_E \varphi$, such that $\varphi \in \mathrm{Irr}(N)$ and $\mathrm{Ker}(\varphi) \supseteq V$, in particular, all the irreducible constituents of ν_E^N are of this form. By [8, (6.11)], each $(\mu_E \varphi)^W$ is irreducible. Now, $\chi_i^W = (\nu_E^{VQ})^W = \nu_E^W = (\nu_E^N)^W$ and, by Frobenius reciprocity,

$$\left(\nu_E^N, \mu_E \varphi\right) = \left(\nu_E, (\mu_E \varphi)_M\right) = (\nu_E, \nu_E \varphi_M) = (1_M, \varphi_M) = \left(1_M^N, \varphi\right) \leqslant 1,$$

because $M \leqslant N$ is a Gelfand subgroup (consider

$$M/VS_{2i} \simeq S_{k-2i} \times S_{n-k} \leqslant S_{n-2i} \simeq N/VS_{2i}).$$

That is, ν_E^L is multiplicity free. Each of its irreducible constituents is of form $\mu_E \varphi$ and these induce irreducibly to W, so we obtain that $\chi_i^W = \nu_E^W$ is multiplicity free, indeed.

By Clifford Theory [8, (6.2)], if $\eta \in \mathrm{Irr}(V)$ and ψ is an irreducible summand of η^W then the constituents of ψ_V are conjugates of η. Therefore, if $i \neq j$ then the characters χ_i^W and χ_j^W share no common irreducible summand unless the underlying η_{E_i} and η_{E_j} are W-conjugates for $E_i = \{2i+1, \dots, k\}$, $E_j = \{2j+1, \dots, k\}$. Suppose $\sigma \in W$ is such that $\eta_{E_i}^\sigma = \eta_{E_j}$. As V is Abelian, we may assume $\sigma \in S_n$. If $n\sigma = n$ then, by (3) and (4)

$$\{1\sigma, \dots, (2i)\sigma\} = \left\{f \mid \nu_f \notin \mathrm{Ker}(\eta_{E_i}^\sigma)\right\} = \left\{f \mid \nu_f \notin \mathrm{Ker}(\eta_{E_j})\right\} = \{1, \dots, 2j\}$$

have the same cardinality, so $i = j$. If $m\sigma = n$, $2i < m < n$, then

$$\nu_{n\sigma} = \nu_m\sigma \in \mathrm{Ker}(\eta_{E_i}^\sigma),$$

so

$$\{1\sigma, \dots, (2i)\sigma\} = \left\{f \mid \nu_f \notin \mathrm{Ker}(\eta_{E_i}^\sigma)\right\} = \left\{f \mid \nu_f \notin \mathrm{Ker}(\eta_{E_j})\right\} = \{1, \dots, 2j\},$$

again and $i = j$. Finally, if $m\sigma = n$, $m \leqslant 2i$ then

$$\nu_{n\sigma} = \nu_m\sigma \notin \mathrm{Ker}(\eta_{E_i}^\sigma),$$

so

$$\begin{aligned}\{n\sigma, (2i+1)\sigma, \dots (n-1)\sigma\} &= \left\{f \mid \nu_f \notin \mathrm{Ker}(\eta_{E_i}^\sigma)\right\} \\ &= \left\{f \mid \nu_f \notin \mathrm{Ker}(\eta_{E_j})\right\} = \{1, \dots, 2j\}\end{aligned}$$

have the same cardinality $n - 2i = 2j$, that is $n = 2i + 2j$, as required.

For the converse, let n be even, $j > i = n/2 - j$ (and $i > 0$ because $k < n - 1$). Then

$$E_i = \{2i+1, \dots, k\}, \quad E_j = \{2j+1, \dots, k\}.$$

Let σ be the transposition

$$(1, 2j+1)(2, 2j+2) \dots (2i, n).$$

By (3), $\nu_{2i}\sigma = \nu_{2i}$ and $\nu_f\sigma = \nu_{f\sigma} + \nu_{2i}$ for $f \neq 2i$. As $\nu_{2i} \notin \mathrm{Ker}(\eta_{E_i})$, so $\nu_{2i} \notin \mathrm{Ker}(\eta_{E_i}^\sigma)$ and

$$\nu_f \in \mathrm{Ker}(\eta_{E_i}) \Leftrightarrow \nu_{f\sigma} \notin \mathrm{Ker}(\eta_{E_i}^\sigma)$$

for $f \neq 2i$. Therefore $\eta_{E_j}^{\sigma} = \eta_{E_i}$ by (4) as

$$\{1, \ldots, 2i-1\}\sigma = \{2j+1, \ldots, n-1\}.$$

If $M_i = I_{VQ}(\eta_{E_i})$ and $M_j = I_{VQ}(\eta_{E_j})$ then

$$\nu_{E_i} \downarrow_{M_j^{\sigma} \cap M} = \nu_{E_j}^{\sigma} \downarrow_{M_j^{\sigma} \cap M} .$$

Using Mackey's theorem (see [8, Problem (5.6)]), we get

$$\begin{aligned} (\chi_i^W, \chi_j^W) &= (\nu_{E_i}^W, \nu_{E_j}^W) = (\nu_{E_i}, \nu_{E_j}^W \downarrow_{M_i}) \\ &= \sum_{W = \cup M_j g M_i} (\nu_{E_i}, (\nu_{E_j}^g \downarrow_{M_j^g \cap M_i})^{M_i}). \end{aligned}$$

Among the summands is

$$\begin{aligned} (\nu_{E_i}, (\nu_{E_j}^{\sigma} \downarrow_{M_j^{\sigma} \cap M_i})^{M_i}) &= (\nu_{E_i}, (\nu_{E_i} \downarrow_{M_j^{\sigma} \cap M_i})^{M_i}) \\ &= (\nu_{E_i} \downarrow_{M_j^{\sigma} \cap M_i}, \nu_{E_i} \downarrow_{M_j^{\sigma} \cap M_i}) = 1, \end{aligned}$$

as required. □

4 Acknowledgments

This paper is an outgrowth from a joint work with Ron Adin and Yuval Roichman [2], but uses quite different methods. Beyond friendship and hospitality, they helped in the development of the present work in several ways both in structure and in content. The author wishes to express gratitude towards them.

REFERENCES

[1] P. Abramenko – J. Parkinson – H. Van Maldeghem: "A classification of commutative parabolic Hecke algebras", *J. Algebra* 385 (2013), 115–133.

[2] R. Adin – P. Hegedüs – Y. Roichman: "Flip actions and combinatorial Gelfand pairs for affine Weyl groups", *J. Algebra* 607 (2022), 5–33.

[3] C.W. Curtis – N. Iwahori – R. Kilmoyer: "Hecke algebras and characters of parabolic type of finite groups with (B, N)-pairs", *Inst. Hautes Études Sci. Publ. Math.* 40 (1971), 81–116.

[4] M.J. Dyer – G.I. Lehrer: "Reflection subgroups of finite and affine Weyl groups", *Trans. Amer. Math. Soc.* 363 (2011), 5971–6005.

[5] The GAP Group: "GAP — Groups, Algorithms, and Programming", v4.11.0 (2020).

[6] C. Godsil – K. Meagher: "Multiplicity-free permutation representations of the symmetric group", *Ann. Comb.* 13 (2010), 463–490.

[7] J.E. Humphreys: "Reflection Groups and Coxeter Groups", *Cambridge University Press*, Cambridge (1990).

[8] M. Isaacs: "Character Theory of Finite Groups", *Academic Press*, London (1976).

[9] G.I. Lehrer: "On incidence structures in finite classical groups", *Math. Z.* 147 (1976), 287–299.

[10] J. Saxl: "On multiplicity-free permutation representations", in: Finite Geometries and Designs, *Cambridge University Press*, Cambridge (1981), 337–353.

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