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Growth of Groups and of Group Endomorphisms

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Abstract

In these notes we recall the classical notions of growth and growth rate for finitely generated groups and the main results in the theory related to Milnor's problem. Then, we describe how one can extend these concepts and results to the general case of group endomorphisms, using the language and features of the algebraic entropy. Finally, we mention the main properties of the algebraic entropy, paying special attention to its additivity with respect to short exact sequences.

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1 Introduction

These are the notes of the mini-course given by the author at the conference GABY 2022 held at the University of Milano-Bicocca in 2022, June 13th–17th. They can be considered also a survey on the recent theory of growth for group endomorphisms and on the additivity of the algebraic entropy with respect to short exact sequences.

The notion of growth for finitely generated groups (see Section 2.2) was introduced by Milnor in the sixties and since then it has become a prominent field of research. In particular, the famous Milnor Problem on group growth [41] was completely solved by Gromov [35] in his celebrated theorem characterizing the finitely generated groups of polynomial growth as those that are virtually nilpotent, and by Grigorchuk [32] by constructing his famous examples of finitely generated groups of intermediate growth. Another central result in this theory is the so-called Milnor–Wolf Theorem [40, 55] establishing that solvable finitely generated groups have either polynomial or exponential growth (so no intermediate growth is allowed). This dichotomy was extended by Chou [9] to elementary amenable finitely generated groups (see Section 2.3 for the details).

Inspired by the work of Kolmogorov and Sinai concerning the measure entropy in ergodic theory, Adler, Konheim and McAndrew [1] introduced the topological entropy for continuous selfmaps of compact spaces, and in a final remark of their paper they proposed a notion of algebraic entropy for endomorphisms of abelian groups. Later, Weiss [54] studied this algebraic entropy, which is suitable for torsion abelian groups, and its connection to the topological entropy, while Peters [44] developed it for automorphisms out of the torsion case. The interest in this topic increased after the work by Dikranjan, Goldsmith, Salce and Zanardo [21], where a rather complete description of the algebraic entropy for endomorphisms of torsion abelian groups was obtained. A further extension of the algebraic entropy and its properties to all abelian groups can be found in [18], while in [15] the algebraic entropy was introduced for the first time for endomorphisms of arbitrary groups (see Section 3.3).

See [45, 52] for the algebraic entropy of continuous endomorphisms of locally compact abelian groups, see [16, 17, 20, 24, 45] for the connection of the algebraic entropy with the topological entropy by means of the Pontryagin duality, and see [3, 12, 13, 20, 38, 53] for the algebraic entropy for (semi)group actions.

In [15], using the language of algebraic entropy, the classical notion of growth is extended to endomorphisms of arbitrary groups, that is, neither necessarily abelian nor finitely generated (see Section 3.1). In particular, considering the growth of the identity automorphism $\mathrm{id}_G\colon G \to G$ of a group G, one gets a notion of growth for arbitrary groups and in [27] the classical Gromov Theorem and Milnor–Wolf Theorem were extended to this setting (see Section 3.2). On the other hand, it is worth to mention here that Xi, Dikranjan, Freni and Toller [56] gave a new proof of the Milnor–Wolf Theorem using the theory of growth and algebraic entropy for group endomorphisms.

In the already cited paper [27], the dichotomy between polynomial and exponential growth, as in the classical Milnor–Wolf Theorem, was proved for endomorphisms of locally finite groups (see Section 4.1). The same dichotomy was already known for endomorphisms of abelian groups, as a consequence of the study offered in [14] of the Pinsker subgroup (see Section 4.2). Moreover, Chou's extension of the Milnor–Wolf Theorem was proved for endomorphisms of elementary amenable groups in [28] (see Section 4.3).

The main property that one wishes to have for the algebraic entropy is the so-called Addition Theorem: namely, for a group G, an endomorphism $\phi\colon G \to G$ and a ϕ-invariant normal subgroup H of G, one expects that

$$h(\phi) = h(\phi\restriction_H) + h(\overline{\phi}_{G/H}), \tag{1.1}$$

where

$$\overline{\phi}_{G/H}\colon G/H \to G/H$$

is the endomorphism induced by ϕ on G/H and $\phi\restriction_H$ is the restriction of ϕ to H. If (1.1) holds for every endomorphism $\phi\colon G \to G$, we briefly say that $AT(G)$ holds.

The Addition Theorem for abelian groups was proved in the torsion case in [21], then it was extended to the general setting in [18] (see Section 5.2). In particular, among others, it is the fundamental property in the Uniqueness Theorem (see Theorem 5.12), which characterizes the algebraic entropy as the unique invariant for abelian groups and their endomorphisms satisfying the Addition Theorem and the continuity with respect to direct limits, and taking suitable values on the right Bernoulli shifts (see Example 5.7) and on the endomorphisms of $\mathbb{Q}^n$ (see the Algebraic Yuzvinski Formula in Theorem 5.10); see Section 5.1 for the basic properties of the algebraic entropy.

The importance of the Addition Theorem comes also from the fact that, together with the property of being continuous with respect to direct limits, it implies that the algebraic entropy is a length function of the category of $\mathbb{Z}[x]$-modules in the sense of Northcott and Reufel [43] and Vámos [51] (see [18, 47] for the details on this connection). In fact, since an endomorphism of an abelian group G induces on G the structure of a $\mathbb{Z}[x]$-module, and vice versa, one can consider the algebraic entropy as an invariant of the category of $\mathbb{Z}[x]$-modules.

In the non-abelian context, a metabelian counterexample to the Addition Theorem was found in [27] (see Example 5.13), by exploiting the knowledge on the classical growth for finitely generated groups (see Section 5.3). Anyway, we conjecture that the Addition Theorem holds for locally virtually nilpotent groups, and so in particular for locally finite groups. In this setting some positive results were obtained in [26, 57] (where also the locally compact case was considered), then covered by the Addition Theorem in [25] (see Theorem 5.19) for finitely quasihamiltonian locally finite groups (e.g., all quasihamiltonian groups, also called Iwasawa groups, and all FC-groups, also called locally finite and normal groups, are finitely quasihamiltonian).

Recently, Shlossberg [49] proved the Addition Theorem for torsion nilpotent groups of nilpotency class 2 (see Theorem 5.21), and also for locally finite groups admitting a fully characteristic finite index simple subgroup (see Theorem 5.22), hence in particular for the group $S_{\mathrm{fin}}(\mathbb{N}_+)$ of all permutations of $\mathbb{N}_+$ with finite support, that is not finitely quasihamiltonian.

2 Growth of finitely generated groups

In this section, we first recall the classical notions of growth type and growth rate for finitely generated groups, and then we state the main classical results we are interested in. We refer to [34] and to the monographs [8, 11, 39].

2.1 Growth of functions

Given two maps $\gamma, \gamma' \colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$, we write $\gamma \preceq \gamma'$ if there exists $C \in \mathbb{N}$ such that $\gamma(n) \leqslant \gamma'(Cn)$ for every $n \in \mathbb{N}$. Moreover, we say that γ and γ' are *equivalent*, and write $\gamma \sim \gamma'$, if $\gamma \preceq \gamma'$ and $\gamma' \preceq \gamma$; indeed, $\sim$ is an equivalence relation.

Example 2.1 Routine computations show that:

(1) for every $\alpha, \beta \in \mathbb{R}_{\geqslant 0}$, $n^\alpha \sim n^\beta$ if and only if $\alpha = \beta$;

(2) for every $a, b \in \mathbb{R}_{>1}$, $a^n \sim b^n$;

(3) for every $d \in \mathbb{N}$, $n^d \preceq e^n$ and $n^d \not\succeq e^n$.

Using the equivalence relation $\sim$, now we recall the following notion of growth for functions $\mathbb{N} \to \mathbb{R}_{\geqslant 0}$, which we later see to be suitable for the case of growth functions of finitely generated groups.

Definition 2.2 A function $\gamma\colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ is called:

- *polynomial* if $\gamma(n) \preceq n^d$ for some $d \in \mathbb{N}_+$;
- *exponential* if $\gamma(n) \sim e^n$;
- *intermediate* if $n^d \preceq \gamma(n)$ for all $d \in \mathbb{N}_+$, $\gamma(n) \preceq e^n$ and $e^n \not\preceq \gamma(n)$.

As in [34], we recall also the following related notions.

Definition 2.3 A function $\gamma\colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ is called:

- *superpolynomial* if $\lim_{n\to\infty} \frac{\log \gamma(n)}{\log n}$ exists and equals ∞;
- *subexponential* if $\lim_{n\to\infty} \frac{\log \gamma(n)}{n}$ exists and equals 0.

Clearly, a function $\gamma\colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ is intermediate if and only if it is superpolynomial and subexponential.

Example 2.4 Here we list some basic examples:

(1) $n \log n$ is polynomial;

(2) for every $\ell \in \mathbb{N}$, $n^\ell e^n$ is exponential;

(3) for $\alpha \in (0,1)$, the functions e^{n^α} are intermediate, pairwise non equivalent;

(4) $e^{\sqrt{n}}$, $e^{n \log n}$, $e^{\sqrt{\log n}}$ and $n^{\log \log n}$ are intermediate;

(5) n^n and $e^{n^{\sin n}}$ are not classifiable.

2.2 Growth type and growth rate

Let G be a finitely generated group and let S be a finite set of generators for G. Let $\ell_S(e_G) := 0$, where e_G is the neutral element of G. For every $g \in G \setminus \{e_G\}$, let

$$\ell_S(g) := \min\left\{n \in \mathbb{N}_+ : g = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \dots s_\ell^{\varepsilon_\ell}, s_i \in S, \varepsilon_i \in \{1,-1\}\right\}$$

be the length of a shortest word representing g in the alphabet $S \cup S^{-1}$, where $S^{-1} := \left\{s^{-1} : s \in S\right\}$.

Remark 2.5 In many cases, it is convenient to take the finite set of generators S for the group G to be symmetric, that is, $S = S^{-1}$. With this assumption, for $g \in G \setminus \{e_G\}$,

$$\ell_S(e_G) = \min\left\{\ell \in \mathbb{N}_+ : g = s_1 s_2 \dots s_\ell, s_i \in S\right\}.$$

Definition 2.6 Let G be a finitely generated group and let S be a finite set of generators for G. The *word metric* on G with respect to S is $d_S\colon G \times G \to \mathbb{N}$, defined by $d_S(g,h) = \ell_S(g^{-1}h)$ for every $g,h \in G$.

The word metric d_S is a left invariant (i.e., $d_S(xg,xh) = d_S(g,h)$ for every $x,g,h \in G$) metric on G. In other words, the action of G on itself by left multiplication is an isometric action with respect to d_S.

For $n \in \mathbb{N}$, let

$$B_S(n) := \{g \in G : \ell_S(g) \leqslant n\} \tag{2.1}$$

be the *ball centered in* e_G *of radius* n in the word metric d_S of G. Clearly, $B_S(0) = \{e_G\}$ and $B_S(1) = S \cup S^{-1} \cup \{e_G\}$. Moreover,

$$B_S(n) \subseteq B_S(n+1) \text{ for every } n \in \mathbb{N}. \tag{2.2}$$

Definition 2.7 Let G be a finitely generated group and let S be a finite set of generators for G. The *growth function* of G with respect to S is

$$\gamma_S\colon \mathbb{N} \to \mathbb{N},\ n \mapsto |B_S(n)|.$$

Remark 2.8 From (2.2) we get that $\gamma_S(n) \leqslant \gamma_S(n+1)$ for every $n \in \mathbb{N}$. Moreover, $\gamma_S(n) \leqslant |S \cup S^{-1} \cup \{e_G\}|^n$ for every $n \in \mathbb{N}$, showing that γ_S is at most exponential.

Routine computations show that $\gamma_S \sim \gamma_{S'}$, for every pair of finite generating sets S, S' for G. This observation allows us to give the following definition of growth type and to notice that it does not depend on S.

Definition 2.9 Let G be a finitely generated group and let S be a finite set of generators for G. The group G has *polynomial* (respectively, *exponential*, *intermediate*, *superpolynomial*, *subexponential*) *growth* if γ_S is polynomial (respectively, exponential, intermediate, superpolynomial, subexponential).

Example 2.10 Let G be a finitely generated group and let S be a finite set of generators for G. Then:

(1) $\gamma_S \sim 1$ precisely when γ_S is bounded, if and only if G is finite;

(2) if G is infinite, $n \preceq \gamma_S$.

The above example shows that the growth of a finitely generated group recalled in Definition 2.9 is not interesting in the finite case and that an infinite finitely generated group has at least polynomial growth (and at most exponential growth by Remark 2.8), that is, denoting by S a finite set of generators for a group G,

$$n \preceq \gamma_S \preceq e^n.$$

Example 2.11 (1) Let $G = \mathbb{Z}$ and $S = \{1\}$ (take $S = \{1, -1\}$ for a symmetric set of generators). Then, for every $n \in \mathbb{N}$,

$$B_S(n) = \{-n, \dots, n\} \quad \text{and} \quad \gamma_S(n) = 2n + 1.$$

So, $\gamma_S \sim n$ and $\mathbb{Z}$ has polynomial growth.

(2) Let $G = \mathbb{Z}^2$ and $S = \{(1,0), (0,1)\}$. Then $\gamma_S(n) = 2n^2 + 2n + 1$ for every $n \in \mathbb{N}$, and so $\gamma_S \sim n^2$ and $\mathbb{Z}^2$ has polynomial growth.

(3) If, for $i = 1, 2$, G_i is a finitely generated group and S_i is a finite set of generators for G_i, then $S = \big(S_1 \times \{e_{G_2}\}\big) \cup \big(\{e_{G_1}\} \times S_2\big)$ is a finite set of generators for $G = G_1 \times G_2$ and $\gamma_S \sim \gamma_{S_1} \cdot \gamma_{S_2}$. Then one can prove by induction that for $d \in \mathbb{N}_+$ and S a finite set of generators for $\mathbb{Z}^d$, $\gamma_S \sim n^d$ and so $\mathbb{Z}^d$ has polynomial growth.

(4) Since every finitely generated abelian group G is isomorphic to $\mathbb{Z}^d \times F$ for some $d \in \mathbb{N}$ and some finite abelian group F, we get that $\gamma_S \sim n^d$ for S a finite set of generators for G, and so also that G has polynomial growth.

Example 2.12 Let again $G = \mathbb{Z}^2$. For $S' = \{(1,0), (0,1), (1,1), (1,-1)\}$, we get a different growth function with respect to γ_S from Example 2.11 (2), that is, $\gamma_{S'}(n) = 4n^2 + 4n + 1$ for every $n \in \mathbb{N}$; anyway, $\gamma_{S'} \sim n^2 \sim \gamma_S$.

Example 2.13 Let $G = \mathbb{F}_2$ be the free group on two generators a, b and $S = \{a, b\}$. Then $\gamma_S(n) = 2 \cdot 3^n - 1$ for every $n \in \mathbb{N}$, so $\gamma_S \sim e^n$ and clearly $\mathbb{F}_2$ has exponential growth.

Definition 2.14 Let G be a finitely generated group and let S be a finite set of generators for G. The *growth rate of* G *with respect to* S is

$$\lambda_S := \lim_{n \to \infty} \frac{\log \gamma_S(n)}{n}.$$

As the sequence $\{\log \gamma_S(n)\}_{n\in\mathbb{N}}$ is subadditive (that is, for every $n, m \in \mathbb{N}$, $\log \gamma_S(n+m) \leqslant \log \gamma_S(n) + \log \gamma_S(m)$), by Fekete Lemma (see [23]) the above limit exists and $\lambda_S = \inf_{n\in\mathbb{N}_+} \frac{\log \gamma_S(n)}{n}$.

It is easy to see (e.g., by changing set of generators in Example 2.13) that λ_S depends on S. Nevertheless, the following equivalence holds, namely, when G has exponential growth λ_S is positive for every finite set of generators S for G and, conversely, if G has either polynomial or intermediate growth, then $\lambda_S = 0$ for every finite set of generators S for G.

Remark 2.15 Let G be a finitely generated group and let S be a finite set of generators for G.

(i) It is straightforward to prove that G has exponential growth if and only if $\lambda_S > 0$. In particular, if G has not exponential growth, then equivalently $\lambda_S = 0$, that is, G has subexponential growth.

(ii) On the other hand, it is non-trivial to see that in case G has not polynomial growth, then G has superpolynomial growth. Anyway this is true and can be deduced from [34, Corollary 8.6].

(iii) In particular, we get that in case G has neither polynomial growth nor exponential growth, necessarily G has intermediate growth.

2.3 Main classical results

Milnor [41] posed its celebrated problem on group growth:

Problem 2.16 (Milnor) *Let G be a finitely generated group and let S be a finite set of generators for G.*

(i) *Is γ_S necessarily equivalent either to n^d for some $d \in \mathbb{N}$ or to e^n?*

(ii) *In particular, is the growth exponent $\lim_{n\to\infty} \frac{\log \gamma_S(n)}{\log n}$ always either a well defined integer or infinity? For which groups is it finite?*

Part (i) was solved by Grigorchuk [32] by constructing his famous examples of finitely generated groups of intermediate growth.

With respect to part (ii), Wolf [55] showed that every nilpotent finitely generated group has polynomial growth. Then Bass [2] and Guivarc'h [36] independently proved the following result. We denote by $\mathrm{rk}(A)$ the free rank of an abelian group A. Moreover, the lower central series of a group G is defined inductively by $\gamma_1(G) := G$ and $\gamma_{n+1}(G) := [\gamma_n(G), G]$ for every $n \in \mathbb{N}_+$; the group G is nilpotent if and only if $\gamma_{c+1}(G) = 1$ for some $c \in \mathbb{N}$, and the minimum such c is the *nilpotency class* of G.

Theorem 2.17 (Bass–Guivarc'h) *Let G be a nilpotent finitely generated group with nilpotency class c and let S be a finite set of generators for G. Then $\gamma_S \sim n^d$, where $d = \sum_{i=1}^{c} \mathrm{rk}\big(\gamma_i(G)/\gamma_{i+1}(G)\big)$.*

In particular, given a nilpotent finitely generated group G with finite set of generators S, not only it has polynomial growth, but more precisely there exists $d \in \mathbb{N}$ with $\gamma_S \sim n^d$.

Recall that a group G is virtually nilpotent if it contains a nilpotent subgroup H having finite index. When G is a virtually nilpotent finitely generated group and H a nilpotent finite index subgroup of G, then H is finitely generated and G and H have the same growth type; more precisely, if S is a finite set of generators for G and S' is a finite set of generators for H, then $\gamma_S \sim \gamma_{S'}$. So, the above results by Wolf and Bass–Guivarc'h immediately extend to virtually nilpotent finitely generated groups.

Part (ii) of the Milnor Problem was completely solved by Gromov [35], by proving what was already conjectured by Milnor [41].

Theorem 2.18 (Gromov) *A finitely generated group has polynomial growth if and only if it is virtually nilpotent.*

Before the work of Gromov, Milnor [40] proved that a solvable finitely generated group of subexponential growth is polycyclic, while Wolf [55] showed that a polycyclic finitely generated group of subexponential growth is virtually nilpotent, so it has polynomial growth. As customary, we call Milnor–Wolf Theorem the following result, which we call also a Dichotomy Theorem.

Theorem 2.19 (Milnor–Wolf) *A solvable finitely generated group has either polynomial or exponential growth.*

The class of elementary amenable groups was introduced by Day [10] as the smallest class of groups containing the finite groups and the abelian groups which is closed under taking subgroups, quotients, group extensions and direct limits. Later Chou [9] showed that this class can be constructed from finite groups and abelian groups by applying only group extensions and direct limits. Moreover, he extended the Milnor–Wolf Theorem to elementary amenable groups.

Theorem 2.20 (Chou) *An elementary amenable finitely generated group has either polynomial or exponential growth.*

3 Growth and algebraic entropy of group endomorphisms

In this section we extend to group endomorphisms the notions of growth type and growth rate.

3.1 Growth of group endomorphisms

For a group G, denote by $\mathcal{F}(G)$ the family of all finite non-empty subsets of G. For $F \in \mathcal{F}(G)$, we let $T_0(\phi, F) := \{e_G\}$ and for $n \in \mathbb{N}_+$,

$$T_n(\phi, F) := F\,\phi(F)\ldots\phi^{n-1}(F)$$

is the n-*th* ϕ-*trajectory* of F. When $e_G \in F$, we get

$$T_n(\phi, F) \subseteq T_{n+1}(\phi, F) \text{ for every } n \in \mathbb{N}. \tag{3.1}$$

Remark 3.1 Compare the definition of $T_n(\phi, F)$ with (2.1) and (3.1) with (2.2): if G is finitely generated and S is a finite set of generators for G, with $F = B_S(1) = S \cup S^{-1} \cup \{e_G\}$ we get

$$B_S(n) = T_n\big(id_G, F\big) \text{ for every } n \in \mathbb{N},$$

where we denote by $id_G \colon G \to G$ the identity automorphism.

Definition 3.2 Let G be a group and let $\phi\colon G \to G$ be an endomorphism. The *growth function* of ϕ with respect to $F \in \mathcal{F}(G)$ is

$$\gamma_{\phi,F}\colon \mathbb{N} \to \mathbb{N},\ n \mapsto |T_n(\phi, F)|.$$

For every $F \in \mathcal{F}(G)$, we have $|F| \leqslant \gamma_{\phi,F}(n) \leqslant |F|^n$ for every $n \in \mathbb{N}_+$, hence the growth of $\gamma_{\phi,F}$ is always at most exponential. From (3.1), in case $e_G \in F$, we get that $\gamma_{\phi,F}(n) \leqslant \gamma_{\phi,F}(n+1)$ for every $n \in \mathbb{N}$.

Remark 3.3 From Remark 3.1 and the comparison of Definition 3.2 with Definition 2.7, we get that in case G is a finitely generated group and S is a finite set of generators for G, letting $F = S \cup S^{-1} \cup \{e_G\}$, then $\gamma_{\mathrm{id}_G,F} = \gamma_S$.

The following problem remains open, even for automorphisms of finitely generated groups.

Problem 3.4 *Let G be a finitely generated group and let $\phi\colon G \to G$ be an endomorphism. For S, S' finite sets of generators for G, is it true that $\gamma_{\phi,S} \sim \gamma_{\phi,S'}$?*

This has positive solution for $\phi = \mathrm{id}_G$ as stated in Section 2.2, but it is open in general. In Section 3.4 we give some partial (positive) answer for automorphisms (see Theorem 3.17) and discuss what is still open (see Problem 3.18).

Anyway, we give the following definition, which extends the classical one recalled in Definition 2.9 in view of Remarks 3.1 and 3.3.

Definition 3.5 ([14, 15]) Let G be a group and let $\phi\colon G \to G$ be an endomorphism. Then:

(a) ϕ has *polynomial growth* if $\gamma_{\phi,F}$ is polynomial for every $F \in \mathcal{F}(G)$;

(b) ϕ has *exponential growth* if there exists $F_0 \in \mathcal{F}(G)$ such that γ_{ϕ,F_0} is exponential;

(c) ϕ has *intermediate growth* if $\gamma_{\phi,F}$ is not exponential for every $F \in \mathcal{F}(G)$ and there exists $F_0 \in \mathcal{F}(G)$ such that γ_{ϕ,F_0} is intermediate.

Several examples are given below.

3.2 Growth of groups

In this subsection, as a first example and application of the notions in the previous section, we recall the results from [27, Page 5]. In view of Definition 3.5, one can extend the concept of growth to any group (not necessarily finitely generated):

Definition 3.6 A group G has *polynomial* (respectively, *intermediate*, *exponential*) growth if id_G has polynomial (respectively, intermediate, exponential) growth.

By applying the Gromov Theorem (Theorem 2.18) and the Milnor–Wolf Theorem (Theorem 2.19) respectively, one can extend them to every group as follows. Recall that a group G is locally virtually nilpotent (respectively, locally virtually solvable) if every finitely generated subgroup of G is virtually nilpotent (respectively, virtually solvable).

Theorem 3.7 ([27]) *Let G be a group. Then:*

(1) *G has polynomial growth if and only if G is locally virtually nilpotent;*

(2) *if G is locally virtually solvable, then G has either polynomial or exponential growth.*

3.3 Algebraic entropy

The following notion in particular extends that of growth rate recalled in Definition 2.14.

Definition 3.8 Let G be a group and let $\phi\colon G \to G$ be an endomorphism. The *algebraic entropy of ϕ with respect to* $F \in \mathcal{F}(G)$ is

$$H(\phi, F) := \lim_{n\to\infty} \frac{\log \gamma_{\phi,F}(n)}{n}.$$

This limit exists because the sequence $\{\log \gamma_{\phi,F}(n)\}_{n\in\mathbb{N}}$ is subadditive, and hence Fekete Lemma [23] applies.

Remark 3.9 Let G be a finitely generated group and let S be a finite set of generators for G. With $F = B_S(1) = S \cup S^{-1} \cup \{e_G\}$, since $\gamma_S = \gamma_{id_G,F}$ as noted in Remark 3.3, we get that $\lambda_S = H(id_G, F)$.

Definition 3.10 Let G be a group and let $\phi\colon G \to G$ be an endomorphism. The *algebraic entropy* of ϕ is

$$h(\phi) := \sup_{F\in\mathcal{F}(G)} H(\phi, F).$$

Remark 3.11 If $F, F' \in \mathcal{F}(G)$ are such that $F \subseteq F'$, then

$$H(\phi, F) \leqslant H(\phi, F').$$

This is useful because it implies that for the computation of $h(\phi)$ one can consider a suitable cofinal subfamily $\mathcal{F}$ of $\mathcal{F}(G)$, with respect to the order $\subseteq$ given by inclusion, and get $h(\phi) = \sup_{F\in\mathcal{F}} H(\phi, F)$. For example, one can take $\mathcal{F} = \{F \in \mathcal{F}(G) : e_G \in F\}$.

It was proved in [15, Proposition 5.4.2] that

$$H(\phi, F) > 0 \text{ if and only if } \gamma_{\phi,F} \text{ is exponential,} \tag{3.2}$$

and so

$$h(\phi) > 0 \text{ if and only if } \phi \text{ has exponential growth.} \tag{3.3}$$

Moreover, it is clear that if ϕ has either polynomial growth or intermediate growth, then $h(\phi) = 0$.

Remark 3.12 The equivalence in (3.2) should be compared with Remark 2.15. In fact, (3.2) implies that if $\gamma_{\phi,F}$ is not exponential then it is subexponential.

On the other hand, we do not know in general whether $\gamma_{\phi,F}$ not polynomial implies $\gamma_{\phi,F}$ superpolynomial. It is true in case ϕ is injective.

Next we list few examples, more are given later on. Moreover, Section 5 is dedicated to the algebraic entropy, starting from its basic properties.

Example 3.13 (see [18], Example 3.1 (a)) For the abelian group $\mathbb{Z}$, we have that $id_{\mathbb{Z}}$ has polynomial growth and $h(id_{\mathbb{Z}}) = 0$.

An endomorphism of $\mathbb{Z}$ is of the form $\mu_k \colon \mathbb{Z} \to \mathbb{Z}$, $x \mapsto kx$, with $k \in \mathbb{Z}$. The case $k = 0$ is trivial. If $k = 1$, then we find the already discussed case $\mu_k = id_{\mathbb{Z}}$, and if $k = -1$, that is, $\mu_k = -id_{\mathbb{Z}}$, analogously one shows that $-id_{\mathbb{Z}}$ has polynomial growth and $h(-id_{\mathbb{Z}}) = 0$. If $|k| > 1$, then $h(\mu_k) = \log |k|$ and μ_k has exponential growth. In particular, no endomorphism of $\mathbb{Z}$ has intermediate growth.

Example 3.14

(1) (see [18, Example 2.5]) If G is an abelian group, then id_G has always polynomial growth, so $h(id_G) = 0$.

(2) If G is a finitely generated group with exponential growth, then id_G has exponential growth, so $h(id_G) > 0$ by (3.3), and hence $h(id_G) = \infty$ by the Logarithmic Law (see Lemma 5.3).

We conclude this section with a description of the growth and the algebraic entropy of the inner automorphisms.

Example 3.15 (see [27], Theorem 3.2) Let G be a group, let $g \in G$ and denote by $\varphi_g \colon G \to G$, $x \mapsto g^{-1}xg$, the inner automorphism of G induced by g. Then:

(1) φ_g has the same growth type of G (i.e., of id_G);

(2) $h(\varphi_g) = h(\mathrm{id}_G)$.

3.4 When G is finitely generated and ϕ is an automorphism

The most interesting case for the purpose of this paper and for the growth of group endomorphisms is when G is finitely generated and ϕ is an automorphism of G. Under these assumptions, one can consider the subgroup $\langle G, \phi\rangle$ of the holomorph $G \rtimes \mathrm{Aut}(G)$ generated by G and ϕ. Clearly, $\langle G, \phi\rangle$ is finitely generated.

The proof of the following result is based on [33, Theorems 1 and 2] concerning the growth of cancellative finitely generated semigroups.

Theorem 3.16 ([28], Proposition 5.2) *Let G be a finitely generated group and let $\phi\colon G \to G$ be an automorphism. Then ϕ has polynomial growth if and only if $\langle G, \phi\rangle$ has polynomial growth.*

More precisely, one can see that, given a finite set S of generators for G with $e_G \in S$, $\gamma_{\phi,S} \sim \gamma_{S\phi^{-1}}$, where $S\phi^{-1}$ is a finite set of generators for $\langle G, \phi\rangle$. So, one can deduce the following important properties by applying also the Gromov Theorem (Theorem 2.18) and the Bass–Guivarc'h Formula (Theorem 2.17); note that in [28, Theorem 1.5] the hypothesis on the group to be elementary amenable can be removed.

Corollary 3.17 *Let G be a finitely generated group and let $\phi\colon G \to G$ be an automorphism of polynomial growth. Then there exists $d \in \mathbb{N}$ such that, for every finite set S of generators for G with $e_G \in S$, $\gamma_{\phi,S} \sim n^d$.*

In particular, for every pair S, S' of finite sets of generators for G containing e_G, $\gamma_{\phi,S} \sim \gamma_{\phi,S'}$.

The following problem remains open in general.

Problem 3.18 *Let G be a finitely generated group, let $\phi\colon G \to G$ be an automorphism and let S, S' be finite sets of generators for G containing e_G. Is it always true that $\gamma_{\phi,S} \sim \gamma_{\phi,S'}$?*

In particular, the validity of the following conjecture would give interesting consequences.

Conjecture 3.19 ([28], Conjecture 5.4) *Let G be a finitely generated group and let $\phi\colon G \to G$ be an automorphism. Then ϕ has exponential growth if and only if $\langle G, \phi\rangle$ has exponential growth.*

See also [56, Conjecture 1.9 and Conjecture 1.11].

4 The Dichotomy Theorem

In this section we discuss the following problem posed in [27, Problem 1.1] and [28, Problem 1.4].

Problem 4.1 *Characterize the groups admitting only endomorphisms either of polynomial growth or of exponential growth.*

In view of (3.3), the strategy in the proofs of the instances of the Dichotomy Theorem described below is to show that, in each of the special cases, the endomorphisms with zero algebraic entropy necessarily have polynomial growth.

4.1 Locally finite groups

Let G be a group and let $\phi\colon G \to G$ be an endomorphism. For $F \in \mathcal{F}(G)$ and $n \in \mathbb{N}$, we let

$$V_n(\phi, F) := \left\langle \phi^i(F) : i \in \{0, \ldots, n\} \right\rangle \text{ and } V(\phi, F) := \left\langle \phi^n(F) : n \in \mathbb{N} \right\rangle.$$

Observe that $V(\phi, F) = \bigcup_{n\in\mathbb{N}} V_n(\phi, F)$ is the smallest ϕ-invariant subgroup of G containing F.

Moreover, $V_0(\phi, F) = \langle F \rangle$ and $T_{n+1}(\phi, F) \subseteq V_n(\phi, F)$ for every $n \in \mathbb{N}$, and if $e_G \in F$, then $V_{n+1}(\phi, F) = \left\langle T_{n+1}(\phi, F) \right\rangle$.

Lemma 4.2 ([27], Lemmas 4.1 and 4.2) *Let G be a group and $\phi\colon G \to G$ an endomorphism. Then:*

(1) $V(\phi, F)$ *is finitely generated if and only if* $V(\phi, F) = V_n(\phi, F)$ *for some* $n \in \mathbb{N}$;

(2) *if* $g \in G$ *and* $V\big(\phi, \{g\}\big)$ *is not finitely generated, then* $H(\phi, \{e_G, g\}) > 0$.

Corollary 4.3 ([27], Corollary 4.4) *Let G be a group and let $\phi\colon G \to G$ be an endomorphism. If $h(\phi) = 0$, then $V(\phi, F)$ is finitely generated for every $F \in \mathcal{F}(G)$.*

The converse implication of Lemma 4.2 does not hold true; indeed, it may occur that each $V(\phi, g)$ is finitely generated while $h(\phi) > 0$: consider a group G of exponential growth; in this case, $V\big(id_G, F\big) = \langle F \rangle$ is finitely generated for every $F \in \mathcal{F}(G)$, while $h\big(id_G\big) = \infty$ by Example 3.14 (2).

On the other hand, the converse implication of Lemma 4.2 holds true assuming that G is locally finite [27, Proposition 4.5], and as a consequence we find a complete solution to [15, Problem 5.2.3].

Theorem 4.4 ([27], Theorem 4.6) *Let* G *be a locally finite group and let* $\phi\colon G \to G$ *be an endomorphism. Then the following conditions are equivalent:*

(a) ϕ *has polynomial growth;*

(b) $h(\phi) = 0$*;*

(c) $V(\phi, F)$ *is finite for every* $F \in \mathcal{F}(G)$.

Theorem 4.4 shows in particular that, if ϕ is an endomorphism of a locally finite group G and ϕ has zero algebraic entropy, then G is a direct limit of finite ϕ-invariant subgroups.

As a consequence of Theorem 4.4 and (3.3), we get that locally finite groups satisfy the condition of Problem 4.1.

Corollary 4.5 ([27], Corollary 4.7) *Let* G *be a locally finite group and let* $\phi\colon G \to G$ *be an endomorphism. Then* ϕ *has either polynomial or exponential growth.*

This solves also [15, Problem 5.4.5] for locally finite groups.

4.2 The Pinsker subgroup

The Dichotomy Theorem for abelian groups was proved in [14] as a consequence of deeper results.

Inspired by the concept of Pinsker algebra for the measure entropy, the Pinsker factor for the topological entropy was introduced by Blanchard and Lacroix [4]; its counterpart for the algebraic entropy was introduced in [14]: the Pinsker subgroup $P(G, \phi)$ for a group G and an endomorphism $\phi\colon G \to G$ is the greatest ϕ-invariant subgroup of G such that $h(\phi\restriction_{P(G,\phi)}) = 0$. For abelian groups this subgroup exists.

In [14] we saw that the Pinsker subgroup $P(G, \phi)$ coincides with the greatest ϕ-invariant subgroup $\mathrm{Pol}(G, \phi)$ of G such that $\phi\restriction_{\mathrm{Pol}(G,\phi)}$ has polynomial growth:

Theorem 4.6 ([14], Main Theorem) *Let* G *be an abelian group and let* $\phi\colon G \to G$ *be an endomorphism. Then* $P(G, \phi) = \mathrm{Pol}(G, \phi)$.

From this result we deduce the above mentioned dichotomy, which is formally stronger.

Theorem 4.7 ([14], Dichotomy Theorem) *Let* G *be an abelian group and let* $\phi\colon G \to G$ *be an endomorphism. Then* ϕ *has either polynomial or exponential growth.*

Indeed, this means that for every $F \in \mathcal{F}(G)$, $H(\phi, F) = 0$ if and only if $\gamma_{\phi,F}$ has polynomial growth.

4.3 Elementary amenable groups

The main result of [28] is the following Dichotomy Theorem for elementary amenable groups, covering both Corollary 4.5 and Theorem 4.7.

Theorem 4.8 ([28], Theorem 1.2) *Let G be an elementary amenable group and let $\phi\colon G \to G$ be an endomorphism. Then ϕ has either polynomial or exponential growth.*

The most interesting case is when G is finitely generated and ϕ is an automorphism. Under these assumptions, something stronger and more precise can be proved by applying Theorem 3.16:

Corollary 4.9 ([28], Theorem 1.3) *Let G be an elementary amenable finitely generated group and let $\phi\colon G \to G$ an automorphism. Then either ϕ has exponential growth or $\langle G, \phi\rangle$ is virtually nilpotent; in the latter case, ϕ has polynomial growth.*

The proof of Theorem 4.8 follows in some sense the proof of the classical result by Chou (see Theorem 2.20), but clearly in many steps it needs original ideas due to the dynamical setting (i.e., the presence of an endomorphism). In [28], first we reduce to automorphisms and moreover the case when G is finitely generated is crucial. Under these assumptions, the Algebraic Yuzvinski Formula (see Theorem 5.10) for the algebraic entropy applies to get one of the main ingredients in the proof of Theorem 4.8.

5 The (non-)additivity of the algebraic entropy

In this section we recall the basic properties of the algebraic entropy, its connection with number theory by means of the Algebraic Yuzvinski Formula, and pay special attention to its additivity with respect to short exact sequences (the so-called Addition Theorem) recalling what it is known in the non-abelian case.

In entropy theory the Addition Theorem is always a fundamental property; in particular, Yuzvinski [58] proved it (indeed, it is alternatively called Yuzvinski's addition formula) for the topological entropy of continuous endomorphisms of compact metrizable groups, and it was recently extended to all compact groups in [22].

5.1 Basic properties of the algebraic entropy

We list here most of the basic properties of the algebraic entropy, for which we refer to [15, Section 5] (see [18, Section 2] in the abelian case).

First we see that the algebraic entropy is an invariant for groups and group endomorphisms:

Lemma 5.1 (Invariance under conjugation) *Let* G, H *be groups and let* $\phi\colon G \to G, \eta\colon H \to H$ *be endomorphisms. If* ϕ *and* η *are conjugated (i.e., there exists an isomorphism* $\xi\colon G \to H$ *such that* $\eta = \xi \circ \phi \circ \xi^{-1}$*), then* $h(\phi) = h(\eta)$*.*

Next we recall the properties of monotonicity with respect to subgroups and quotients:

Lemma 5.2 (Monotonicity) *Let* G *be a group, let* $\phi\colon G \to G$ *be an endomorphism and let* H *be a* ϕ*-invariant subgroup of* G*. Then:*

(1) $h(\phi) \geqslant h(\phi\restriction_H)$;

(2) *if* H *is also normal,* $h(\phi) \geqslant h(\overline{\phi}_{G/H})$.

The following property is another typical property of entropy functions.

Lemma 5.3 (Logarithmic Law) *Let* G *be a group and let* $\phi\colon G \to G$ *be an endomorphism. For every* $k \in \mathbb{N}_+$, $h(\phi^k) = k\,h(\phi)$*. If* ϕ *is an automorphism, then* $h(\phi^k) = |k|\,h(\phi)$ *for every* $k \in \mathbb{Z} \setminus \{0\}$*.*

Corollary 5.4 *Let* G *be a group and let* $\phi\colon G \to G$ *be an endomorphism.*

(1) $h(\phi) = 0$ *if and only if* $h(\phi^k) = 0$ *for some* $k \in \mathbb{N}_+$*;*

(2) $h(\phi) = \infty$ *if and only if* $h(\phi^k) = \infty$ *for some* $k \in \mathbb{N}_+$*.*

Moreover, in case $\phi = \phi^k$ for some $k \in \mathbb{Z}$, $|k| > 1$ (e.g., for $\phi = id_G$), either $h(\phi) = 0$ or $h(\phi) = \infty$.

Lemma 5.5 (Continuity) *Let* G *be a group and let* $\phi\colon G \to G$ *be an endomorphism. If* G *is a direct limit of* ϕ*-invariant subgroups* $\{G_i : i \in I\}$*, then* $h(\phi) = \sup_{i \in I} h(\phi\restriction_{G_i})$*.*

The following is a first instance of the Addition Theorem in a special case.

Lemma 5.6 (Weak Addition Theorem) *Let G be a group and $\phi\colon G \to G$ an endomorphism. If $G = G_1 \times G_2$, $\phi = \phi_1 \times \phi_2$ with $\phi_i\colon G_i \to G_i$ an endomorphism, $i = 1, 2$, then $h(\phi_1 \times \phi_2) = h(\phi_1) + h(\phi_2)$.*

The following example is the main one in entropy theory, that is, the Bernoulli shift.

Example 5.7 ([15], Example 5.2.1) *Let K be a group and $G = \bigoplus_{\mathbb{N}} K$. The right Bernoulli shift $\beta_K\colon G \to G$ is defined by*

$$(x_0, x_1, x_2 \ldots) \mapsto (e_G, x_0, x_1, \ldots).$$

Then

$$h(\beta_K) = \log |K|,$$

with the usual convention that $\log |K| = \infty$, if $|K|$ is infinite.

5.2 The abelian case

For abelian groups it is known that the Addition Theorem holds.

Theorem 5.8 ([18], Theorem 1.1) *If G is an abelian group, then $AT(G)$ holds.*

The following fundamental property of the algebraic entropy in the abelian context was used in the proof of the above Addition Theorem and also in the proof of Theorem 4.8.

Proposition 5.9 ([18], Proposition 3.7) *Let G be an abelian group and let $\phi\colon G \to G$ be an endomorphism. Denote by $D(G)$ the divisible hull of G and by $\widetilde{\phi}\colon D(G) \to D(G)$ the extension of ϕ to $D(G)$. Then $h(\phi) = h(\widetilde{\phi})$.*

In fact, the Addition Theorem in the torsion case was proved in [21] and then one can restrict to the torsion free case by using the fact that the torsion part $t(G)$ of an abelian group G is fully invariant and the quotient $G/t(G)$ is torsion free. Another reduction is to the case of torsion free abelian groups of finite free rank. At this point the divisible hull of such a group G is isomorphic to $\mathbb{Q}^n$ with $n = rk(G)$, and Proposition 5.9 applies.

At this stage, the relevant part of the proof of the Addition Theorem is based on the so-called Algebraic Yuzvinski Formula (Theorem 5.10), directly proved in [30]. To state the Algebraic Yuzvinski Formula, we need to recall the notion of Mahler measure, playing an important role in number theory and arithmetic geometry.

For a primitive polynomial with integer coefficients

$$f(t) = a_0 + a_1 t + \ldots + a_k t^k \in \mathbb{Z}[t],$$

let $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ be the roots of $f(t)$ taken with their multiplicity. The Mahler measure of $f(t)$ is

$$m(f(t)) := \log|a_k| + \sum_{|\alpha_i|>1} \log|\alpha_i|.$$

For $g(t) \in \mathbb{Q}[t]$ the characteristic polynomial of ϕ, there exists a smallest $s \in \mathbb{N}_+$ such that $sg(t) \in \mathbb{Z}[t]$ (so $sg(t)$ is primitive); the *Mahler measure of* ϕ is $m(\phi) := m\big(sg(t)\big)$.

Theorem 5.10 (Algebraic Yuzvinski Formula) *Let* $n \in \mathbb{N}_+$ *and let* $\phi\colon \mathbb{Q}^n \to \mathbb{Q}^n$ *be an endomorphism. Then* $h(\phi) = m(\phi)$.

This equality is the counterpart of the same result for the topological entropy proved by Yuzvinski [59].

The Algebraic Yuzvinski Formula establishes a connection between the algebraic entropy and the Mahler measure, so that the celebrated Lehmer Problem in number theory can be stated in terms of the values of the algebraic entropy.

Problem 5.11 *Is* $\mathfrak{L} := \inf\big(\{h(\phi) : \phi \in \mathrm{End}(G), G$ *abelian group*$\} \setminus \{0\}\big)$ *equal to* 0*?*

By [18, Theorem 1.5], $\mathfrak{L} = \inf\big(\{h(\phi) : \phi \in \mathrm{Aut}(\mathbb{Q}^n), n \in \mathbb{N}_+\} \setminus \{0\}\big)$.

For more properties and a deeper discussion on this topic, see [18] and [15, Section 5.5]. In particular, a positive answer to Problem 5.11 would imply that

$$\big\{h(\phi) : \phi \in \mathrm{End}(G), G \text{ abelian group}\big\} = \mathbb{R}_{\geqslant 0} \cup \{\infty\},$$

that is, each real number could be realized as the algebraic entropy of some group endomorphism. On the other hand, [18, Theorem 1.7] shows that $\mathfrak{L} > 0$ precisely when for every abelian group G and every endomorphism $\phi\colon G \to G$ with $h(\phi)$ finite, there exists $F \in \mathcal{F}(G)$ such that $h(\phi) = H(\phi, F)$.

We conclude this part by recalling the so called Uniqueness Theorem, showing that the algebraic entropy is a so natural invariant that it is the unique possible one with the properties listed above. For an abelian group G, we denote by $\mathrm{End}(G)$ the ring of its endomorphisms.

Theorem 5.12 ([18], Theorem 1.3) *The algebraic entropy* h *is the unique collection* $h = \{h_G : G$ *abelian group*$\}$ *of functions*

$$h_G \colon \mathrm{End}(G) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$$

such that:

(1) *if* $\phi \in \mathrm{End}(G)$ *and* $\eta \in \mathrm{End}(H)$ *are conjugated, then* $h_G(\phi) = h_H(\eta)$ (*i.e.,* h *is invariant under conjugation*);

(2) *if* G *is an abelian group,* $\phi \in \mathrm{End}(G)$ *and* G *is a direct limit of* ϕ*-invariant subgroups* $\{G_i : i \in I\}$*, then* $h_G(\phi) = \sup_{i \in I} h_{G_i}(\phi \restriction_{G_i})$*;*

(3) *for every abelian group* G, $\mathrm{AT}(G)$ *holds;*

(4) $h_{\bigoplus_{\mathbb{N}} K}(\beta_K) = \log|K|$ *for any finite abelian group* K*;*

(5) $h_{\mathbb{Q}^n}(\phi) = m(\phi)$ *for every* $n \in \mathbb{N}_+$ *and every* $\phi \in \mathrm{End}(\mathbb{Q}^n)$.

This is the counterpart of the Uniqueness Theorem proved by Stoyanov [50] for the topological entropy of continuous endomorphisms of compact groups.

5.3 The Addition Theorem in the non-abelian case

The next example shows that the Addition Theorem does not hold in general, even for metabelian (and so solvable) groups.

Example 5.13 ([27], Example 2.7) *Consider* $G = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \rtimes \mathbb{Z}$ *the lamplighter group and* $H = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$*. It is known that* G *has exponential growth, and so* $h(\mathrm{id}_G) = \infty$ *by* Example 3.14 (2)*. On the other hand,* H *and* $G/H \simeq \mathbb{Z}$ *are abelian groups, hence* $h(\mathrm{id}_H) = 0$ *and* $h(\mathrm{id}_{G/H}) = 0$*, as recalled in* Example 3.14 (1)*. In particular,* $\mathrm{AT}(G)$ *does not hold.*

This example answers [15, Question 5.2.12(b)], and so it also answers [15, Problem 5.2.10]. Anyway, we believe in the validity of the following conjecture.

Conjecture 5.14 *If* G *is a locally virtually nilpotent group, then* $\mathrm{AT}(G)$ *holds.*

In particular, we conjecture that *if* G *is a locally finite group, then* $\mathrm{AT}(G)$ *holds*. This case was studied in several papers under additional assumptions, as we describe below.

Remark 5.15 In the class of locally finite groups the computation of the algebraic entropy becomes more comfortable. Indeed, denoting by

$$\mathcal{F}_s(G) := \{F \leqslant G : F \text{ is finite}\} \subseteq \mathcal{F}(G)$$

the family of all finite subgroups of a group G, we have that G is locally finite precisely when $\mathcal{F}_s(G)$ is cofinal in $\mathcal{F}(G)$ with respect to the order given by the inclusion. So, for a locally finite group G and an endomorphism $\phi\colon G \to G$, we have

$$h(\phi) = \sup\left\{H(\phi, F) : F \in \mathcal{F}_s(G)\right\}$$

(see Remark 3.11).

In [25], the Addition Theorem was proved in the special case of locally finite groups that are finitely quasi-hamiltonian (see Definition 5.16 and Theorem 5.19). Indeed, it is convenient that not only $F \in \mathcal{F}(G)$ is a subgroup, but also that each $T_n(\phi, F)$ is a subgroup of G for every F in a cofinal subfamily of $\mathcal{F}_s(G)$. For a group G, let

$$\mathcal{F}_C(G) := \left\{F \in \mathcal{F}_s(G) : FE = EF \text{ for all } E \in \mathcal{F}_s(G)\right\} \subseteq \mathcal{F}_s(G).$$

Definition 5.16 A group G is *finitely quasihamiltonian* if $\mathcal{F}_C(G)$ is cofinal in $\mathcal{F}_s(G)$.

Remark 5.17 Recall that a group G is *quasihamiltonian* if all its subgroups are permutable; the quasihamiltonian groups are called also *Iwasawa groups* because the structure of those groups was described by Iwasawa [37] (some gaps in the proof were filled by Napolitani [42]). Clearly, if G is quasihamiltonian, then $\mathcal{F}_C(G) = \mathcal{F}_s(G)$; hence, *every quasihamiltonian group is finitely quasihamiltonian*. Every torsion finitely quasihamiltonian group is locally finite.

Another class of groups with this property is that of FC-*groups*, that is, groups in which each element has only finitely many conjugates. Indeed, by [46, Theorem 14.5.8], a group G is a torsion FC-group if and only if G is locally finite and normal, that is, every finite subset of G is contained in a normal finite subgroup of G: in other words, the family of all finite normal subgroups of a group G, which is contained in $\mathcal{F}_C(G)$, is cofinal in $\mathcal{F}(G)$. Therefore, *every torsion* FC-*group is finitely quasihamiltonian*. We refer to [25, Example 2.1] for examples witnessing in particular that the above two implications cannot be reversed in general, and that there are locally finite groups that are not finitely quasihamiltonian (e.g., the group $\mathcal{S}_{fin}(\mathbb{N}_+)$ of permutations of $\mathbb{N}_+$ with finite support).

For a quasihamiltonian locally finite group G, we have seen that $\mathcal{F}_C(G)$ is cofinal in $\mathcal{F}(G)$, so Remark 3.11 and [25, Lemma 2.4] give the following result.

Proposition 5.18 *Let G be a quasihamiltonian locally finite group and let $\phi\colon G \to G$ be an endomorphism. Then*

$$h(\phi) = \sup\left\{H(\phi, F) : F \in \mathcal{F}_C(G)\right\},$$

where, for every $F \in \mathcal{F}_C(G)$, each $T_n(\phi, F)$ is a subgroup of G.

This property is one of the main ingredients in the proof of the following Addition Theorem.

Theorem 5.19 ([25], Theorem 1.4) *If G is a finitely quasihamiltonian locally finite group, then $AT(G)$ holds.*

This result extends a consequence of [26, Corollary 7.2] for torsion FC-groups and one of the main results from [57], namely, that $AT(G)$ holds for every locally finite group G which is a quasihamiltonian FC-group. The proof of Theorem 5.19 is inspired by ideas in [12]. Moreover, it is shorter than that in [21] and it follows a different path. The proofs of the mentioned results from [26] and [57] use a third different approach inspired by that in [6, 7, 31, 48].

Recently Shlossberg [49] proved the following instance of the Addition Theorem for another class of locally finite groups.

Theorem 5.20 ([49], Theorem 4.6) *If G is a torsion nilpotent group of nilpotency class 2, then $AT(G)$ holds.*

The proof of this result follows partially the one of Theorem 5.19, but it is based also on the following interesting reduction, answering [15, Question 5.2.11(c) and Question 5.2.12(c)]. For a group G, an endomorphism $\phi\colon G \to G$ and a ϕ-invariant normal subgroup H of G, we write $AT(G, \phi, H)$ in case $h(\phi) = h\big(\phi\restriction_H\big) + h\big(\overline{\phi}_{G/H}\big)$.

Theorem 5.21 ([49], Theorem 4.6) *Let $\mathfrak{X}$ be a class of solvable groups closed under taking subgroups and quotients.*

(1) $AT(G)$ *holds for every* $G \in \mathfrak{X}$, *in case for every* $G \in \mathfrak{X}$ *and every endomorphism* $\phi\colon G \to G$, $AT\big(G, \phi, G'\big)$ *holds;*

(2) *if* $\mathrm{AT}\big(G, \phi, Z(G)\big)$ *holds for every nilpotent group* $G \in \mathfrak{X}$ *and every automorphism* $\phi\colon G \to G$*, then* $\mathrm{AT}(G, \phi, H)$ *holds for every nilpotent group* $G \in \mathfrak{X}$*, every automorphism* $\phi\colon G \to G$ *and every* ϕ*-stable normal subgroup* H *of* G*.*

Finally, the following result shows in particular that the Addition Theorem holds for the group $\mathcal{S}_{\mathtt{fin}}(\mathbb{N}_+)$, which is not covered by Theorem 5.19 as it is locally finite but not finitely quasihamiltonian.

Theorem 5.22 ([49], Corollary 5.3) *If* G *is a locally finite group admitting a fully characteristic finite index simple subgroup, then* $\mathrm{AT}(G)$ *holds.*

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