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Sum of the Powers of the Orders of Elements in Finite Abelian Groups

Subhrajyoti Saha

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Abstract

Let G be a finite group and let $\psi(G)$ denote the sum of element orders of G; in general $\psi^l(G)$ denotes the the sum of the l-th powers of the element orders G where l is a positive integer. We further generalise this by introducing $\Psi_l(G)$ for negative integers l. Motivated by the recursive formula for $\psi(G)$, we consider a finite abelian group G and obtain a similar formula for $\psi^l(G)$ and $\Psi_l(G)$ for $l \in (0,\infty) \cap \mathbb{Z}$ and $l \in (-\infty, 0) \cap \mathbb{Z}$ respectively.

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1 Introduction

Let G be a finite group. For any non-empty subset S of G, let $\psi(S)$ denote the sum of element orders of S that is

$$\psi(S) = \sum_{x \in S} o(x)$$

where $o(x)$ denotes the order of x in G. This has been introduced in [2] and it is also proved that if C is a finite cyclic group of order same as that of G, then $\psi(G) \leqslant \psi(C)$ and the equality holds if and only if G is cyclic. In [4] and [6] explicit formulas for computing $\psi(G)$

were obtained in case G is abelian. Later in [1] this has further been generalised by introducing the function

$$\psi^{l}(G) = \sum_{x \in G} o(x)^{l} \tag{1.1}$$

for every positive integer l with $\psi^{1} = \psi$. In this paper we further generalise this concept by considering $l \in (-\infty, 0) \cap \mathbb{Z}$ and modifying (1.1) accordingly. For any non-empty subset S of G and $l \in \mathbb{Z} \setminus \{0\}$, we define

$$R_{l}(S) = \sum_{x \in S} o(x)^{l}$$

and if $l \in (-\infty, 0) \cap \mathbb{Z}$ then we write

$$\Psi_{l}(S) = |S|^{-l} R_{l}(S).$$

Note that $l \in (0, \infty) \cap \mathbb{Z}$ then $R_{l}(S) = \psi^{l}(S)$ as defined in [1]. If we take $l \in (-\infty, 0) \cap \mathbb{Z}$, then for any subgroup S of G we find that $\Psi_{l}(S)$ is an integer since $o(x)$ divides $|S|$ for all $x \in S$. Motivated by the formulas in [4], in this paper we consider a finite abelian group G and obtain a similar recursive formula for computing $\psi^{l}(G)$ and $\Psi_{l}(G)$ for $l \in (0, \infty) \cap \mathbb{Z}$ and $l \in (-\infty, 0) \cap \mathbb{Z}$ respectively.

Throughout this paper, p denotes a prime number and $\varphi(n)$ denotes the Euler totient function of the positive integer n. A cyclic group of order n is denoted by C_n. For a group G, the notation $\exp(G)$ denotes the exponent of G which is the smallest positive integer z such that $g^{z} = 1_G$ for all $g \in G$ where 1_G is the identity element of G; without any ambiguity we will denote this identity element as 1. Unless mentioned otherwise, $\psi^{l}(G)$ will implicitly denote that $l \in (0, \infty) \cap \mathbb{Z}$ and similarly $\Psi_{l}(G)$ will implicitly denote that $l \in (-\infty, 0) \cap \mathbb{Z}$.

2 Preliminaries

In this paper we first state some commonly used results; we also generalise some of these results.

Lemma 1 (see [4], Lemma 1.1) *Let n be any positive integer. Then C_n has exactly $\varphi(d)$ elements of order d for each divisor d of n, and hence we have $\psi(C_n) = \sum_{d|n} d\varphi(d)$.*

Theorem 2 (see [3], Lemma 2.5) *Let* H *and* K *be two finite groups. Then we have* $\psi^l(H \times K) \leqslant \psi^l(H)\psi^l(K)$*, with equality if and only if* $\gcd(|H|, |K|) = 1$.

Note that Theorem 2 is a generalisation of [1, Lemma 2.1]. We prove an analogous result in the next theorem for Ψ_l. Our proof is motivated by the proof of [1, Lemma 2.1].

Theorem 3 *Let* H *and* K *be two finite groups and* $l \in \mathbb{Z} \setminus \{0\}$. *Then*

$$R_l(H \times K) \leqslant R_l(H)R_l(K)$$

and the equality if and only if $\gcd(|H|, |K|) = 1$.

Proof — Note that $o((h,k)) \leqslant o(h)o(k)$ for $(h,k) \in H \times K$. Then

$$\begin{aligned} R_l(H \times K) &= \textstyle\sum_{h \in H} \sum_{k \in K} o((h,k))^l \\ &\leqslant \textstyle\sum_{h \in H} \sum_{k \in K} o(h)^l o(k)^l \\ &= \left(\textstyle\sum_{h \in H} o(h)^l\right) \left(\textstyle\sum_{k \in K} o(k)^l\right) \\ &= R_l(K)R_l(K). \end{aligned}$$

In order to see the equality, note that, $\gcd(|H|, |K|) = 1$ if and only if $o\big((g,h)\big) = o(g)o(h)$ for each $h \in H$ and $k \in K$. In that case, following the previous part of the proof, it is straight forward to see that $R_l(H \times K) = R_l(H)R_l(K)$ if and only if $\gcd(|H|, |K|) = 1$. □

3 Explicit formulas for finite abelian groups

In this section, we obtain explicit recursive formulas for $\psi^l(G)$ and $\Psi_l(G)$ where G is a finite abelian group. We present this in different cases. This format is inspired by Section 2 of [4]. We start from a finite cyclic group which will lead to the direct product of a finite cyclic p-group and a p-group. Finally, we will consider a the most general case of finite abelian groups. We will conclude this section by discussing the case $\Psi_{-1}(G)$ as a consequence of our general results. The proofs of our results in this section are motivated by the methods used in [4]. We begin with the following result which is Lemma 2.6 of [4].

Lemma 4 *Let* $H \simeq C_{p^{r_1}} \times C_{p^{r_2}} \times \ldots \times C_{p^{r_n}}$ *where* $1 \leqslant r_1 \leqslant r_j$ *for all* j *with* $2 \leqslant j \leqslant n$. *Then for any* $i \in \{1, \ldots, r_1\}$, *there are* $\left(p^i\right)^n - \left(p^{i-1}\right)^n$ *elements of* H *of order* p^i.

3.1 Finite cyclic groups

Let G be a cyclic group of order n. Then we know that

$$G \simeq C_{m_1} \times \ldots \times C_{m_k},$$

where the $m_1, \ldots, m_k$ are coprime to each other and $n = m_1 \ldots m_k$. The explicit formulas for $\psi(G)$ are obtained for two special cases in [4, Proposition 2.1 and Proposition 2.2]. We generalise these results for $\psi^l(G)$ and $\Psi_l(G)$. Recall that $\psi^l = R_l$ for $l \in (0, \infty) \cap \mathbb{Z}$.

Lemma 5 *Let* G *be a cyclic group of order* p^n *where* p *is a prime number and* n *is a positive integer, then for any non-zero integer* l *we have*

(i) $R_l(G) = 1 + \frac{p-1}{p} \sum_{r=1}^{n} p^{r(l+1)}$, *and*

(ii) $\Psi_l(G) = p^{-nl}(1 + \frac{p-1}{p} \sum_{r=1}^{n} p^{r(l+1)})$.

Proof — (i) Using Lemma 1, we get $R_l(G) = 1 + \sum_{r=1}^{n} \varphi(p^r)p^{lr}$. We know that $\varphi(p^r) = p^r(1 - 1/p)$, thus a straight forward calculation yields the required result.

(ii) This follows directly from the definition of Ψ_l and by using the previous part. □

Lemma 6 *Let* G *be a cyclic group of order* $s = p_1^{r_1} \ldots p_k^{r_k}$ *where the* p_i *are distinct primes with* $r_i \geqslant 1$ *for* $i = 1, \ldots, k$. *Then*

(i) $R_l(G) = \prod_{i=1}^{k} \left(1 + \frac{p-1}{p} \sum_{r=1}^{n} p^{r(l+1)}\right)$, *and*

(ii) $\Psi_l(G) = s^{-l} \prod_{i=1}^{k} \left(1 + \frac{p-1}{p} \sum_{r=1}^{n} p^{r(l+1)}\right)$.

Proof — (i) We know that

$$G \simeq C_{p_1^{r_1}} \times \ldots \times C_{p_k^{r_k}}$$

and p_i are all distinct primes. Hence for any element

$$x = (x_1, \ldots, x_k) \in G,$$

we have $o(x) = o(x_1) \dots o(x_k)$. So, by applying Lemma 5 and Theorem 3 repeatedly, we arrive at the required result.

(ii) This follows directly from the definition of Ψ_l and by using the previous part. □

The next corollary immediately follows from a straight forward calculation using Lemmas 5 and 6 and putting $l = -1$.

Corollary 7 *Let* G *be a cyclic group of order* s.

(i) *If* $s = p^n$ *where* p *is a prime number and* n *is a positive integer, then* $\Psi_{-1}(G) = (n+1)p^n - np^{n-1}$.

(ii) *If* $s = p_1^{r_1} \dots p_k^{r_k}$ *where the* p_i *are distinct primes with* $r_i \geqslant 1$, *then* $\Psi_{-1}(G) = \prod_{i=1}^{k} \Big((r_i+1)p^{r_i} - r_i p_i^{r_i - 1} \Big)$.

3.2 Direct product of a cyclic p-group and a p-group

In this section, we obtain a recursive formula for $\psi^l(G)$ and $\Psi_l(G)$ where G is a direct product of a finite cyclic p-group and any p-group. We begin with the following result (see [4], Lemma 2.3).

Lemma 8 *Let* $G = H \times K$ *where* H *and* K *are* p*-groups. Then for any* (x_1, x_2) *in* G, $o\big((x_1, x_2)\big) = \max\{o(x_1), o(x_2)\}$.

We now prove the following which is a generalisation of [4, Proposition 2.4]. We follow the same methods used in [4].

Proposition 9 *Let* $G = C_{p^r} \times H$ *where* $r \geqslant 1$, *and* H *is a* p*-group with* $\exp(H) \geqslant p^r$. *Let* N_j *be the number of elements in* H *that have order* p^j. *Then*

(i)

$$R_l(G) = \begin{cases} p^r R_l(H) + \displaystyle\sum_{i=2}^{r} \left\{ \left(p^i - p^{i-1}\right) \left[\left(p^{li} - 1\right) + \sum_{j=1}^{i-1} \left(p^{li} - p^{lj}\right) N_j \right] \right\} \\ \qquad + (p-1)(p^l - 1), \quad \textit{if } r > 1 \\ \\ pR_l(H) + (p-1)(p^l - 1), \quad \textit{if } r = 1 \end{cases}$$

(ii)

$$\Psi_l(G) = \begin{cases} p^{2r}\Psi_l(H) \\ \quad + p^r|H| \sum_{i=2}^{r} \left\{ \left(p^i - p^{i-1}\right) \left[\left(p^{li} - 1\right) + \sum_{j=1}^{i-1} \left(p^{li} - p^{lj}\right) N_j \right] \right\} \\ \quad + p^r|H|(p-1)(p^l - 1), & \textit{if } r > 1 \\ \\ p^2\Psi_l(H) + p|H|(p-1)(p^l - 1), & \textit{if } r = 1 \end{cases}$$

Proof — (i) G is a finite group whose elements are of the for (x, y) where $x \in C_{p^r}$ and $y \in H$. We now partition G based on the order of the elements in the first component. In particular we have

$$F_k = \left\{ (x_1, x_2) \in G \mid o(x_1) = p^k \right\}$$

and $G = \bigcup_{k=0}^{r} F_k$. Since $F_i \cap F_j = \emptyset$ for $i \neq j$, we have

$$R_l(G) = \sum_{k=0}^{r} R_l\left(F_k\right).$$

Now let $x_1 \in C_{p^r}$ with $o(x_1) = p^i$ for some i with $0 \leqslant i \leqslant r$. For each such x_1, define $F_{i,x_1} = \{(x_1, x_2) \mid x_2 \in H\}$. Then we have

$$F_i = \bigcup_{\substack{x_1 \in C_{p^r} \\ o(x_1) = p^i}} F_{i,x_1} \qquad \text{and} \qquad R_l(F_i) = \sum_{\substack{x_1 \in C_{p^r} \\ o(x_1) = p^i}} R_l\left(F_{i,x_1}\right)$$

for $i = 0, 1, \ldots, r$. Lemma 4 shows that there are $(p^i - p^{i-1})$ elements of order p^i. As a result,

$$R_l\left(F_i\right) = (p^i - p^{i-1}) R_l\left(F_{i,x_1}\right).$$

Taking $i = 0$, we have $F_0 = F_{0,1}$ and thus, $R_l\left(F_0\right) = R_l(H)$. For $i = 1$, each element (x_1, x_2) in F_{1,x_1} has order same as $o(x_2)$ except for $(x_1, 1)$ which has order p. Thus $R_l(F_{1,x_1}) = R_l(H) + p^l - 1$. So

$$R_l(F_1) = (p-1)(R_l(H) + (p^l - 1)).$$

Case 1: $r = 1$.

Then $R_l(G) = R_l(F_0) + R_l(F_1) = pR_l(H) + (p-1)(p^l - 1)$.

Case 2: $r > 1$.

Let $i \in \{2, \ldots, r\}$. For $(x_1, x_2) \in F_{i,x_1}$ with $o(x_2) = p^j$, we have $o\big((x_1, x_2)\big) = p^i$ if $j < i$ and $o\big((x_1, x_2)\big) = p^j$ if $j \geqslant i$. If $r' = \exp(H)$ then

$$R_l\left(F_{i,x_1}\right) = \textstyle\sum_{j=0}^{i-1} p^{li} N_j + \sum_{j=i}^{r'} p^{lj} N_j$$

$$= \left(1 + \sum_{j=1}^{i-1} p^{li} N_j\right) + \sum_{j=i}^{r'} p^{lj} N_j + \left(p^{li} - 1\right)$$

$$= \left(1 + \sum_{j=1}^{r'} p^{lj} N_j\right) + \left(p^{li} - 1\right) + \sum_{j=1}^{i-1} \left(p^{li} - p^{lj}\right) N_j$$

$$= R_l(H) + \left(p^{li} - 1\right) + \sum_{j=1}^{i-1} \left(p^{li} - p^{lj}\right) N_j$$

Thus we have

$$\sum_{i=2}^{r} R_l\left(F_i\right) = \left(p^r - p\right) R_l(H) + \sum_{i=2}^{r} \left(\left(p^{li} - 1\right) + \sum_{j=1}^{i-1} \left(p^{li} - p^{lj}\right) N_j\right).$$

Now

$$R_l(G) = R_l\left(F_0\right) + R_l\left(F_1\right) + \sum_{i=2}^{r} R_l\left(F_i\right)$$

and the result follows from a straight forward calculation using

$$R_l\left(F_0\right) + R_l\left(F_1\right) = pR_l(H) + (p-1)(p^l - 1).$$

(ii) This follows directly from the definition of Ψ_l and by using the previous part. □

3.3 Finite abelian groups

We can now state how to compute $\psi_l(G)$ and $\Psi_l(G)$ for any finite abelian group G. This result is analogous to [4, Proposition] and we follow the same method. In view of Theorem 2 and 3, the following is a direct application of Proposition 9 and Lemma 4. Recall that $\psi^l(G) = R_l(G)$ for $l \in (0,\infty) \cap \mathbb{Z}$ and $\Psi^l(G) = |G|^{-l}R_l(G)$ for $l \in (-\infty, 0) \cap \mathbb{Z}$. Further note from the introduction that $\psi^l(G)$ will implicitly denote that $l \in (0,\infty) \cap \mathbb{Z}$ and similarly $\Psi_l(G)$ will implicitly denote that $l \in (-\infty, 0) \cap \mathbb{Z}$.

Theorem 10 *Let* G *be a finite abelian group with* $G \simeq H_1 \times \ldots \times H_k$ *where each* H_i *is an abelian* p_i*-group and the* p_i *are distinct primes for* $i = 1, \ldots, k$. *Then*

$$\psi^l(G) = \psi^l(H_1) \ldots \psi^l(H_k), \quad \Psi_l(G) = \Psi_l(H_1) \ldots \Psi_l(H_k)$$

where $\psi^l(H_i)$ *and* $\Psi_l(H_i)$ *for* $i = 1, \ldots, k$ *are computed as follows:*

(i) *If* $H_i \simeq C_{p_i^n}$ *then*

$$\psi^l(H_i) = 1 + \frac{p_i - 1}{p_i} \sum_{r=1}^{n} p_i^{r(l+1)}$$

and

$$\Psi_l(G) = p_i^{-nl} \left(1 + \frac{p_i - 1}{p_i} \sum_{r=1}^{n} p_i^{r(l+1)} \right).$$

(ii) *If* $H_i \simeq C_{p_i^{r_1}} \times C_{p_i^{r_2}} \times \ldots \times C_{p_i^{r_n}}$, *where* $1 \leqslant r_1 \leqslant r_2 \leqslant \ldots \leqslant r_n$, *and* $r_1 + \ldots + r_n = r$ *then* $\psi^l(H_i)$ *and* $\Psi_l(H_i)$ *can be determined recursively as follows*

(iia) *If* $r_1 > 1$ *then*

$$\begin{aligned}\psi^l(H_i) = {} & p_i^{r_1} R_l(C_{p_i^{r_2}} \times \ldots \times C_{p_i^{r_n}}) \\ & + \sum_{z=2}^{r_1} \left\{ \left(p_i^z - p_i^{z-1}\right) \left[\left(p_i^{lz} - 1\right) + \sum_{j=1}^{z-1} \left(p_i^{lz} - p_i^{lj}\right) N_j \right] \right\} \\ & + (p_i - 1)(p_i^l - 1), \quad N_j = \left(\left(p_i^j\right)^{n-1} - \left(p_i^{j-1}\right)^{n-1} \right),\end{aligned}$$

and

$$\Psi_l(H_i) = p_i^{-rl} \left[\begin{aligned} & p_i^{r_1} R_l(C_{p_i^{r_2}} \times \ldots \times C_{p_i^{r_n}}) \\ & + \sum_{z=2}^{r_1} \left\{ \left(p_i^z - p_i^{z-1}\right) \left[\left(p_i^{lz} - 1\right) + \sum_{j=1}^{z-1} \left(p_i^{lz} - p_i^{lj}\right) N_j \right] \right\} \\ & + (p_i - 1)(p_i^l - 1), \quad N_j = \left(\left(p_i^j\right)^{n-1} - \left(p_i^{j-1}\right)^{n-1} \right), \end{aligned} \right]$$

(iib) *If* $r_1 = 1$ *then*

$$\psi^l(H_i) = p_i\psi^l(C_{p_i^{r_2}} \times \ldots \times C_{p_i^{r_n}}) + (p_i - 1)(p_i^l - 1),$$

and

$$\Psi_l(H_i) = p_i^{-rl}\left(p_i\psi^l(C_{p_i^{r_2}} \times \ldots \times C_{p_i^{r_n}}) + (p_i - 1)(p_i^l - 1)\right)$$

4 Further comments for $l = -1$

We conclude this paper by introducing some results related to Ψ_{-1}. We begin with some examples computed in GAP [5].

4.1 Some examples

We implement Theorem 10 in GAP [5] and checked the correctness of the formulas by running different examples against the results obtained by the brute force method where we run the sum over all elements in a group. We record the run time (measured in seconds on a MacBook Pro with 2.2 GHz 6-Core Intel Core i7 processor and 16 GB of RAM) for computing Ψ_{-1} for some groups of size > 100000 in Table 1.

Table 1: Runtime for computing $\Psi_{-1}(G)$

G	Size of the group	Runtime in seconds	
		Using Theorem 10	Using brute force method
$C_{590625} \times C_5$	2953125	< 1	20.845
$C_{48600} \times C_{120}$	5832000	< 1	54.718
$C_{1944} \times C_{648} \times C_6$	7558272	< 1	69.003
$C_{210912} \times C_{338}$	71288256	< 1	612.248

4.2 Some further comments

We begin with the following lemma.

Lemma 11 *For any finite group* G *with* $|G| > 1$, *we have* $R_{-1}(G) < \psi(G)$.

Proof — For any $x \in G$ with $x \neq 1$ we have $o(x)^{-1} < o(x)$ and thus the result follows since $\sum_{x\in G} o(x)^{-1} < \sum_{x \in G} o(x)$ if $|G| > 1$. □

The following corollary is now immediate from Lemma 11 and the main theorem of [2].

Corollary 12 *For all non cyclic groups* G *with* $|G| = n > 1$, *we have* $\Psi_{-1}(G) < n\psi(C_n)$.

Using our results in Section 3 we now prove the following.

Theorem 13 *If* G *be an abelian group order* n, *then* $\Psi_{-1}(G) < n\Psi_{-1}(C_n)$.

Proof — Let $G \simeq H_1 \times \ldots \times H_k$ where each H_i is an abelian p_i-group and each p_i is distinct prime for $i = 1, \ldots, k$. Let

$$H_i \simeq C_{p_i^{r_{i,1}}} \times C_{p_i^{r_{i,2}}} \times \ldots \times C_{p_i^{i,r_{n_i}}}$$

where $r_i = r_{i,1} + r_{i,2} + \ldots + r_{i,n_i}$ and $n = |G| = p_i^{r_i} \ldots p_k^{r_k}$. Letting $l = -1$ in Theorem 10 we see that

$$\left(p_i^{-z} - 1\right) + \sum_{j=1}^{z-1} \left(p_i^{-z} - p_i^{-j}\right) N_j < 0 \quad \text{and} \quad (p_i - 1)(p_i^{-1} - 1) < 0.$$

As a result, using Corollary 7, we can show inductively that

$$R_{-1}(H_i) < p_i^{r_{i,1}+\ldots+r_{i,n_i}} \left((r_{i,n_i} + 1) - r_{i,n_i} p_i^{-1}\right).$$

A further direct calculation shows that

$$R_{-1}(H_i) < p^{r_i}\left((1 - p_i^{-1}) r_{i,n_i} - 1\right).$$

Since $r_{i,n_i} \leqslant r_i$, again using Corollary 7, we have $R_{-1}(H_i) < \Psi_{-1}(C_{p^{r_i}})$. Finally Theorem 10 shows that $\Psi_{-1}(G) < n\Psi_{-1}(C_n)$. □

We conclude this paper with the following remark. Note that Theorem 13 is an analogous result of [2, Main Theorem] where it is shown that $\psi(G) \leqslant \psi(C_n)$ where $n = |G|$. However this does not hold for Ψ_{-1}. For example if we take $G = C_{3528} \times C_{24}$ which is a group of order 84672 then a direct computation (using Theorem 10) shows that $\Psi_{-1}(G) = 14976864$ which is greater than $\Psi_{-1}(C_{84672}) = 2757888$.

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Subhrajyoti Saha
School of Mathematics
Monash University
Melbourne (Australia)
e-mail: subhrajyotimath@gmail.com