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Central Extensions and Groups with Quotients Periodic Infinite *

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Abstract

For an arbitrary family of groups without involutions and any Abelian group $\mathcal{D}$ we construct a group $A_{\mathcal{D}}(G)$ such that the center of $A_{\mathcal{D}}(G)$ coincides with $\mathcal{D}$, and the quotient group of the group $A_{\mathcal{D}}(G)$ by the subgroup $\mathcal{D}$ coincides with the n-periodic product of the given family of groups. In particular, as an application, 2-generated non-simple and non-periodic Hopfian groups are constructed, any proper non-trivial quotient of which is infinite periodic. The construction is based on some modification of the method used by S.I. Adian for a positive solution of the known problem on the existence of non-commutative analogues of the additive group of rational numbers.

Mathematics Subject Classification (*2020*): 20F50, 20E22, 20E06

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1 Introduction and results

The concept of the periodic product of period n for a given family of groups $\{G_i\}_{i\in I}$ was introduced by S.I. Adian in [2]. These products solve the well-known problem of A.I. Maltsev, posed in 1948, about the existence of an operation on the family of groups different from the classical operations of free and direct products, and satisfying all natural properties of these operations. These product operations of

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groups are exact and have the properties of associativity and heredity on subgroups. They are denoted by

$$G = \prod_{i \in I}^{n} G_i \qquad (\star)$$

and were constructed for odd periods $n \geqslant 665$ and any given family of groups $\{G_i\}_{i \in I}$ (see also [5]). The essence of the construction presented in [2] is that, based on the free product $\prod_{i \in I}^{*} G_i$ of a given family of non trivial groups, periodic words are classified by rank in the same way just as it was done in [4] for words in the alphabet of a free group, adding rank by rank all new defining relations of the form $A^n = 1$ with elementary periods A of rank α. The question is how to define these elementary periods A of rank α inductively. The concept of an elementary period of rank α, as well as all related concepts for rank α, are defined by simultaneous induction by analogy with the concepts of the same name constructed in the monograph [4] for free Burnside groups. In the next paper [3], S.I. Adian proved an interesting criterion for the simplicity of n-periodic products of groups: the periodic product of a given family of groups $\{G_i\}_{i \in I}$ that do not contain an involution is a simple group if and only if $G_i^n = G_i$ for each factor G_i of this product. This criterion of simplicity made it possible for the first time to indicate series of finitely generated infinite simple groups in a variety different from the variety of all groups (see [22], Problem 23).

Later in [6], a certain natural generalization of the simplicity criterion for n-periodic products was proved, their so-called CEP-subgroups were described, and a sufficient condition to be Hopfian was found.

In [7] (see also [9]), the characteristic property of n-periodic products is obtained. It turned out that the free product of a given family of groups has a unique normal subgroup with certain properties, the quotient group with respect to which is isomorphic to the n-periodic product of this family. This property was used to show that almost all subgroups of n-periodic products contain a free Burnside subgroup of infinite rank, and are even uniformly non-amenable. In 2016, [8] the C^*-simplicity of n-periodic products for a wide class of groups is proved. In particular, it turned out that n-periodic products of finite or cyclic groups are C^*-simple groups. Note that the question of the existence of C^*-simple groups without free subgroups of rank 2 was posed by de la Harpe in 2007.

In another paper [1], Adian constructed finitely generated non-Abelian groups $A(m,n)$ in which the intersection of any two non-trivial subgroups is infinite (solution of the Kontorovich problem on existence of non-Abelian analogues of the rational numbers group). The group $A(m,n)$ is a central extension of the free Burnside group $B(m,n)$ of rank m and fixed odd period $n \geqslant 665$ by an infinite cyclic group. By definition a free Burnside group $B(m,n)$ of period n and rank m has the following presentation:

$$B(m,n) = \langle a_1, a_2, \ldots, a_m \mid X^n = 1 \rangle,$$

where X runs through the set of all words in the alphabet

$$\{a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_m^{\pm 1}\}.$$

The definition of the group $A(m,n)$ by generators and defining relations is obtained from the definition of the group $B(m,n)$ as a result of adding to the alphabet of generators $\{a_1, a_2, \ldots, a_m\}$ a new letter d commuting with all a_i and replacing each relation $A^n = 1$ by $A^n = d$, where $\{A^n = 1 \mid A \in \mathcal{E}\}$ is the independent system of defining relations for $B(m,n)$ constructed in Chapter VI of the Adian's monograph [4]. Using modifications of the method for constructing the group $A(m,n)$, it was shown in 2018 [10] that any countable Abelian group D can be verbally embedded as a center into some m-generated group A so that the quotient group A/D is isomorphic to the free Burnside group $B(m,n)$ for any fixed odd exponent n. In the same paper, the free groups of the variety defined by the identity $[x^n, y] = 1$ were described, and the Schur multiplier of the group $B(m,n)$ was calculated. We emphasize that central extensions of free Burnside groups and their Schur multipliers were also studied in [11] and [23].

In this paper, for an arbitrary odd period $n \geqslant 665$ we propose a further modification of the method of central extensions of free Burnside groups and use it to prove that any Abelian group D can be embedded as a center into a group A such that the quotient group A/D is isomorphic to an n-periodic product of an arbitrary given family of groups without involutions (we note that based on [5] this result can also be extended to arbitrary families of groups at least one of which contains an element of odd order).

Theorem 1.1 *For an arbitrary family of groups without involutions and any* (*countable*) *Abelian group* $\mathcal{D}$ *there exists a group* $A_{\mathcal{D}}(G)$ *such that:*

(1) *the center of the group* $A_{\mathcal{D}}(G)$ *coincides with* $\mathcal{D}$*;*

(2) *the quotient of* $A_{\mathcal{D}}(G)$ *by the subgroup* $\mathcal{D}$ *coincides with the* n*-periodic product of the given family of groups.*

If all groups of a given family are cyclic groups of order n, then the n-periodic product coincides with the free Burnside group of the corresponding rank (see Theorem 9 of [2]). Therefore, Theorem 1.1 implies Theorem 1 from [10].

Corollary 1.2 (Theorem 1 [10]) *For any* $m > 1$ *and odd* $n \geqslant 665$ *and for any Abelian group* $\mathcal{D}$ *one can construct by generating and defining relations an* m*-generated group* $A_{\mathcal{D}}(m, n)$ *such that:*

(1) *the identity* $[x^n, y] = 1$ *holds in* $A_{\mathcal{D}}(m, n)$*;*

(2) *a verbal subgroup of the group* $A_{\mathcal{D}}(m,n)$ *corresponding to the word* x^n *coincides with the Abelian group* $\mathcal{D}$*;*

(3) *the center of the group* $A_{\mathcal{D}}(m, n)$ *coincides with* $\mathcal{D}$*;*

(4) *the quotient of* $A_{\mathcal{D}}(m, n)$ *by the subgroup* $\mathcal{D}$ *is the free Burnside group* $B(m, n)$*.*

Corollary 1.1 also strengthens Theorem 31.2 in [23] (see also [11]).

In 1999, P. de la Harpe posed the following question in the "Kourovka Notebook": "Find an explicit embedding of the additive group of rational numbers $\mathbb{Q}$ into a finitely generated group"; see [17], Question 14.10 b). Such an embedding was proposed by V. Mikaelian in [19]. Later, Mikaelian [21],[20] and J. Belk, J. Hyde and F. Matucci [14] even pointed out embeddings of the group of rational numbers into a finitely defined group. Here we give new examples of such an embedding while maintaining the Hopf property (note that the group of rational numbers is Hopf). Recall that a group is *almost periodic* if it is not periodic, but any proper non-trivial quotient is an infinite periodic group.

Theorem 1.3 *There is a Hopfian group* $A_{\mathbb{Q}}(G)$ *(with an arbitrary fixed number* $m > 1$ *of generators) explicitly given by generators and defining relations such that:*

(1) *the center of* $A_{\mathbb{Q}}(G)$ *coincides with the additive group of rational numbers* $\mathbb{Q}$*;*

(2) *the group* $A_{\mathbb{Q}}(G)$ *is almost periodic;*

(3) *the group* $A_{\mathbb{Q}}(G)$ *is residually periodic.*

Replacing in our construction (see Section 4 below) the group of rational numbers by the quasi-cyclic group C_{p^∞}, we obtain a group with the same properties (1)–(3) which, in addition, is residually periodic of bounded exponent. For comparison, note that according to [12],[13] the free Burnside group of odd exponent $n \geqslant 1003$ is approximated by *Tarski's monsters*, that is, by infinite periodic groups of bounded exponent, all of whose proper subgroups are cyclic. In [15],[16],[18],[26] *just infinite groups* (that is, infinite groups all of whose proper quotients are finite) are constructed and studied. But examples of non-periodic almost infinite periodic groups apparently have not been explicitly constructed before. However, the existence of such groups with significantly more generators can be deduced from the existence of a minimal subgroup of finite index in infinite free Burnside groups of finite rank. As noted in [24] "the task to construct a finitely presented, residually periodic (of bounded exponent) group which is not residually finite should be difficult".

2 Central extensions

As mentioned in the previous section, some modification of the definition of groups $A(m, n)$ from [1] was proposed in [10], which made it possible to construct a group, whose center coincides with an arbitrary countable Abelian group $\mathcal{D}$ and the quotient by the subgroup $\mathcal{D}$ is isomorphic to the free Burnside group $B(m, n)$. We will show how this modification can be developed further and, for an arbitrary Abelian group $\mathcal{D}$, we will construct a group whose center coincides with $\mathcal{D}$ and whose quotient group by the subgroup $\mathcal{D}$ is isomorphic to the n-periodic product of a given family of groups.

By definition the periodic product of exponent n of a family $\{G_i\}_{i\in I}$ of groups without involutions is the group $G = \prod_{i\in I}^{n} G_i$ obtained from the free product $F = \prod_{i\in I}^{*} G_i$ by adding to the set of its defining relations all the defining relations of the form $A^n = 1$, where A is an elementary period of some rank in a certain classification of periodic words of a free product F. Now denote by $\mathcal{E}$ the set of all elementary periods chosen in accordance with the definition of [4], Chapter VI, Section 2.1. This set $\mathcal{E}$ is countable. We fix some numbering and let $\mathcal{E} = \{A_j \mid j \in \mathbb{N}\}$ (where $\mathbb{N}$ is the set of natural numbers).

We also fix an arbitrary at most countable Abelian group $\mathcal{D}$ given by generators and defining relations:

$$\mathcal{D} = \langle d_1, d_2, \dots, d_s, \dots \mid r = 1,\ r \in \mathcal{R} \rangle, \tag{2.1}$$

where $\mathcal{R}$ is some set of words in the group alphabet $d_1, d_2, \dots, d_i, \dots$

Assume that each group G_i has a system of generators $\{g_{ij}\}_{j \in K_i}$. Denote by $A_{\mathcal{D}}(G)$ the group given by the following system of two types generators

$$\{g_{ij}\}, i \in I, j \in K_i, \tag{2.2}$$

and

$$d_1, d_2, \dots, d_s, \dots \tag{2.3}$$

and having the following system of tree types defining relations:

$$r = 1, \text{ for all } r \in \mathcal{R}, \tag{2.4}$$

$$g_{ij} d_s = d_s g_{ij}, \tag{2.5}$$

$$A_s^n = d_s \tag{2.6}$$

for all $i \in I, j \in K_i$, $s \in \mathbb{N}$ and $A_s \in \mathcal{E}$. Moreover, if the set

$$\{d_1, d_2, \dots, d_s, \dots\}$$

is finite and consists of, say, k elements, then for all $s > k$ we define

$$A_s^n = d_k. \tag{2.7}$$

From the relations (2.6) it follows that if the free product

$$F = \prod_{i \in I}^{*} G_i$$

is an m-generated group, then the group $A_{\mathcal{D}}(G)$ also is an m-generated group. For the groups $A_{\mathcal{D}}(G)$ in Section 3 we will prove the following main theorem.

Theorem 2.1 *For any abelian group* $\mathcal{D}$ *with representation* (2.1) *and for a group* $A_{\mathcal{D}}(G)$ *with defining relations* (2.4)–(2.7) *we have:*

(1) *the center of the group* $A_{\mathcal{D}}(G)$ *coincides with* $\mathcal{D}$*;*

(2) *the quotient of* $A_{\mathcal{D}}(G)$ *by* $\mathcal{D}$ *is the* n*-periodic product of the given family of groups* $\{G_i\}_{i\in I}$ *for any odd* $n \geqslant 665$.

Finally, we draw the reader's attention to a certain freedom in constructing the groups $A_{\mathcal{D}}(G)$. First, the numbering order of the elementary periods $A_j \in \mathcal{E}$, $j \in \mathbb{N}$ is free. Secondly, we also have freedom for the choosing of an Abelian group $\mathcal{D}$ in (2.1). Thus, by virtue of defining relations (2.6), for a fixed group $\mathcal{D}$ we can obtain different groups $A_{\mathcal{D}}(G)$. Moreover, Theorem 2.1 holds for each of them.

3 Proof of Theorem 1.1 and Theorem 2.1

To justify Theorem 1.1 it is obviously sufficient to prove Theorem 2.1.

For words in the alphabet (2.2)–(2.3) of the group $A_{\mathcal{D}}(G)$ we construct generalized analogues of the concepts that were constructed and studied in Chapters I–V of the monograph [4].

We denote by $\mathcal{R}_\alpha^D$ the set of all words of the form Qd, where d belongs to $\mathcal{D}$, Q is a word in the alphabet (2.2) belonging to the set $\mathcal{R}_\alpha$, and the group $\mathcal{D}$ has the presentation (2.1). Similarly,

$$\mathcal{N}_\alpha^D, \mathcal{P}_\alpha^D, \mathcal{K}_\alpha^D, \mathcal{L}_\alpha^D, \mathcal{M}_\alpha^D, \overline{\mathcal{M}}_\alpha^D, \mathcal{A}_\alpha^D$$

denote the set of words of the form Qd, where $d \in \mathcal{D}$, and Q belongs, respectively, to the set $\mathcal{N}_\alpha$, $\mathcal{P}_\alpha$, $\mathcal{K}_\alpha$, $\mathcal{L}_\alpha$, $\mathcal{M}_\alpha$, $\overline{\mathcal{M}}_\alpha$, $\mathcal{A}_\alpha$.

We will consider only such occurrences in words Qd whose bases are contained in Q, i.e. are words in the alphabet (2.2). The concepts of periodic, integer, half-integer and elementary word of rank α, generating occurrences of rank α, support kernel of rank α and all the concepts that were defined in [4, Chapter I, Paragraphs 4.3–4.10] do not change. The set of elementary periods is defined exactly as they were defined in [2] for a given n-periodic product of the family $\{G_i\}_{i\in I}$, and we denote by $\mathcal{E}$ the set of elementary periods for which the conditions of [4], Chapter VI, Section 2.1, hold. In particular, by simultaneous induction on the rank α we define on the set $\mathcal{R}_\alpha^{\mathcal{D}}$

the relation $\stackrel{\alpha}{\sim}$ of generalized equivalence of rank α and the generalized coupling operation $[X, Y]_\alpha^{\mathcal{D}}$ of rank α. At the same time, similarly to relations (12) and (13) from [10], it is proved that the relation $\stackrel{\alpha}{\sim}$ satisfies the relations

$$P \stackrel{\alpha}{\sim} Q \Leftrightarrow \exists d \,\forall d' \left(Pd' \stackrel{\alpha}{\sim} Q(dd')\right), \tag{3.1}$$

and

$$Qd \stackrel{\alpha}{\sim} Qd' \Rightarrow d = d' \text{ in } \mathcal{D}, \tag{3.2}$$

where P, Q are words in the alphabet (2.2), $d, d' \in \mathcal{D}$, (dd') is the product of d and d' in the Abelian group $\mathcal{D}$.

On this basis, we construct an auxiliary group $\Gamma^{\mathcal{D}}(G, \alpha)$ whose elements are the equivalence classes on which the set $\mathcal{R}_\alpha^{\mathcal{D}}$ is decomposed by the equivalence relation $\stackrel{\alpha}{\sim}$, and the group operation coincides with the coupling operation of rank α. By analogy with items 1.4 and 1.5 of Chapter VI of [4], we check that $\Gamma^{\mathcal{D}}(G, \alpha)$ is a group with respect to the specified operation. Let us describe this group using the generators and defining relations. To do this, denote by $A_{\mathcal{D}}(G, \alpha)$ the group with generators $g_{ij}, d_1, d_2, \dots, d_s, \dots$ $(i \in I, j \in K_i)$ and a system of defining relations of the types (2.4), (2.5) and (2.6) for all $A_s \in \bigcup_{t=1}^{\alpha} \mathcal{E}_t$.

Lemma 3.1 *For any words $X, Y \in \mathcal{R}_\alpha^{\mathcal{D}}$ the following assertions are equivalent:*

(1) $X = Y$ *in* $A_{\mathcal{D}}(G, \alpha)$.

(2) $X \stackrel{\alpha}{\sim} Y$.

(3) $X = Y$ *in* $\Gamma^{\mathcal{D}}(G, \alpha)$.

This lemma is proved similarly to Lemma 3 in [10]. All that is needed in its proof is to replace the group $A_{\mathcal{D}}(m, n, \alpha)$ by $A_{\mathcal{D}}(G, \alpha)$, and to replace the group $\Gamma^{\mathcal{D}}(m, n, \alpha)$ by $\Gamma^{\mathcal{D}}(G, \alpha)$.

Denote by $\mathcal{Z}$ the subgroup of the group $A_{\mathcal{D}}(G)$ generated by the elements $\{d_j \mid j \in \mathbb{N}\}$.

Lemma 3.2 *The subgroup $\mathcal{Z}$ coincides with the center of the group $A_{\mathcal{D}}(G)$, and the quotient group $A_{\mathcal{D}}(G)/\mathcal{Z}$ is isomorphic to the given n-periodic product G.*

Proof — By virtue of the relations (2.5), $\mathcal{Z}$ is contained in the center of the group $A_{\mathcal{D}}(G)$. According to the relations (2.6) and (2.7), the quotient of the group $A_{\mathcal{D}}(G)$ by the subgroup $\mathcal{Z}$ is a given n-periodic product G. Moreover, by Theorem 8 from [2], the center of any n-periodic product is trivial. Therefore, the group $G = A_{\mathcal{D}}(G)/\mathcal{Z}$ has a trivial center, so $\mathcal{Z}$ coincides with the center of the group $A_{\mathcal{D}}(G)$. □

Thus, according to Lemma 3.2, the subgroup $\mathcal{Z}$ generated by the elements $\{d_j \,|\, j \in \mathbb{N}\}$ coincides with the center of the group $A_{\mathcal{D}}(G)$, while the quotient group $A_{\mathcal{D}}(G)/\mathcal{Z}$ is isomorphic to the group G. Therefore, Theorem 2.1 will be proved if we show that the Abelian group $\mathcal{D}$ with the same generators $\{d_j \,|\, j \in \mathbb{N}\}$ is embedded in the group $A_{\mathcal{D}}(G)$ and thus coincides with $\mathcal{Z}$.

First, we make sure that $\mathcal{D}$ is embedded in the group $\Gamma^{\mathcal{D}}(G,\alpha)$ for any rank α. Suppose that for some elements $d', d'' \in \mathcal{D}$ the equivalence $d' \overset{\alpha}{\sim} d''$ takes place. Then $d' \overset{\gamma}{\sim} d''$ for any rank of $\gamma \geqslant \alpha$ since, by definition, $d', d'' \in \mathcal{R}^{\mathcal{D}}_{\gamma}$. Hence, due to the relation (3.2), we immediately obtain that $d' = d''$ in the Abelian group $\mathcal{D}$, that is, $\mathcal{D}$ is embedded in the group $\Gamma^{\mathcal{D}}(G,\gamma)$. Hence, by Lemma 3.1, it follows that the Abelian group $\mathcal{D}$ is embedded in the group $A_{\mathcal{D}}(G,\gamma)$ for any rank $\gamma \geqslant \alpha$. Since the set of defining relations (2.4)–(2.7) of the group $A_{\mathcal{D}}(G)$ is the union of the sets of defining relations of the groups $A_{\mathcal{D}}(G,\gamma)$ for all $\gamma \geqslant \alpha$, then $\mathcal{D}$ also embedded in the group $A_{\mathcal{D}}(G)$. Theorem 2.1 is proved.

4 Proof of Theorem 1.3

Consider the periodic product of an arbitrary family of groups $\{G_i\}_{i\in I}$ of period p, where p and $n \geqslant 665$ are coprime odd numbers. According to the simplicity criterion for an n-periodic product (see Theorem 1 in [3]), the periodic product is a simple group if and only if $G_i^n = G_i$ for each factor G_i of product. Hence, $G = \prod_{i\in I}^{n} G_i$ is a simple group because p and n are coprime. Let us construct the group $A_{\mathcal{D}}(G)$ by choosing the additive group of rational numbers as the Abelian group D:

$$\mathbb{Q} = \langle d_1, d_2 \dots, d_i, \dots \mid d_i^i = d_{i-1}, i \geqslant 2 \rangle. \tag{4.1}$$

By Theorem 2.1, the center of the group $A_{\mathbb{Q}}(G)$ coincides with $\mathbb{Q}$, and the quotient group by the subgroup $\mathbb{Q}$ coincides with the simple group G. Since the subgroup $\mathbb{Q}$ is torsion-free, the group $A_{\mathbb{Q}}(G)$ is not periodic. Moreover, since the quotient group $A_{\mathbb{Q}}(G)/\mathbb{Q} = G$ is simple, $\mathbb{Q}$ is a maximal normal subgroup of $A_{\mathbb{Q}}(G)$ containing all its proper normal subgroups. Assume that N is some non-trivial normal subgroup. Then N is contained in $\mathbb{Q}$, and the quotient group $\mathbb{Q}/N$ is periodic. On the other hand, any element x of the group $A_{\mathbb{Q}}(G)$ can be represented by gq, where g is a word in the generators (2.2) and $q \in \mathbb{Q}$. By virtue of Theorem 7 [2], in the quotient group $A_{\mathbb{Q}}(G)/\mathbb{Q}$ the element g is either conjugate to some element of one of the factors $\{G_i\}_{i\in I}$, or conjugate to a degree of some element of order n. In the first case, in the quotient group $A_{\mathbb{Q}}(G)/\mathbb{Q}$, the element g has order p. In any case, we obtain that in the group $A_{\mathbb{Q}}(G)$ the element g^{pn} belongs in $\mathbb{Q}$. Then

$$x^{pn} = (gq)^{pn} = g^{pn}q^{pn} \in \mathbb{Q}$$

(since q is from the center of $A_{\mathbb{Q}}(G)$). This means that the pn-th power of every element from the quotient $A_{\mathbb{Q}}(G)/N$ belongs in the subgroup $\mathbb{Q}/N$. Obviously, the group $\mathbb{Q}/N$ is periodic. Thus, firstly, the quotient group $A_{\mathbb{Q}}(G)/N$ is a periodic group, and secondly, the groups $A_{\mathbb{Q}}(G)/N$ and $A_{\mathbb{Q}}(G)$ are not isomorphic ($A_{\mathbb{Q}}(G)$ is not periodic). Hence the group $A_{\mathbb{Q}}(G)$ is Hopfian, and all its proper quotients are infinite periodic groups. Finally, the group $A_{\mathbb{Q}}(G)$ is residually periodic because its unique maximal normal subgroup $\mathbb{Q}$ is residually periodic and $A_{\mathbb{Q}}(G)/\mathbb{Q}$ is periodic.

Now we point out an explicit system of elementary periods $\{A_s\}_{s\in\mathbb{N}}$, which participate in the relations (2.6). First of all, to simplify the reasoning, we modify the definition (4.1) of the group $\mathbb{Q}$ by "doubling" the system of generators:

$$\mathbb{Q} = \langle d_1, d_2 \dots, d_i, \dots \mid d_{2(i+1)}^i = d_{2i},\ d_{2i+1} = d_{2i}, i \geqslant 1\rangle. \quad (4.2)$$

It is clear that generating elements with even indices also generate the entire group of rational numbers. Consider two copies G_1, G_2 of the cyclic group of an arbitrary fixed simple period $p > 2$ generated by a and b respectively. Let $G = G_1 *^n G_2$ be the n-periodic product of these groups. In the alphabet a, b one can construct an infinite word that does not contain the cube of any non-empty word (see, for example, [25]). Denote the initial segments of such a word of length k by $C_k = C_k(a, b)$. It follows from the definition of elementary periods (see [4], Chapter I, item 4.10) that if a cyclically irreducible word does not contain the cube of any word, then it is an elementary period of rank 1. In particular, all words C_k are elementary periods of rank 1. So, $C_k \in \mathcal{E}_1 \subset \mathcal{E}$ and the words C_k satisfy the conditions of [4], Chapter VI, Section 2.1 (in this case, we only need to verify that the words C_k are not conjugate in the free product of the groups $G_1 * G_2$). The elements of the set $\mathcal{E}$ are numbered as follows: $A_{2k} = C_k$ for $k = 1, 2, \ldots$, and the remaining elementary periods are numbered arbitrarily with odd indices. Constructing the group $A_{\mathbb{Q}}(G_1 *^n G_2)$ with the system of defining relations (2.4)–(2.7), where the group $\mathbb{Q}$ and the set $\mathcal{E}$ are defined as above, we obtain an explicit embedding of the group of rational numbers in the 2-generated group $A_{\mathbb{Q}}(G_1 *^n G_2)$ with the properties indicated in Theorem 1.3.

Theorem 1.3 is proved.

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