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Locally Maximal Subgroups and the Normalizer Condition in p-Groups

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Abstract

This work is a continuation of the investigation of a locally nilpotent p-group satisfying the normalizer condition by imposing certain conditions on locally maximal subgroups, where $p \neq 2$. A sufficient condition is obtained for making every abelian-by-elementary abelian normal subgroup of such a group to be abelian. If in addition the group in question is hyperabelian, then it is abelian, where $p \geqslant 5$. In the general case if a locally nilpotent p-group satisfies the mentioned condition above ($p \neq 2$), then it contains a unique maximal normal abelian subgroup.

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1 Introduction

This is a continuation of the study of a locally nilpotent p-group G which satisfies the normalizer condition and/or is a Fitting group in order to search for imperfectness conditions and to obtain some information about its inner structure when G is perfect (see [1, 2, 3, 5]). In [2, 3], it was shown that if G is a Fitting p-group satisfying the normalizer condition and if in every homomorphic image of G certain (w, V)-maximal subgroups satisfy the $(**)$-condition (see below for definitions), then, under certain conditions, G cannot be perfect (see [2, Theorem 1.1] and [3, Theorem 1.1]). Now it follows from Theorem 1.5 (see below) that a group G satisfying the hypotheses of these theorems is actually abelian. In this work it is shown that in a locally nilpotent p-group satisfying the normalizer condition only and whose locally maximal subgroups have large normalizers (see definitions below), every normal abelian-by-elementary abelian subgroup is abelian, where $p \geqslant 5$ (see Theorem 1.1 (b)). If in addition G is hyperabelian, then it is abelian (see Theorem 1.3 and Corollary 1.4). In the general case ($p \neq 2$), G contains a unique maximal normal abelian subgroup (see Theorem 1.1 (a)).

But before stating the main results it will be suitable to recall some of the definitions and notations given in [2] and [3] since they form the basis of this work. Let G be a group, $w \in G \setminus 1$ and V be a finitely generated subgroup of G with $w \notin V$. Then the ordered pair (w, V) is called a $(*)$-*pair* in G (note that in [2, 3]; that is in the definition of $\Lambda(w, V)$, $w \in G \setminus Z(G)$ but in the present definition there is no such a restriction on w, the only restriction is that $w \neq 1$). A subgroup E of G which is maximal with respect to the condition that

$$w \notin E \text{ but } V \leqslant E$$

is called (w, V)-*maximal* or *maximal at* (w, V) and if (w, V) is not mentioned, then it is called *locally maximal*. In addition let the following be defined.

$$E^*(w, V) = \{E : E \text{ is a } (w, V)- \text{ maximal subgroup of } G\}$$

and

$$W^*(w, V) = \{\mathrm{Core}_G(E) : E \in E^*(w, V)\}.$$

An element E of $E^*(w, V)$ is said to satisfy the $(**)$-*property*, if

$$N_G(E) = N_G(E')$$

and (w, V) is said to satisfy $(**)$ if every element of $E^*(w, V)$ satisfies it. On the other hand if

$$N_G\big(EC_G(E)\big) \leqslant N_G(E),$$

then E is said to have a *large normalizer*.

Obviously $EC_G(E) \leqslant N_G(E) \leqslant N_G\big(EC_G(E)\big)$. So if $N_G(E)$ is large, then $N_G(E) = N_G\big(EC_G(E)\big)$. Put $N = N_G(E)$. Now if G satisfies the normalizer condition, N is large and $N \neq G$, then $EC_G(E) \neq N$. Indeed if $EC_G(E) = N$, then $N_G(N) = N_G(EC_G(E)) = N_G(E) = N$, which cannot happen by the normalizer condition. This fact will be used without further notice.

Furthermore if E satisfies $(**)$, then N is large (see Lemma 4.1), which shows that the first property is stronger than the second one.

In a locally nilpotent group G a locally maximal subgroup E behaves similar to a maximal subgroup M of G (if M exists), since $M \lhd G$ and so G/M is cyclic of order p. Also $N_G(E)/E$ is (locally) cyclic by Lemma 2.1, provided $p \neq 2$. Moreover,

$$N_G(M) = G = N_G(MC_G(M))$$

and so $N_G(M)$ is large. Every subgroup of a Dedekind group satisfies $(**)$ since in this group every subgroup is normal and if it has odd exponent, then it is abelian by [12, 5.3.7].

Again let (w, V) be a $(*)$-pair in G. If there exists a proper subgroup L of G such that

$$w \in \langle V, y \rangle \text{ for every } y \in G \setminus L,$$

then (w, V, L) is called a $(*)$-*triple* in G. This situation occurs when $\langle E^*(w, V) \rangle \neq G$. In this case L can be any proper subgroup of G containing $\langle E^*(w, V) \rangle$. This case was studied in [5]. By means of it a new characterization of a barely transitive p-group was given (see [5, Theorem 1.2 (a)]). Furthermore G cannot be generated by normal abelian subgroups (see [1, Lemma 2.2]); as was shown in [5], if G is minimal non-hypercentral or barely transitive, then $(*)$-triples exist. Thus it follows that if either $E^*(w, V)$ contains locally maximal subgroups whose normalizers are large or $\langle E^*(w, V) \rangle \neq G$, then G cannot be generated by normal abelian subgroups (this author knows of no perfect locally nilpotent p-group other than McLain's characteristically simple group $M(\mathbb{Q}, F)$ [12, 12.1.9] which can be generated by normal abelian subgroups).

As usual if a group G is solvable (nilpotent), then its derived length (nilpotent class) is denoted by $d(G)$ $(c(G))$. If $d(G) = 2$, then G is called *metabelian*. Also $\exp(G) = \max\{|g| : g \in G\}$ is called the *exponent* of G. It may be expressed as $\exp(G) < \infty$ or $\exp(G) = \infty$ according as it is finite or infinite, respectively. A group is called *hyperabelian* if it has an ascending normal series with abelian factors (see [12, p.365]).

Definitions and notations are standard and may be found in [7, 8, 9, 10, 11, 12].

Theorem 1.1 *Let* G *be a locally finite* p*-group satisfying the normalizer condition, where* $p \neq 2$*. Suppose that in every homomorphic image* H *of* G *every* $(*)$*-pair* (w_H, V_H) *satisfying* $W^*(w, V) = 1$ *has a* (w_H, V_H)*-maximal subgroup whose normalizer is large. Then the following hold.*

(a) *Every abelian-by-elementary abelian normal nilpotent subgroup of* G *is abelian. In particular* G *contains a unique maximal normal abelian subgroup.*

(b) *If* $p \geqslant 5$*, then every abelian-by-elementary abelian normal subgroup of* G *is abelian.*

Theorem 1.2 *Let* G *be a solvable* p*-group satisfying the normalizer condition, where* $p \geqslant 5$*. Suppose that in every homomorphic image* H *of* G *every* $(*)$*-pair* (w_H, V_H) *satisfying* $W^*(w, V) = 1$ *has a* (w_H, V_H)*-maximal subgroup whose normalizer is large. Then* G *is abelian.*

Theorem 1.3 *Let* G *be an hyperabelian* p*-group satisfying the normalizer condition, where* $p \geqslant 5$*. Suppose that in every homomorphic image* H *of* G *every* $(*)$*-pair* (w_H, V_H) *satisfying* $W^*(w, V) = 1$ *has a* (w_H, V_H)*-maximal subgroup whose normalizer is large. Then* G *is abelian.*

Corollary 1.4 *Let* G *be a locally finite* p*-group satisfying the normalizer condition, where* $p \geqslant 5$*. Suppose that every proper normal subgroup of* G *is solvable and in every homomorphic image* H *of* G *every* $(*)$*-pair* (w_H, V_H) *satisfying* $W^*(w, V) = 1$ *has a* (w_H, V_H)*-maximal subgroup whose normalizer is large. Then* G *is abelian.*

Proof — Assume that G is not abelian. Then also G is not solvable by Theorem 1.2 and so G is perfect. Then G has an ascending normal series

$$1 = M_0 \lhd M_1 \lhd \ldots M_\alpha \lhd M_{\alpha+1} \lhd \ldots M_\lambda = G$$

with $G = \bigcup_{\alpha<\lambda} M_\alpha$ since a minimal normal subgroup of G is abelian. But since $M_{\alpha+1}/M_\alpha$ is solvable for every $\alpha < \lambda$, the above series can be refined into an ascending normal series whose factors are abelian and so it follows that G is hyperabelian. But now since G must be abelian by Theorem 1.3, we get a contradiction and so G must be abelian. □

The following is a complete characterization of the group given in [3, Theorem 1.1].

Theorem 1.5 *Let* G *be a Fitting* p*-group satisfying the normalizer condition, where* $p \neq 2$*. Suppose that in every homomorphic image* H *of* G *every* $(*)$*-pair* (w_H, V_H) *has a* (w_H, V_H)*-maximal subgroup satisfying* $(**)$*. Then* G *is abelian.*

In the following simple example for $p = 3$, the hypothesis of Theorem 1.2 is not satisfied.

Example Let $A = C_3$ and $C = C_{3^\infty}$ be a cyclic group of order 3 and a locally cyclic 3-group respectively and let $G = C \text{ wr } A$ be the standard wreath product. Then $G = [C]A$, where $[C]$ is the base group. Let

$$A = \langle a \rangle \quad \text{and} \quad C = \langle c_i : c_0 = 1 \text{ and } c_{i+1}^3 = c_i \text{ for every } i \geqslant 0 \rangle.$$

Thus $[C] = C_0 \times C_1 \times C_2$, where $C_i = C_{a^i} = C$ for $i = 0, 1, 2$. Each $f \in [C]$ is a function $f : A \to C$ with $f(a^i) \in C_i$, which is the i^{th} component of f. $[C]$ is an abelian group under point-wise multiplication; that is, for $f, g \in [C]$ and $y \in A$, $fg(y) = f(y)g(y)$. We define an action of A on $[C]$ as follows.

$$f^y(a) = f\big(ay^{-1}\big) \text{ for every } a, y \in A.$$

For example if $f = (c_0, c_1, c_2)$, then

$$f^a(1) = f\big(a^2\big) = c_2,\ f^a(a) = f(1) = c_0,\ f^a\big(a^2\big) = f(a) = c_1.$$

Thus $f^a = (c_2, c_0, c_1)$ an so every entry of f is moved one step to the right.

The correspondence $f \to f^y$ defines an automorphism of $[C]$; that is $(fg)^y = f^y g^y$ and so a monomorphism of A into $\text{Aut}\big([C]\big)$, which we identify with A. The semidirect product of $[C]$ with A is called the wreath product of C with A and denoted by $C \text{ wr } A = [C]A$.

Let $f, g \in [C]$ and $a, b \in A$. Then

$$(f, a)(g, b) = \big(fg^{a^{-1}}, ab\big).$$

In particular

$$(f, a)^{-1} = \big((f^{-1})^a, a^{-1}\big).$$

If we identify $(f, 1)$ and $(1, a)$ with f and a respectively, then $(f, a) = (f, 1)(1, a)$ becomes fa.

Let $w = (c_1, c_1, c_1)$ and consider the $(*)$-pair $(w, 1)$, where $|c_1| = p$. Let $E \in E^*(w, 1)$ and let $N = N_G(E), L = C_G(E)$. First suppose $E \leqslant [C]$. Then $L = [C]$ since $w \notin E$ and a cannot centralize any subgroup $\neq 1$ of $[C]$ not containing w. In this case $N = L$ and $\mathrm{Core}_G(E) = 1$.

Next suppose that $ga \in E$ for some $g \in [C]$. Then since $G = [C]\langle ga \rangle$, it follows that $E = ([C] \cap E)\langle ga \rangle = \langle ga \rangle$ since a cannot normalize any subgroup of $[C]$. Thus $\langle ga \rangle\langle w \rangle \leqslant L$. Since $N = (N \cap [C])\langle fa \rangle \leqslant L$, it follows again that $N = L$. Also obviously $\mathrm{Core}_G(E) = 1$. Thus we see that $W^*(w, 1) = 1$ but $(w, 1)$ cannot satisfy the hypothesis of Theorem 1.2.

2 First properties of G given in Theorem 1.1

Lemma 2.1 *Let G be a locally finite p-group and let (w, V) be a $(*)$-pair in G, where $w \in G \setminus 1$. Let $E \in E^*(w, V)$. Then $N_G(E)/E$ is either (locally) cyclic or $p = 2$ and isomorphic to a (locally) quaternion group.*

Proof — Put $N = N_G(E)$ and define $\overline{N} = N/E$. Let $\overline{A}$ be a finite abelian subgroup of $\overline{N}$. Assume if possible that $\overline{A}$ is not cyclic. Then $\overline{A}$ contains an elementary abelian subgroup $\langle \overline{a} \rangle \times \langle \overline{b} \rangle$. But since E is (w, V)-maximal, we must have $w \in \langle a \rangle E$ and $w \in \langle b \rangle E$. Hence $w \in \langle a \rangle E \cap \langle b \rangle E = E$, but this is impossible since $w \notin E$. Therefore every finite abelian subgroup of $\overline{N}$ is cyclic. In this case every finite subgroup of $\overline{N}$ is cyclic or isomorphic to a generalized quaternion group by [7, Theorem 5.4.10 (ii)]. Therefore either $\overline{N}$ is (locally) cyclic or isomorphic to a 2-group which is isomorphic to a (locally) quaternion group. □

Lemma 2.2 *Let G be an infinitely generated locally nilpotent group and let (w, V) be a $(*)$-pair in G, where $w \in G \setminus 1$. If $W^*(w, V) = 1$, then the following hold.*

(a) $Z(G) \neq 1$.

(b) *Let $E \in E^*(w, V)$ and put $N = N_G(E)$. Suppose that N/E is (locally) cyclic. If $N \lhd G$, then it is abelian. If in addition G satisfies the normalizer condition and N is large, then G is (locally) cyclic and $E = 1$. In particular if $w \notin Z(G)$, then $N \ntriangleleft G$.*

Proof — (a) Assume if possible that $Z(G) = 1$. Now G contains a proper normal subgroup $N \neq 1$ since a minimal normal subgroup of G is contained in $Z(G)$ by [12, 12.1.6]. Let $Q = \{1 < L < N : L \lhd G\}$. Let Q be partially ordered by saying that if for $L_1, L_2 \in Q$, $L_1 \geqslant L_2$, then $L_1 \preccurlyeq L_2$. Then it is easy to check that $(Q, \preccurlyeq)$ is a partially ordered set. Assume if possible that Q has a maximal element L_0. Then since $L_0 \leqslant L$ for every $L \in Q$ which is comparable with L_0, it follows that L is a minimal normal subgroup of G and so $L_0 \leqslant Z(G)$. But since $L_0 \neq 1$ and $Z(G) = 1$ this is a contradiction. Therefore Q cannot have a maximal element. Therefore there exists a chain

$$L_1 \preccurlyeq L_2 \preccurlyeq \dots L_\alpha \preccurlyeq \dots$$

of elements of Q whose upper bound does not belong to Q by Zorn's Lemma. Since this upper bound is $\bigcap_{\alpha \geqslant 1} L_\alpha$, it must be equal to the trivial group 1. Now if $w \in VL_\alpha$ for all $\alpha \geqslant 1$, then there exists a $v_1 \in V$ and a $\beta \geqslant 1$ so that $v_1^{-1}w \in L_\alpha$ for all $\alpha \geqslant \beta$ since V is finite. Then since $v_1^{-1}w = 1$, it follows that $w = v_1$, which is a contradiction since $w \notin V$. Therefore there exists an $\alpha \geqslant 1$ so that $w \notin VL_\alpha$. Clearly, then there exists an $E \in E^*(w, V)$ such that $VL_\alpha \leqslant E$. But since $1 \neq L_\alpha \lhd G$ and $W^*(w, V) = 1$, this is a contradiction. Therefore the assumption is false and so $Z(G) \neq 1$.

(b) Suppose that $N \lhd G$. Since N/E^g is (locally) cyclic for every g in G, there is natural homomorphism

$$N \to \prod (N/E^g)_{g \in G}$$

given by $y \to (yE^g)_{g \in G}$ with kernel $E^* = \bigcap_{g \in G} E^g$. Hence it follows that N/E^* is abelian. Since $W^*(w, V) = 1$ by hypothesis, $E^* = 1$ and therefore N is abelian.

Now suppose that N is large and satisfies the normalizer condition. Then $N = N_G(EC_G(E))$. Also $N = EC_G(E)$ since N is abelian. But since G satisfies the normalizer condition, this is possible only if $N = G$ and so G is abelian. Then $E = 1$ since $E^* = 1$ and so $N/E = N$ is (locally) cyclic. The last assertion is a trivial consequence of the first one. □

Lemma 2.3 *Let* G *be a locally finite* p*-group and let* (w, V) *be a* $(*)$*-pair in* G *such that* $W^*(w, V) = 1$*, where* $w \in G \setminus 1$*. Assume that there exists an* $E \in E^*(w, V)$ *having a large normalizer. If* $EC_G(E)/E$ *is infinite, then* $N_G(E)$ *is self-normalizing. In particular if* G *satisfies the normalizer condition, then* G *is locally cyclic and* $E = 1$*.*

Proof — Put $N = N_G(E)$. Assume that $EC_G(E)/E$ is infinite, then $N = EC_G(E)$ since N/E is (locally) cyclic by Lemma 2.1. Hence

$$N_G(N) = N_G\big(EC_G(E)\big) = N_G(E) = N$$

since N is large and so $N = N_G(N)$, which means that N is self-normalizing. Now if G satisfies the normalizer condition, then this is possible only if $E = 1$ and G is (locally) cyclic by Lemma 2.2 (b). □

Lemma 2.4 *Let G be a locally finite p-group and (w, V) be a $(*)$-pair in G, where $w \in G \setminus 1$. Let $E \in E^*(w, V)$ and put $N = N_G(E)$. Suppose that N/E is (locally) cyclic. Let A be a normal abelian subgroup of G with $Z(G) \leqslant A$. Let $R = N \cap A$ and $D = R \cap E$. Then the following hold.*

(a) *Let $t \in G$ and $U \leqslant Z(G)$. If t normalizes UE, then t normalizes $EC_G(E)$.*

(b) *Suppose that $W^*(w, V) = 1$. Let $L = N_G(EC_G(E)$. Let $a \in A \setminus N$ with $N^a = N$. If $a^p \in R$, then a normalizes $EC_G(E)$ and so $a \in L$. In particular if N is large then $A \cap N_G(N) \leqslant N$.*

Proof — If G is abelian, then there is nothing to prove. Therefore in both cases we may suppose that G is not abelian.

(a) Assume that t normalizes UE. Let $C = C_G(E)$. Then $C = C_G(UE)$ since $U \leqslant Z(G)$. Since t normalizes UE, it must also normalize its centralizer C. Clearly then t normalizes CE since $U \leqslant C$ and so (a) is verified.

(b) Suppose that $W^*(w, V) = 1$. Then $Z(G) \cap E = 1$ since $\mathrm{Core}_G(E) = 1$ but also $Z(G) \neq 1$ by Lemma 2.2 (a). Therefore $\Omega_1(R) \leqslant \langle z \rangle D$ for some $z \in Z(G)$ with $|z| = p$ since N/E is (locally) cyclic. Assume that $a \in A \setminus N$ with $N^a = N$ and $a^p \in R$. Put $H = \langle a \rangle D$ and $\overline{H} = H/D$. Since N/E is (locally) cyclic, $[R, E] \leqslant D$ and so $[\overline{R}, \overline{E}] \leqslant \overline{D} = 1$. Hence

$$1 = [\overline{a}^p, \overline{E}] = [\overline{a}, \overline{E}]^p$$

by [7, Lemma 2.2.2 (i)] since $\overline{a}^p \in \overline{R}$, $[\overline{R}, \overline{E}] = 1$, $a \in A$, $[\overline{a}, \overline{E}] \leqslant \overline{A}$ and A is abelian. Thus $[\overline{a}, \overline{E}]$ has order $\leqslant p$ and so is contained in $\langle \overline{z} \rangle \overline{D}$ since $[\overline{a}, \overline{E}] \leqslant N$. Clearly then $[a, E] \leqslant \langle z \rangle E$ since $D \leqslant E$ and so a normalizes $\langle z \rangle E$. Then since a normalizes $EC_G(E)$ by (a), it follows that $a \in L$. The last assertion follows from the first one since N is large means $N = L$. □

Lemma 2.5 *Let G be a locally finite p-group, (w, V) be a $(*)$-pair in G, where $w \in G \setminus 1$. Suppose that $W^*(w, V) = 1$ and let $E \in E^*(w, V)$.*

Let B *be a normal abelian-by-elementary abelian subgroup of* G *and* A *be a normal abelian subgroup of* G *contained in* B *such that* B/A *is elementary abelian and* $Z(G) \leqslant A$*. Put* $N = N_G(E)$, $R = N \cap B$, $D = R \cap E$ *and suppose that* N/E *is (locally) cyclic. Furthermore suppose that there exists a* $t \in B \setminus N$ *with* $N^t = N$ *and* $t^p \in N$*. Put* $T = \langle t \rangle R$, $H = TN$ *and* $D^* = \mathrm{Core}_H(D)$*. Then the following hold.*

(a) R/D *is (locally) cyclic and*

$$R/D^* \leqslant Z(N/D^*).$$

Also $Z(G) \neq 1$ *and* $Z(G) \cap E = 1$*. Therefore* $\Omega_1(R/D^*) \leqslant \langle z \rangle D/D^*$*, where* $\langle z \rangle$ *is the unique subgroup of order* p *in* $Z(G)$*.*

(b) *Suppose that* N *is large. Then*

$$C_{T/D^*}(R/D^*) = R/D^* \text{ and so } C_{H/D^*}(R/D^*) = N/D^*.$$

Thus $Z(T/D^*) \leqslant R/D^*$ *and* $Z(T/D^*) \cap E/D^* = 1$ *and so is (locally) cyclic.*

Proof — Clearly B is not abelian by Lemma 2.4 (b) by the choice of t. Now $T = B \cap H$ and so $T \lhd H$. Then also $R \lhd H$ since $R = T \cap N$ and $N \lhd H$. Also $D \lhd N$ since $E \lhd N$. Put $\overline{H} = H/D^*$.

(a) Obviously R/D is (locally) cyclic since N/E has this property. $[R, N]$ is normal in H since $R, N \lhd H$ and is contained in E since N/E is (locally) cyclic. Clearly then $[R, N] \leqslant D^*$ and so $[\overline{R}, \overline{N}] = 1$, which implies that $\overline{R} \leqslant Z(\overline{N})$.

Next $Z(G) \neq 1$ by Lemma 2.2 (a) and $Z(G) \cap E = 1$ since $\mathrm{Core}_G(E) = 1$. Therefore if $z \in Z(G)$ with $|z| = p$, then $\Omega_1(\overline{R}) \leqslant \langle \overline{z} \rangle \overline{D}$ since R/D is (locally) cyclic.

(b) Now suppose that N is large. Assume if possible that $[\overline{t}, \overline{R}] = 1$. Then

$$1 = [\overline{t}^p, \overline{N}] = [\overline{t}, \overline{N}]^p$$

since $t^p \in R$ and $\overline{R} \leqslant Z(\overline{N})$. Therefore $[\overline{t}, \overline{N}]$ is a subgroup of order $\leqslant p$ of $\overline{R}$. Clearly then $[\overline{t}, \overline{N}] \leqslant \Omega_1(\overline{R}) \leqslant \langle \overline{z} \rangle \overline{D} \leqslant \langle \overline{z} \rangle \overline{E}$ by (a) and thus t normalizes $\langle z \rangle E$. But then since t normalizes $EC_G(E)$ by Lemma 2.4 (a) and N is large we have $t \in N$, which is a contradiction. Therefore $C_{\overline{T}}(\overline{R}) = \overline{R}$. Since $\overline{R} \leqslant Z(\overline{N})$, it follows that $C_{\overline{H}}(\overline{R}) = \overline{N}$. In particular now $Z(\overline{T}) \leqslant \overline{N}$. Then also $Z(\overline{T}) \cap \overline{D} = 1$ and so $Z(\overline{T})$ is (locally) cyclic since $Z(\overline{T}) \cap \overline{D} \lhd \overline{H}$ and so is trivial by definition of D^*. □

Lemma 2.6 (see [3], Lemma 2.7) *Let G be a locally finite p-group and let (w, V) be a $(*)$-pair in G such that $W^*(w, V) = 1$, where $w \in G \setminus 1$. Assume that there exists an $E \in E^*(w, V)$ such that $N_G(E)/E$ is (locally) cyclic and $N_G(E)$ is large. Furthermore let B be a normal nilpotent subgroup of G with $c(B) < p$ and A be a normal abelian subgroup of G contained in B with that B/A is elementary abelian and $Z(G) \leqslant A$. If $B \cap N_G(N_G(E)) \setminus N_G(E) \neq 1$ whenever $B \nleqslant N_G(E)$, then B is abelian.*

Proof — Assume that B is not abelian. Then $B \nleqslant N_G(E)$. For if $B \leqslant N_G(E)$, then $B' \leqslant E$ since N/E is (locally) cyclic by Lemma 2.1. But then since $\mathrm{Core}_G(E) = 1$, we must have $B' = 1$, which is a contradiction. Therefore there exists a $t \in B \setminus N_G(E)$ with $N_G(E)^t = N_G(E)$ and $t^p \in N_G(E)$. As before put

$$N = N_G(E), R = N \cap B, D = R \cap E \text{ and } T = \langle t \rangle R$$

and $H = TN$. Let $D^* = \mathrm{Core}_H(D)$ and put $\overline{H} = H/D^*$. Then $\overline{H} = \langle \overline{t} \rangle \overline{N}$. Also $\overline{R} \leqslant Z(\overline{N})$ by Lemma 2.5 (a).

Let $y \in N$. Then

$$1 = [\overline{y}, \overline{t}^p] = \prod_{k=1}^{p} [\overline{y},_k \overline{t}]^{\binom{p}{k}}$$

since $\overline{t}^p \in \overline{R}$ and $\overline{R} \leqslant Z(\overline{N})$. Also $\langle \overline{t} \rangle \overline{R}/\overline{R}$ is elementary abelian, which implies that $\exp\big([\overline{R}, \overline{t}]\big) \leqslant p$ by [3, Lemma 2.6] since $c < p$. Using this in the above equality we get

$$1 = [\overline{y}, \overline{t}]^p [\overline{y},_p \overline{t}]$$

Moreover $[\overline{y},_p \overline{t}] = 1$ since $c < p$. Using this above we get finally

$$1 = [\overline{y}, \overline{t}]^p$$

Here since y is any element of N, it follows that $\exp\big([\overline{N}, \overline{t}]\big) \leqslant p$ and so $[\overline{N}, \overline{t}] \leqslant \langle \overline{z} \rangle \overline{E}$ by Lemma 2.5 (a). Clearly this implies that $[\overline{E}, \overline{t}] \leqslant \langle \overline{z} \rangle \overline{E}$ and then t normalizes $\langle z \rangle E$. But then since t normalizes $EC_G(E)$ by Lemma 2.4 (a) and N is large, it follows that $t \in N$, which is a contradiction. Therefore the assumption is false and so B must be abelian. □

Lemma 2.7 *Let G be a locally finite p-group satisfying the normalizer condition, where $p \neq 2$. Suppose that in every homomorphic image H of G*

for every $(*)$*-pair* (w_H, V_H) *there exists an* $E_H \in E^*(w_H, V_H)$ *having a large normalizer, where* $w_H \in H \setminus 1$*. Let* B *be a normal metabelian subgroup of* G *and* A *be a normal abelian subgroup of* G *contained in* B *such that* B/A *is elementary abelian. Then* $B' \nleqslant Z(B)$*.*

Proof — Clearly $B' \neq 1$ since B is metabelian. Assume if possible that $B' \leqslant Z(B)$. Then $c(B) \leqslant 2$. Let $1 \neq w \in B'$ and V be a finite subgroup of G with $w \notin V$. Thus (w, V) is a $(*)$-pair in G and there is an element of $E^*(w, V)$ having a large normalizer by hypothesis. Now if $W^*(w, V) = 1$, then B is abelian by Lemma 2.6 since $c(B) \leqslant 2 < p$. Therefore we may suppose that $W^*(w, V) \neq 1$.

Let M be a maximal element of $W^*(w, V)$, which exists by [3, Lemma 2.1 (b)] and put $\overline{G} = G/M$. Then $(\overline{w}, \overline{V})$ is a $(*)$-pair in $\overline{G}$ since $w \notin M$ and also $W^*(\overline{w}, \overline{V}) = 1$. In addition there is $\overline{E} \in E^*(\overline{w}, \overline{V})$ having a large normalizer by hypothesis. But now since $c(\overline{B}) \leqslant 2$, $\overline{B}$ must be abelian by the first case and then $\overline{B} \leqslant \overline{N}$ by Lemma 2.4 (b), where $N = N_G(E)$. Then $B' \leqslant E$ since N/E is abelian but this is impossible since $w \notin E$ and so the proof is complete. □

3 Proofs of Theorems 1.1 and 1.2

Lemma 3.1 *Let* G *be a locally finite* p*-group satisfying the normalizer condition, where* $p > 2$*. Suppose that in every homomorphic image* H *of* G *every* $(*)$*-pair* (w_H, V_H) *satisfying* $W^*(w, V) = 1$ *has an* E_H *in* $E^*(w_H, V_H)$ *having a large normalizer, where* $w_H \in H \setminus 1$*. Let* B *be a normal metabelian subgroup of* G *and* A *be a normal abelian subgroup of* G *contained in* B *such that* B/A *is elementary abelian and* $Z(G) \leqslant A$*. Then* B/B^p *is abelian.*

Proof — Assume that B/B^p is not abelian. Put $\overline{G} = G/B^p$. Then $\overline{B}$ is nilpotent of class $\geqslant 2$ by [11, Corollary to Theorem 7.18]. Next put $Q = \overline{G}/\gamma_3(\overline{B})$. Then

$$\left(\overline{B}/\gamma_3(\overline{B})\right)' \leqslant Z\left(\overline{B}/\gamma_3(\overline{B})\right)$$

and so $c\left(\overline{B}/\gamma_3(\overline{B})\right) \leqslant 2$.

Since Q satisfies the hypothesis of the Lemma 2.7, there is $\overline{w} \in \overline{B}'$ such that $\overline{w}\gamma_3(\overline{B}) \in \left(\overline{B}/\gamma_3(\overline{B})\right)' \setminus Z\left(\overline{B}/\gamma_3(\overline{B})\right)$. Also there exists a finite subgroup $\overline{V}$ of $\overline{G}$ such that $\left(\overline{w}\gamma_3(\overline{B}), \overline{V}\gamma_3(\overline{B})/\gamma_3(\overline{B})\right)$ is a $(*)$-pair in Q. Now if $W^*\left(\overline{w}\gamma_3(\overline{B}), \overline{V}\gamma_3(\overline{B})/\gamma_3(\overline{B})\right) = 1$, then since $Z(G) \neq 1$

by Lemma 2.2 (a), $E^*\big(\overline{w}\gamma_3(\overline{B}), \overline{V}\gamma_3(\overline{B})/\gamma_3(\overline{B})\big)$ has an element having a large normalizer by hypothesis. Therefore $\overline{B}/\gamma_3(\overline{B})$ is abelian by Lemma 2.6 and by hypothesis since $p > 2$, so that $\overline{B}' \leqslant \gamma_3(\overline{B})$. Clearly this is possible only if $\overline{B}$ is abelian since $\overline{B}$ is nilpotent and then $B' \leqslant B^p$, contrary to our assumption. Therefore the assumption is false and so $B' \leqslant B^p$ in this case.

Next suppose that $W^*(\overline{w}\gamma_3(\overline{B}), \overline{V}\gamma_3(\overline{B})/\gamma_3(\overline{B})) \neq 1$ and choose a maximal element $\overline{M}/\gamma_3(\overline{B})$ in $W^*(\overline{w}\gamma_3(\overline{B}), \overline{V}\gamma_3(\overline{B})/\gamma_3(\overline{B})$. Consider

$$\overline{G}/\gamma_3(\overline{B})/(\overline{M}/\gamma_3(\overline{B})) \simeq \overline{G}/\overline{M}.$$

Now, using this isomorphism, we have $W^*(\overline{w}\overline{M}, \overline{V}\overline{M}/\overline{M}) = 1$. But since $\overline{B}\overline{M}/\overline{M}$ has class $\leqslant 2$, this group must be abelian by Lemma 2.6, as in the first case, and then $\overline{B}' \leqslant \overline{M}$ and hence $\overline{w} \in \overline{M}$ since $\overline{w} \in \overline{B}'$. But since $\overline{M} = \mathrm{Core}_{\overline{G}}(\overline{E})$ for some $\overline{E} \in E^*(\overline{w}, \overline{V})$ and $\overline{w} \notin \overline{E}$, this is a contradiction and so the proof is complete. □

Lemma 3.2 *Let* B *be a metabelian* p*-group and* A *a normal abelian subgroup of* B *such that* B/A *is elementary abelian and* $\exp(B)' \leqslant p$. *Let* $t \in B \setminus A$. *Then the following hold.*

(a) *If* $|t| = p$, *then* $[B^p, t] \leqslant C_B(t)$.

(b) *If* $|t| > p$, *then* $[B^p, t, t] \leqslant C_B(t)$.

Proof — (a) Let $y \in B$. Then

$$[y^p, t] \equiv [y, t]^p \mod \gamma_2(H)^p \gamma_p(H), \tag{1}$$

and this gives

$$[y^p, t] \equiv 1 \mod \gamma_p(H), \tag{2}$$

since $\exp(B') \leqslant p$ by hypothesis by [9, VIII.1.1, Lemma (b)], where $H = \langle [y, t], t \rangle$. Moreover, $c(\langle t \rangle A) \leqslant p$ by [6, Lemma 4.2.1 (ii)] since $|t| = p$, which means that $\gamma_p(\langle t \rangle A) \leqslant Z(\langle t \rangle A)$. Then in particular $\gamma_p(H) \leqslant Z(H)$ since $H \leqslant \langle t \rangle A$. Therefore

$$[y^p, t] \leqslant C_B(t).$$

Now since

$$[x^p y^p, t] = [x^p, t][y^p, t]$$

for every $x, y \in B$ due to $\exp(B/A) = p$ and A is abelian, it follows that $[x^p y^p, t] \in C_B(t)$ and so (a) follows.

(b) Let $Z = Z\big(\langle t\rangle A\big)$. Then $Z \lhd B$, since $B' \leqslant A$ and so $\langle t\rangle A \lhd B$. Also $t^p \in Z$. Put $\overline{B} = B/Z$. Now (2) takes the form

$$[\overline{y}^p, \overline{t}] \equiv 1 \mod \gamma_p(\overline{H}),. \tag{3}$$

Also $c\big(\langle\overline{t}\rangle\overline{A}\big) \leqslant p$ since $\overline{t}^p = 1$ and so $\gamma_p(\langle\overline{t}\rangle\overline{A}) \leqslant Z\big(\langle\overline{t}\rangle\overline{A}\big)$. Then in particular $\gamma_p(\overline{H}) \leqslant Z(\overline{H})$. Clearly then $[\overline{y}^p, \overline{t}] \in Z(\overline{H})$ by (3) and so is centralized by $\overline{t}$; that is, $[\overline{y}^p, \overline{t}, \overline{t}] = 1$. Since $\overline{y}$ is any element of $\overline{B}$, it follows as in the first case that $[\overline{B}^p, \overline{t}, \overline{t}] = 1$. Taking the inverse images we get $[B^p, t, t] \leqslant Z \leqslant C_B(t)$. □

Lemma 3.3 *Let G be a locally finite p-group satisfying the normalizer condition, where $p > 2$. Suppose that in every homomorphic image H of G every $(*)$-pair (w_H, V_H) satisfying $W^*(w, V) = 1$ has an $E_H \in E^*(w_H, V_H)$ having a large normalizer, where $w_H \in H \setminus 1$. If B is a metabelian normal subgroup of G, then B' cannot be radicable abelian.*

Proof — Assume that B' is radicable abelian and put $Q = B'$. First assume if possible that $[Q, F] \leqslant C_Q(F)$ for every finite subgroup F of B. Since $Q = C_Q(F)[F, Q]$ by [11, Lemma 3.29.1], it follows that $Q \leqslant C_Q(F)$ for every finite subgroup F of B. Clearly then $Q \leqslant Z(B)$ and so $c(B) \leqslant 2$. But now since $p \neq 2$ and G satisfies the normalizer condition, B must be ablian by Lemma 2.6, which is a contradiction. Therefore there exists a finite subgroup V of G with $[V, Q] \nleqslant C_Q(V)$.

Put $C = C_Q(V)$. Then $Q = C[V, Q]$. Now $[Q, V]C/C$ is radicable abelian since Q is. So if it is finite, then it is trivial and then $[Q, V] \leqslant C$, which is impossible by the choice of V. Therefore $[Q, V]C/C$ is infinite. Clearly then also $[Q, V]C(V \cap Q)/C(V \cap Q)$ is infinite since V is finite. Therefore there exists a $w \in [Q, V] \setminus C(V \cap Q)$. Then in particular $w \notin VC$. Indeed if $w \in VC$, then $w \in VC \cap Q = C(V \cap Q)$, which is impossible. Thus (w, V) is a $(*)$-pair in G.

Now, if $W^*(w, V) = 1$, there exists an $E \in E^*(w, V)$ so that $N_G(E)$ is large. Then $Q \leqslant N_G(E)$ by Lemma 2.4 (b) and then $[Q, V] \leqslant E$ since N/E is (locally) cyclic. But since $w \in [Q, V] \setminus E$, this is a contradiction. Therefore the assumption is false and so Q cannot exists.

Next suppose that $W^*(w, V) \neq 1$. Choose a maximal element M in $W^*(w, V)$ and put $\overline{G} = G/M$. Then since $W^*(\overline{w}, \overline{V}) = 1$, there exists an $\overline{R} \in E^*(\overline{w}, \overline{V})$ whose normalizer is large and so $\overline{Q} \leqslant \overline{N_G(R)}$ by Lemma 2.4 (b). This means that $Q \leqslant N_G(R)$ and hence $[Q, w] \leqslant E$, which gives a contradiction as in the first case and so it follows that Q cannot be radicable abelian. □

Proof of Theorem 1.1 — Let G be a locally finite p-group satisfying the normalizer condition, where $p \neq 2$. Suppose that in every homomorphic image H of G every $(*)$-pair (w_H, V_H) satisfying $W^*(w,V)=1$, where $w_H \in H \setminus 1$, has a (w_H, V_H)-maximal subgroup whose normalizer is large. Let B be a normal abelian-by-elementary abelian subgroup and A be a normal abelian subgroup of G contained in B with B/A is elementary abelian and $Z(G) \leqslant A$.

(a) Assume that B is nilpotent but not abelian. Then $[B', B] < B'$ since B is nilpotent. Let $\overline{G} = G/[B', B]$. Then $1 \neq \overline{B}' \leqslant Z(\overline{B})$ and so $c(\overline{B}) \leqslant 2$. Choose a $w \in B'$ with $\overline{w} \neq 1$ and let V be a finite subgroup of G with $\overline{w} \notin \overline{V}$. If $W^*(\overline{w}, \overline{V}) = 1$ then $\overline{B}$ is abelian by Lemma 2.6. But then since $1 \neq \overline{w} \in \overline{B}' = 1$, we get a contradiction. Therefore $W^*(\overline{w}, \overline{V}) \neq 1$.

Let M be a maximal element of $W^*(\overline{w}, \overline{V})$ and consider $\overline{G}/\overline{M}$. Because of isomorphism we may consider $Q = G/M$. Then $(wM, VM/M)$ is a $(*)$-pair in Q and $W^*(wM, VM/M) = 1$. Since the hypothesis holds in Q, it follows that BM/M is abelian and so $B' \leqslant M$. But this gives another contradiction since $w \notin M$. Therefore the assumption is false and so B must be abelian. Thus every normal nilpotent abelian-by-elementary subgroup of G is abelian.

Next let K, L be two normal abelian subgroups G. Let $H = KL$. Then H is nilpotent of class $cH) \leqslant 2$. Let A be a largest normal abelian subgroup of G contained in H with $K \cap L \leqslant A$ and let $B/A = \Omega_1(H/A)$. Then B is nilpotent and abelian-by-elementary abelian and so is abelian by the first part of the proof . Clearly then $B = A$ and this means that $B = H$ since H is nilpotent. Therefore G contains a unique maximal normal abelian subgroup.

(b) Let p be a prime $\geqslant 5$. Assume if possible that B is not abelian. Then $\exp(B/A) = p$. Also B' cannot be radicable abelian by Lemma 3.3. Therefore $(B')^p < B'$. Let $\overline{G} = G/(B')^p$. Then $\exp(\overline{B}') = p$ and so $\overline{B}$ is not abelian. Also $\overline{G}$ satisfies the hypothesis of G. Therefore without loss of generality we may replace $\overline{G}$ with G and suppose that $\exp(B') = p$.

By Lemma 2.7 there exists a $w \in B' \setminus Z(B)$. Let V be a finite subgroup with $w \notin V$. Therefore (w, V) is a $(*)$-pair in G. Suppose first that $W^*(w, V) = 1$. Let $E \in E^*(w, V)$ have a large normalizer. Put $N = N_G(E)$, $R = N \cap B$, $D = R \cap E$. Then N/E is (locally) cyclic by Lemma 2.1. Moreover, $A \leqslant R$ by Lemma 2.4 (b) and $Z(G) \cap E = 1$ since $\mathrm{Core}_G(E) = 1$. Since B is not abelian, there is $t \in B \cap N_G(N) \setminus N$ (see the proof of Lemma 2.6). As before put $T = \langle t \rangle R$ and $H = TN$.

Also define $D^* = \mathrm{Core}_H(D)$ and put $\overline{H} = H/D^*$. Then $\overline{R} \leqslant Z(\overline{N})$ and $\overline{R}/\overline{D}$ is (locally) cyclic by Lemma 2.5 (a).

First suppose that $|\bar{t}| = p$. Then $[\overline{B}^p, \bar{t}] \leqslant C_{\overline{B}}(\bar{t})$ by Lemma 3.2 (a). Then also $[\overline{B}', \bar{t}] \leqslant C_{\overline{B}}(\bar{t})$ since $B' \leqslant B^p$ by Lemma 3.1 and then

$$[\overline{B}', \bar{t}, \bar{t}] = 1.$$

Then in particular $[\overline{T}', \bar{t}, \bar{t}] = 1$ since $T \leqslant B$ and so $\gamma_4(\overline{T}) = 1$. Now since $c(\overline{T}) \leqslant 3 < 5$, it follows that $\overline{T}$ is abelian by Lemma 2.6 and so $\bar{t} \in C_{\overline{B}}(\overline{R})$. But since $C_{\overline{B}}(\overline{R}) = \overline{R}$ by Lemma 2.5 (b), it follows that $\bar{t} \in \overline{R}$ which is a contradiction.

Next suppose that $|\bar{t}| > p$. Then $[\overline{B}^p, \bar{t}, \bar{t}] \leqslant C_{\overline{B}}(\bar{t})$ by Lemma 3.2 (b). Hence $[\overline{B}^p, \bar{t}, \bar{t}, \bar{t}] = 1$ and hence also $[\overline{B}', \bar{t}, \bar{t}, \bar{t}] = 1$ by Lemma 3.1. Then since $\overline{T}' \leqslant \overline{B}'$, it follows that $[\overline{T}', \bar{t}, \bar{t}, \bar{t}] = 1$. Obviously the last equality implies that $c(\overline{T}) < 5$, and so applying Lemma 2.6 one more time it follows that $\overline{T}$ is abelian and so $\bar{t} \in C_{\overline{T}}(\overline{R}) = \overline{R}$ and this gives another contradiction since $\bar{t} \notin \overline{N}$. Therefore the assumption is false and so B must be abelian.

Next assume that $W^*(w, V) \neq 1$. In this case choose a maximal element M of $W^*(w, V)$ and define $\overline{G} = G/M$. Then $(\overline{w}M, \overline{V}M/M)$ is a $(*)$-pair in $\overline{G}$ since $w \notin M$ and $W^*(\overline{w}M, \overline{V}M/M) = 1$. Therefore $E^*(\overline{w}M, \overline{V}M/M)$ contains an element having a large normalizer by hypothesis. Also $\exp(\overline{B}') = p$. Clearly then we get another contradiction as in the first case. Thus, the assumption is false and so B must be abelian. □

Proof of Theorem 1.2 — Let G be a solvable p-group satisfying the normalizer condition, where $p \geqslant 5$. Suppose that in every homomorphic image H of G every $(*)$-pair (w_H, V_H) satisfying $W^*(w, V) = 1$ has a (w_H, V_H)-maximal subgroup whose normalizer is large. Assume that G is not abelian. By Lemma 2.7 there exists a $w \in G' \setminus Z(G)$. Let V be a finite subgroup of G such that $w \notin V$ and consider the $(*)$-pair (w, V). First suppose that $W^*(w, V) = 1$. Then there exists $E \in E^*(w, V)$ whose normalizer is large. Put $N = N_G(E)$. Then $N = N_G\big(EC_G(E)\big)$ by hypothesis.

Now suppose that G' is abelian. Assume first that $W^*(w, V) = 1$. Then $G' \leqslant N$ by Lemma 2.4 (b) since $p \neq 2$ and then $N \lhd G$. But then since G is (locally) cyclic by Lemma 2.2 (b), we get a contradiction. Therefore $W^*(w, V) \neq 1$.

Choose a maximal element $M \in W^*(w, V)$, which exists by [3, Lemma 2.1 (b)], and consider $\overline{G} = G/M$. Then $(\overline{w}, \overline{V})$ is a $(*)$-pair in $\overline{G}$ since $M = \mathrm{Core}_G(R)$ for some $R \in E^*(w, V)$ and $w \notin R$. Moreover $W^*(\overline{w}, \overline{V}) = 1$, since

$$E^*(\overline{w}, \overline{V}) = \{\overline{T} : T \in E^*(w, V) \text{ and } M \leqslant T\}$$

(see [1, Lemma 4.2]). In addition there exists an $\overline{S} \in E^*(\overline{w}, \overline{V})$ whose normalizer is large. Therefore applying Lemma 2.4 (b) again gives $\overline{G}' \leqslant N_{\overline{G}}(\overline{S})$ and so $N_{\overline{G}}(\overline{S}) \lhd \overline{G}$. Clearly then $\overline{G}$ is locally cyclic as above and this implies that $\overline{G}' \leqslant \overline{S}$. But since $\mathrm{Core}_{\overline{G}}(\overline{S}) = 1$, it follows that $\overline{G}' = 1$ and then $G' \leqslant M$. But since $w \notin S$, this gives another contradiction. Therefore we may suppose that G' is not abelian.

Put $\overline{G} = G/G''$. Then $\overline{G}'$ is non-trivial and abelian. Also $\overline{G}$ satisfies the hypothesis of the theorem. Therefore $\overline{G}$ is abelian as in the first case. Thus $\overline{G}' = 1$ and this implies that $G' \leqslant G''$, which is a contradiction since G' is not abelian. Therefore the assumption that G is not abelian is false and so G must be abelian, which completes the proof of the theorem. □

Proof of Theorem 1.3 — Let G be an hyperabelian p-group satisfying the normalizer condition, where $p \geqslant 5$. Suppose that in every homomorphic image H of G every $(*)$-pair (w_H, V_H) with $W^*(w, V) = 1$ has a locally maximal subgroup whose normalizer is large. Assume that G is not abelian. Then also G can not be solvable by Theorem 1.2. Let A be the unique maximal normal abelian subgroup of G which exists by Theorem 1.1 (a). Then $1 \neq A \neq G$ since G is not abelian. In the same way G/A contains a unique maximal normal abelian subgroup, say, U/A such that $1 \neq U/A \neq G/A$ since G is hyperabelian but not solvable. Let $B/A = \Omega_1(U/A)$. Then $B \lhd G$ and is not abelian by definition of A and $\exp(B/A) = p$. But since every abelian-by-elementary abelian normal subgroup of G must be abelian by Theorem 1.1 (b) this is a contradiction. Therefore the assumption is false and so G is abelian, which completes the proof of the theorem. □

4 Proof of Theorem 1.5

In this section we give a complete characterization of [3, Theorem 1.1]. But first a lemma is needed.

Lemma 4.1 *Let* G *be a locally finite* p*-group satisfying the normalizer condition, where* $p \neq 2$ *and let* (w, V) *be a* $(*)$*-pair in* G*, where* $w \in G \setminus 1$*. Let* $E \in E^*(w, V)$ *satisfy* $(**)$ *and let* $N = N_G(E)$*. Then* N *is large.*

Proof — By hypothesis $N = N_G(E')$. We claim that $N = N_G\big(EC_G(E)\big)$. Put $C = C_G(E)$. Then $C \leqslant N$. Since N/E is (locally) cyclic by Lemma 2.1 and since $C \cap E \leqslant Z(C)$, it follows that $C/Z(C)$ is (locally) cyclic. Let $F/Z(C)$ be a finite subgroup of $C/Z(C)$. Since $F/Z(C)$ is cyclic, $F = \langle f, Z(C)\rangle$ for some $f \in F$ and so F is abelian. Since F is any subgroup of C with $|FZ(C) : Z(C)|$ is finite , it follows that C is abelian. Now since

$$N_G(CE) \leqslant N_G((CE)') = N_G(E') = N,$$

it follows that $N_G(CE) \leqslant N$. But also $N \leqslant N_G(CE)$ since $N = N_G(E)$. Therefore we get the equality $N_G(CE) = N$ and so it follows that N is large. □

Proof of Theorem 1.5 — Let G be a Fitting p-group satisfying the normalizer condition, where $p \neq 2$. Then G is generated by normal nilpotent subgroups. Suppose that in every homomorphic image H of G every $(*)$-pair (w_H, V_H) has a (w_H, V_H)-maximal subgroup satisfying $(**)$. Clearly, then in every homomorphic image of G every $(*)$-pair has a locally maximal subgroup whose normalizer is large by Lemma 4.1.

Now assume if possible that G is not abelian. Then we can choose a finite non-abelian subgroup F of G. Put $M = \langle F^G \rangle$. Then M is nilpotent since G is a Fitting group. Also $\exp(M) < \infty$ by [11, Corollary to Theorem 2.26]. Next let A be a maximal normal abelian subgroup of G contained in M such that $Z(M) \leqslant A$ and let $B/A = \Omega_1\big(Z(M/A)\big)$. Then B is not abelian by the maximality of A since B is also normal in G. Also $\exp(B) < \infty$. But now since G satisfies the hypothesis of Theorem 1.1, B must be abelian, which is a contradiction. Therefore F must be abelian. Since F is any finite subgroup of G, it follows that G is abelian and so the proof of the theorem is complete. □

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