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Rational Class Sizes and Their Implications for the Structure of Finite Groups

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Abstract

This paper considers the implications of rational class sizes for the structure of finite groups. Ever since Itô introduced the notion of a conjugate type vector in 1953, the problem of unraveling the connections between the set of conjugacy class sizes and the structure of a finite group has been widely studied. In this study, we set out to consider a similar problem concerning the set of rational class sizes of finite groups.

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1 Introduction

There are interesting instances of recognizing structural properties of a finite group, including solvability, nilpotency, etc. based on the set of conjugacy class sizes. The survey [1] is an interesting read that brings together many results about the influence of the conjugacy class sizes, namely the set $cs(G)$, on the structure of a finite group G.

In this study, we set out to consider a similar problem concerning the set $cs_{rat}(G)$ consisting of the sizes of rational classes of a finite group G. With regard to sets that can possibly occur as $cs_{rat}(G)$ for a finite group G, we will show in Theorem 3.5 that if G is a finite group with only two rational class sizes, then either $cs_{rat}(G) = \{1, p-1\}$, for some prime p, or $cs_{rat}(G) = \{1, 2^n\}$ for some $n \geqslant 1$.

With regard to identifying structural properties based on rational class sizes, we will show in Theorem 3.10 that finite groups with exactly two rational class sizes are nilpotent. Another instance of recognizing nilpotency is Theorem 3.8, where we prove that if $cs_{rat}(G)$ consists entirely of powers of 2, then G is nilpotent.

2 Basic definitions, notations, and results

Let G be a finite group and let ϵ be a primitive $|G|$-th root of unity. Let $\sigma \in Gal(\mathbb{Q}(\epsilon)/\mathbb{Q})$. Then there exists $n \in \mathbb{Z}$ such that $(n, |G|) = 1$ and $\sigma(\epsilon) = \epsilon^n$. This gives rise to the following action of $Gal(\mathbb{Q}(\epsilon)/\mathbb{Q})$ on G. For $g \in G$ and $\sigma \in Gal(\mathbb{Q}(\epsilon)/\mathbb{Q})$ we define $g^\sigma = g^n$.

Next, we consider the following action of $G \times Gal(\mathbb{Q}(\epsilon)/\mathbb{Q})$ on G. For $(g, \sigma) \in G \times Gal(\mathbb{Q}(\epsilon)/\mathbb{Q})$ and $h \in G$ we define:

$$h^{(g,\sigma)} = (h^g)^\sigma$$

The *rational classes* of G are the orbits of the above action of $G \times Gal(\mathbb{Q}(\epsilon)/\mathbb{Q})$ on G. If $g \in G$, the conjugacy class of g and the rational class of g will be denoted by g^G and g^G_{rat}, respectively. Also, we will use $cs_{rat}(G)$ to denote the set that comprises the sizes of rational classes of G.

One can observe that the rational class of $g \in G$ consists of all the elements that are conjugate to g or to an m-th power of g, where m is coprime to the order of g. An element $g \in G$ is called a *rational element* if it is conjugate to the m-th power of g, for all m coprime to the order of g. In other words, g is a rational element precisely when $g^G = g^G_{rat}$.

Example 2.1 *Let* p *be a prime and let* G *be the cyclic group of order* p. *Then* $cs_{rat}(G) = \{1, p-1\}$

Example 2.2 *Let* $G = S_n$ *be the symmetric group of degree* $n \geqslant 3$. *Then* $cs(G) = cs_{rat}(G)$. *This follows from the fact that the* k*-th power on an* n*-cycle is also an* n*-cycle if and only if* $(k, n) = 1$.

Example 2.3 *The Dihedral group* D_8 *and the Quaternion group* Q_8 *are nonisomorphic groups of order* 8 *that share the same set of rational class sizes. More precisely,* $cs_{rat}(Q_8) = cs_{rat}(D_8) = \{1, 2\}$.

Lemma 2.4 *Let* G *be a finite group, and let* $g \in G$*. Then the size of the rational class of* g *could be calculated as follows:*

$$\left|g^G_{rat}\right| = \phi\left(|\langle g\rangle|\right)\left|G : N_G\left(\langle g\rangle\right)\right|$$

where ϕ *represents Euler's totient function.*

Proof — To calculate the rational class of an element $g \in G$, we note that there exist exactly $|G : N_G(\langle g\rangle)|$ cyclic subgroups that are conjugate to $\langle g\rangle$, and each such cyclic subgroup has exactly $\phi\left(|\langle g\rangle|\right)$ generators. □

Lemma 2.5 *Let* G *be a finite group and let* $g \in G$ *be such that* $|g^G_{rat}|$ *is odd. Then* $g^2 = 1$*.*

Proof — By Lemma 2.4 it follows that $\phi\left(|\langle g\rangle|\right)$ must be an odd number, and thus $\phi\left(|\langle g\rangle|\right) = 1$. Therefore $o(g) \in \{1, 2\}$, and thus we have $g^2 = 1$. □

Lemma 2.6 *Let* G *be a finite group and let* $g \in G$ *be an element of order* n*. Then we have*

$$\left|g^G\right| \;\Big|\; \left|g^G_{rat}\right| \;\Big|\; \phi(n)\left|g^G\right|$$

Proof — Let $g \in G$. The rational class of g equals

$$\bigcup_{\langle y\rangle = \langle g\rangle} y^G,$$

which clearly contains g^G. Firstly, we note that if an element $y \in G$ has the property $\langle y\rangle = \langle g\rangle$, then $C_G(g) = C_G(y)$, and so, $|y^G| = |g^G|$. Thus, the size of the rational class of g is a multiple of the size of the conjugacy class containing g. Next, we observe that by Lemma 2.4, we have

$$|g^G_{rat}| = \phi\left(|\langle g\rangle|\right)\left|G : N_G(\langle g\rangle)\right|,$$

which divides $\phi(|\langle g\rangle|)\left|G : C_G(g)\right|$, since $C_G(g) \leqslant N_G\left(\langle g\rangle\right)$. □

Corollary 2.7 *Let* H *be a normal subgroup of a finite group* G*. If* $x \in H$ *then* $\left|x^H_{rat}\right|$ *divides* $\left|x^G_{rat}\right|$*.*

Proof — Let $x \in H$. We have

$$\left|H : N_H\left(\langle x\rangle\right)\right| = \left|H : H \cap N_G\left(\langle x\rangle\right)\right| = \left|HN_G\left(\langle x\rangle\right) : N_G\left(\langle x\rangle\right)\right|,$$

which divides $\left|G : N_G\left(\langle x\rangle\right)\right|$. Therefore $\left|x_{rat}^H\right| = \phi\left(|\langle x\rangle|\right)\left|H : N_H\left(\langle g\rangle\right)\right|$ divides $|x_{rat}^G| = \phi\left(|\langle x\rangle|\right)\left|G : N_G\left(\langle x\rangle\right)\right|$, as desired. □

Corollary 2.8 *Let* H *be a normal subgroup of a finite group* G. *If* $x \in G$, *then* $\left|(xH)_{rat}^{G/H}\right|$ *divides* $\left|x_{rat}^G\right|$.

Proof — Let $x \in G$. We note that $N_G\left(\langle x\rangle\right)H/H \leqslant N_{G/H}\left(\langle xH\rangle\right)$. Therefore

$$\left|G/H : N_{G/H}\left(\langle xH\rangle\right)\right| \quad | \quad \left|G/H : N_G(\langle x\rangle)H/H\right| = \left|G : N_G\left(\langle x\rangle\right)H\right|$$

which divides $\left|G : N_G\left(\langle x\rangle\right)\right|$. Also, we note that $|\langle xH\rangle|$ divides $|\langle x\rangle|$, and, considering that Euler's function preserves divisibility, $\phi\left(|\langle xH\rangle|\right)$ divides $\phi\left(|\langle x\rangle|\right)$. Therefore

$$\left|(xH)_{rat}^{G/H}\right| = \phi\left(|\langle xH\rangle|\right)\left|G/H : N_{G/H}\left(\langle xH\rangle\right)\right|$$

divides $\phi\left(|\langle x\rangle|\right)\left|G : N_G\left(\langle x\rangle\right)\right| = \left|x_{rat}^G\right|$. □

A well known useful fact about the sizes of conjugacy classes is that if x and y are elements of a finite group G with $C_G(x)C_G(y) = G$, then $(xy)^G = x^G y^G$. The following lemma is a similar result for rational classes in a finite group.

Lemma 2.9 *Let* x *and* y *be two elements of coprime order in a finite group* G. *If* $xy = yx$ *and* $N_G\left(\langle x\rangle\right)N_G\left(\langle y\rangle\right) = G$, *then* $(xy)_{rat}^G = x_{rat}^G y_{rat}^G$.

Proof — Let $t \in (xy)_{rat}^G$. Then $t = ((xy)^m)^g$, where $(m, |G|) = 1$ and $g \in G$. Since $xy = yx$ and x and y have coprime orders, it follows that $t = (x^m)^g(y^m)^g$. Hence, $t \in x_{rat}^G y_{rat}^G$ and $(xy)_{rat}^G \leqslant x_{rat}^G y_{rat}^G$. Conversely, let $t \in x_{rat}^G y_{rat}^G$. Therefore, there exists $g, h \in G$ such that

$$t = (x^i)^g(y^j)^h,\ (i, |G|) = 1,\ (j, (|G|) = 1.$$

Now, note that $t = (x^i)^g(y^j)^h$ is conjugate to the element $(x^i)^{gh^{-1}}(y^j)$. Using the assumption that $N_G\left(\langle x\rangle\right)N_G\left(\langle y\rangle\right) = G$, it is possible to write $gh^{-1} = ab$, where $a \in N_G\left(\langle x\rangle\right)$ and $b \in N_G\left(\langle y\rangle\right)$. It follows that

$$(x^i)^{gh^{-1}}(y^j) = (x^i)^{ab}(y^j)$$

is conjugate to $(x^i)^a(y^j)^{b^{-1}} \in \langle x\rangle\langle y\rangle = \langle xy\rangle$ (the equality holds since x and y are elements of coprime order). So we have shown $\langle xy\rangle$ contains a conjugate of t and hence $t \in \langle xy\rangle^k$ for some $k \in G$. Finally, we

note that

$$o(t) = o\big((x^i)^a (y^j)^{b^{-1}}\big) = o(xy),$$

since $a \in N_G\big(\langle x\rangle\big)$, $b \in N_G\big(\langle y\rangle\big)$, $\big(i, (o(x)\big) = 1$, $\big(j, (o(y)\big) = 1$, $xy = yx$ and $\big(o(x), o(y)\big) = 1$. This shows that $t \in (xy)^G_{rat}$. □

Corollary 2.10 *Let x and y be two elements of coprime order in a finite group G such that $[x, y] = 1$ and $|x^G_{rat}|$ and $|y^G_{rat}|$ are coprime. Then we have $(xy)^G_{rat} = x^G_{rat} y^G_{rat}$.*

Proof — Let $x, y \in G$ be such that $|x^G_{rat}|$ and $|y^G_{rat}|$ are coprime. Then using Lemma 2.4, the indices $\big|G : N_G\big(\langle x\rangle\big)\big|$ and $\big|G : N_G\big(\langle y\rangle\big)\big|$ must also be coprime. Thus $N_G\big(\langle x\rangle\big)N_G\big(\langle y\rangle\big) = G$. The result now follows from Lemma 2.9. □

3 The influence of rational class sizes on the structural properties

Proposition 3.1 *Let G be a finite group. We have $cs_{rat}(G) = \{1\}$ if and only if G is an elementary abelian 2-group.*

Proof — Let G be a finite group such that $cs_{rat}(G) = \{1\}$. Using Lemma 2.4, we see that every non-identity element of G has to be an involution, and thus G is an elementary abelian 2-group. Conversely, let G be an elementary abelian 2-group. If $g \in G$ is a non-identity element then it must be an involution. We also note that $\big|G : N_G\big(\langle g\rangle\big)\big| = 1$. Now using Lemma 2.4, we get $|g^G_{rat}| = 1$, and hence $cs_{rat}(G) = \{1\}$. □

Lemma 3.2 *Let G be a finite group such that $cs_{rat}(G)$ consists entirely of odd numbers. Then we must have $cs_{rat}(G) = \{1\}$.*

Proof — Let G be a finite group such that $cs_{rat}(G)$ entirely consists of odd numbers. Let $g \in G$ be an arbitrary element. Then by assumption the rational class of g has odd size, and by Lemma 2.5, we must have $g^2 = 1$. This implies G is an elementary abelian 2-group and hence we must have $cs_{rat}(G) = \{1\}$. □

Lemma 3.3 *Let G be a finite 2-group. Then $cs_{rat}(G)$ consists of powers of 2.*

Proof — This follows immediately from Lemma 2.4. □

Lemma 3.4 *Let G be a finite group with two rational class sizes. Suppose $cs_{rat}(G) = \{1, m\}$. If p is a prime common divisor of $|G|$ and m, then $p = 2$.*

Proof — Let $P \in Syl_p(G)$. Pick $g \in Z(P)$ such that $o(g) = p$. Then

$$\left|g^G_{rat}\right| = \phi(p)\left|G : N_G\left(\langle g\rangle\right)\right| = (p-1)\left|G : N_G\left(\langle g\rangle\right)\right|.$$

If $p > 2$ the above equality cannot hold true since p divides $|g^G_{rat}|$ but $p \nmid (p-1)\left|G : N_G\left(\langle g\rangle\right)\right|$. Thus we have $p = 2$. □

Theorem 3.5 *Let G be a finite group with exactly two rational class sizes. Then either $cs_{rat}(G) = \{1, p-1\}$, for some prime p, or $cs_{rat}(G) = \{1, 2^n\}$ for some $n \geqslant 1$.*

Proof — Suppose we have $cs_{rat}(G) = \{1, m\}$, where m is not a power of 2 and $m+1$ is not a prime. Let $Q \in Syl_2(G)$ and let $a \in Q$. Suppose we have $o(a) = 2^n$. Then $|a^G_{rat}| = 2^{n-1}\left|G : N_G\left(\langle a\rangle\right)\right|$. Using Lemma 3.4, we see that $\left|G : N_G\left(\langle a\rangle\right)\right|$ has to be a power of 2, and consequently, $|a^G_{rat}|$ is a power of 2. Now since $|a^G_{rat}| \in \{1, m\}$ and m is not a power of 2, it follows that $|a^G_{rat}| = 1$, and hence $a^2 = 1$. This shows $Q \leqslant Z(G)$ is elementary abelian.

Next, we note that by Lemma 3.2, m has to be even, and since m is not a power of 2, it follows that there exists an odd prime q such that $q|m$. Since q is odd it follows by Lemma 3.4 that $q \nmid |G|$. Let $g \in G$ be such that $|g^G_{rat}| = m$. Then

$$m = |g^G_{rat}| = \phi\big(o(g)\big)\left|G : N_G\left(\langle g\rangle\right)\right|,$$

which implies $q \,\big|\, \phi\big(o(g)\big) \,\big|\, \phi\big(|G|\big)$. Since $q \,\big|\, \phi\big(|G|\big)$ and $q \nmid |G|$, there exists a prime p such that $p \,|\, |G|$ and $q \,|\, p-1$. We note that p and q are both odd primes, and so $q \neq p-1$. Let $P \in Syl_p(G)$, and pick $h \in Z(P)$ such that $o(h) = p$. We have

$$|h^G_{rat}| = (p-1)\left|G : N_G\left(\langle h\rangle\right)\right|.$$

Since p is an odd prime we have $p-1 > 1$ and thus

$$|h^G_{rat}| = (p-1)\left|G : N_G\left(\langle h\rangle\right)\right| = m.$$

Now if $\langle h\rangle \trianglelefteq G$, then it follows that $m = p-1$, which cannot happen. Thus $\langle h\rangle$ is not normal in G. Using Lemma 3.4, we see that the

index $\left|G : N_G\left(\langle h\rangle\right)\right| \neq 1$ has to be a power of 2, which contradicts our earlier observation that the Sylow 2-subgroup is central, and this completes the proof. □

Lemma 3.6 *Let* G *be a nonabelian finite group and let* $g \in G$ *such that* $|g^G_{rat}|$ *is a prime power greater than 1. Then* G *is not simple.*

Proof — Let $g \in G$ such that $|g^G_{rat}|$ is a prime power greater than 1. Then as we have seen in Lemma 2.6, $|g^G|$ divides $|g^G_{rat}|$, and therefore $|g^G|$ is also a prime power. Now if $|g^G| = 1$, then g, which is a nontrivial element, lies in the center of G, and thus $Z(G) \neq 1$, and considering G is not abelian, we conclude G is not simple. If $|g^G|$ is a prime power greater than 1, then by the Burnside's result G is not simple. □

Proposition 3.7 *Let* G *be a finite group. If all rational class sizes of* G *are either odd numbers or powers of 2, then* G *is solvable.*

Proof — Assume by way of contradiction that G is a counterexample of minimal order. Let N be a maximal normal subgroup of G. Then by Corollary 2.7, $|x^N_{rat}|$ divides $|x^G_{rat}|$, for every element $x \in N$. Therefore all rational class sizes of N are either odd numbers or powers of 2, and by minimality of G, it follows that N is solvable. Next we consider G/N. Since N is a maximal normal subgroup, it follows that G/N is a simple group, and by assumption, since G is not solvable, G/N must be a nonabelian simple group. Using Corollary 2.8, it follows that all rational class sizes of G/N are either odd numbers or powers of 2. If all rational class sizes of G/N are odd numbers, then by Lemma 3.2, we must have $cs_{rat}(G/N) = \{1\}$ which implies G/N is abelian, which is a contradiction. Therefore there exists an element in G/N whose size of rational class is a power of 2 greater than 1. Using Lemma 3.6, it follows that G/N is not simple, which is a contradiction. □

Theorem 3.8 *Let* G *be a finite group such that* $cs_{rat}(G)$ *consists entirely of powers of 2. Then* G *is nilpotent.*

Proof — Let G be a counterexample of minimum order. Then $G \neq 1$. Suppose $Z(G) \neq 1$. Then by Corollary 2.8, $G/Z(G)$ satisfies the hypothesis and hence is nilpotent. This implies G is nilpotent, which is a contradiction. Therefore $Z(G) = 1$. Let M be a maximal normal subgroup of G. By Proposition 3.7, we know that G is solvable, and hence $|G : M| = p$, for some prime p. This implies G/M is an abelian

group of order p. Next, we observe that by 2.7, the subgroup M satisfies the hypothesis and hence is nilpotent. We notice that $M \neq 1$, for otherwise G would be abelian which is not the case. Thus $Z(M) \neq 1$. Let N be a minimal normal subgroup of G with $N \leqslant Z(M)$. Let x be a nontrivial element of N. We observe that $M \leqslant C_G(x)$ and thus, we have

$$\left|x^G\right| = \left|G : C_G(x)\right| \mid |G : M| = p.$$

Since $Z(G) = 1$, x is not a central element and we have $|x^G| = p$. Using Lemma 2.6, it follows that $|x^G_{rat}|$ is divisible by p; however, by assumption $|x^G_{rat}|$ is a power of 2. This implies $p = 2$. Considering G is a solvable group, we have that N is elementary abelian and we can assume $|N| = q^m$, for some prime q and for some m. Since $N \leqslant Z(M)$ we consider the action of G/M on N and using the orbit-stabilizer theorem together with the fact that $Z(G) = 1$, it follows that the size of the orbit of any nonidentity element of N is 2, and hence N is of odd order. Let $y \in G$ be a 2-element such that $y \notin M$. Suppose $C_G(y)$ contains a Sylow q-subgroup Q of G. Then Q contains a conjugate of N so $N \leqslant Q \leqslant C_G(y)$, a contradiction. This implies $q \mid |G : C_G(y)|$ and thus $q \mid |y^G_{rat}|$, which is the desired contradiction. □

Proposition 3.9 *Let G be a finite group of odd order. Suppose G has exactly two rational class sizes. Then $cs_{rat}(G) = \{1, p-1\}$, where p is an odd prime that divides $|G|$. Furthermore, G is a p-group.*

Proof — Suppose we have $cs_{rat}(G) = \{1, m\}$. Let p be a prime divisor of $|G|$, $P \in Syl_p(G)$, and pick $g \in Z(P)$ such that $o(g) = p$. Then we have

$$|g^G_{rat}| = \phi(p)\left|G : N_G\left(\langle g\rangle\right)\right| = (p-1)\left|G : N_G\left(\langle g\rangle\right)\right|.$$

Since p is an odd prime, $|g^G_{rat}| \neq 1$, and thus $m = |g^G_{rat}|$. Assume $\langle g\rangle$ is not normal in G. Then $\left|G : N_G\left(\langle g\rangle\right)\right| \neq 1$. Also since $g \in Z(P)$, it follows that $p \nmid \left|G : N_G\left(\langle g\rangle\right)\right|$. Let q be a prime divisor of the index $\left|G : N_G\left(\langle g\rangle\right)\right|$. Then q divides both $|G|$ and m, and by Lemma 3.4, we must have $q = 2$. This cannot happen since G is a group of odd order. Thus we must have $\langle g\rangle \trianglelefteq G$. This implies

$$m = |g^G_{rat}| = (p-1)\left|G : N_G\left(\langle g\rangle\right)\right| = p - 1.$$

In particular $p = m + 1$, and G is a p-group. □

The following result is an analogue of Itô's result in [2], where it is

shown that a finite group of conjugate rank 1 is nilpotent.

Theorem 3.10 *Let* G *be a finite group with exactly two rational class sizes. Then* G *is nilpotent.*

Proof — Let G be a counterexample with $|G|$ minimum. Then Theorem 3.5 alongside Theorem 3.8 yields that $cs_{rat}(G) = \{1, p-1\}$, where $p-1$ is not a power of 2. Let $O \in Syl_2(G)$ and suppose a is a non-trivial element of O. If $o(a) = 2^n$, then

$$|a^G_{rat}| = 2^{n-1}\big|G : N_G\big(\langle a\rangle\big)\big|.$$

Using Lemma 3.4, we see that $\big|G : N_G\big(\langle a\rangle\big)\big|$ has to be a power of 2, and thus $|a^G_{rat}|$ is a power of 2. Now since $|a^G_{rat}| \in \{1, p-1\}$ and $p-1$ is not a power of 2, it follows that $|a^G_{rat}| = 1$, and hence a is an involution and $a \in Z(G)$. This shows $O \leqslant Z(G)$ is elementary abelian. As the result, we have $G = O \times H$, where H is a group of odd order. Since G is not nilpotent, H is not nilpotent. We also note that $cs_{rat}(G) = cs_{rat}(H)$, since O is an elementary abelian 2-group. Therefore by minimality we must have $G = H$; in particular, G has odd order. Now using Proposition 3.9, it follows that G is nilpotent. This contradiction completes the proof. □

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