Advances in Group Theory and Applications
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13 (2022), pp. 41–53 ISSN: 2499-1287
DOI: 10.32037/agta-2022-003

Realising a Finite Group as a Subgroup of a Product of Two Groups of Permutation Matrices

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(*Received May 24, 2021; Accepted Sept. 12, 2021 — Communicated by I.Ya. Subbotin*)

Abstract

In this paper we prove that any finite group of order n can be viewed as the group of the solutions of a certain matrix equation $XB = BY$, where the unknowns X, Y are two permutation matrices of order n and $(1+k)n+2$ respectively and where $k \in \mathbb{N}$ is given by Cayley's theorem. Moreover, we show that G is isomorphic to a certain subgroup formed by permutation matrices of order $(1+k)n$ obtained by permuting all the rows of the identity matrix $I_{(1+k)n}$.

Mathematics Subject Classification (2020): 15A24, 55P10

Keywords: permutation matrix; graded commutative $\mathbb{Q}$-algebra; group of homotopy self-equivalences

1 Introduction

Let $\mathcal{P}(n)$ denote the group of permutation matrices of degree n. For a given matrix B, let us consider the group Ω_B of the pairs (X, Y) in $\mathcal{P}(n) \times \mathcal{P}(m)$ which are solutions of the matrix equation $XB = BY$. Obviously, Ω_B is finite group as $\mathcal{P}(n)$ and $\mathcal{P}(m)$ are finite and it is worth noting that if $\lambda \in \mathbb{Q}$ and $(X, Y) \in \Omega_B$, then the pair $(\lambda X, \lambda Y)$ needs not be in Ω_B although that we have $(\lambda X)B = B(\lambda Y)$ since $\lambda X, \lambda Y$ are not permutation matrices for $\lambda \neq 1$.

A subgroup H of $\mathcal{P}(n)$ is called *realisable* if each element $M \in H$ is obtained by permuting the rows of the identity matrix I_n using a permutation $\tau \in S_n$ satisfying $\tau(i) \neq i$ for all $1 \leqslant i \leqslant n$. Here S_n denotes the symmetric group of order n.

Recall that, by Cayley's theorem, any finite group G of order n is isomorphic to a realisable subgroup, denoted by $\mathcal{C}_G$, of $P(n)$ via the map

$$G = \{g_1, \dots, g_n\} \to S_n \simeq P(n)$$

$$g_j \mapsto \sigma_j = \begin{pmatrix} g_1 & g_2 & \dots & g_n \\ g_j & \sigma_j(g_2) & \dots & \sigma_j(g_n) \end{pmatrix} \leftrightarrow M_j$$

where M_j is the matrix obtained by permuting all the rows of the identity matrix I_n using σ_j.

Following the idea developed in [1] and inspired by the works done in [4] and [2] regarding the so-called Kahn's realisability problem of groups (see [5] and [7] for more details), this paper is devoted to answer the question whether a given finite group G can occur as a group on the form Ω_B and whether G can be embedded in $\mathcal{P}(m)$, where $m > n$, as a realisable subgroup. For this purpose we shall assign to G a matrix B_G and a realisable subgroup $\mathcal{A}_G$ of $\mathcal{P}\big((1+k)n+2\big)$, where k is given by the decomposition of the permutation σ_2 into product of disjoint cycles, i.e. $\sigma_2 = \tau_1\tau_2\dots\tau_k$ and we shall define Ω_{B_G} as a certain subgroup of $\mathcal{A}_G \times \mathcal{C}_G$.

The group $\mathcal{A}_G$ and the matrix B_G are defined using the framework of rational homotopy theory [6] and the ideas developed in [3] and [1]. More precisely, $\mathcal{A}_G$ is defined in terms of the cohomology of a certain a free commutative cochain $\mathbb{Q}$-algebra associated with the group G and B_G is related to its differential.

In this paper we establish the following result.

Theorem 1 *For any finite group G of order n, there exists a matrix B_G such that G is isomorphic to the group Ω_{B_G} of the solutions of the matrix equation $XB_G = B_GY$, where the unknowns X, Y are two permutation matrices belonging to the groups $\mathcal{A}_G$ and $\mathcal{C}_G$ respectively.*

Corollary 2 *Any finite group G of order n is isomorphic to a realisable subgroup of $\mathcal{P}\big((1+k)n\big)$.*

2 Main results

2.1 Definition of the group $\mathcal{A}_G$

Let us start by recalling the main construction in [1] on which this work is based. Indeed, let $G = \{g_1, g_2, \dots, g_n\}$ be a finite group of order n and let S_n be the symmetric group. By Cayley's theorem there is a monomorphism

$$\Psi : G \to S_n \qquad g_j \mapsto \sigma_j : g_k \longrightarrow g_j g_k \qquad 1 \leqslant k \leqslant n$$

For $2 \leqslant j \leqslant n$, write $\sigma_j = \begin{pmatrix} 1 & 2 & \dots & n \\ j & \sigma_j(2) & \dots & \sigma_j(n) \end{pmatrix}$ and let

$$\sigma_2 = \Big(1\,2\,\sigma_2(2)\dots\sigma_2^{\kappa_2}(2)\Big)\Big(i_1\,\sigma_2(i_1)\,\dots\,\sigma_2^{\kappa_{i_1}}(i_1)\Big)\dots\Big(i_k\,\sigma_2(i_k)\dots\sigma_2^{\kappa_{i_k}}(i_k)\Big)$$

be the decomposition of σ_2 into a product of cycles.

Recall that in [1] we constructed a free commutative cochain $\mathbb{Q}$-algebra

$$\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{g_i \in G}), \partial\big)$$

where the degrees of the elements in this graded algebra are

$$|x_1| = 8, \qquad |x_2| = 10, \qquad |w_j| = 40$$

and where the differential is given by:

$$\begin{aligned}
\partial(x_1) &= \partial(x_2) = \partial(w_j) = 0, \quad \partial(y_1) = x_1^3 x_2, \quad \partial(y_2) = x_1^2 x_2^2, \quad \partial(y_3) = x_1 x_2^3 \\
\partial(z_j) &= w_j^3 + w_j w_{\sigma_{j+1}(1)} x_2^4 + \sum_{\tau=1}^{k} w_j w_{\sigma_{j+1}(i_\tau)} x_2^4 + u + x_1^{15}, \qquad 1 \leqslant j \leqslant n-1 \\
\partial(z_n) &= w_n^3 + w_n w_1 x_2^4 + \sum_{\tau=1}^{k} w_n w_{i_\tau} x_2^4 + u + x_1^{15}
\end{aligned} \tag{2.1}$$

where $u = y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6$, and we proved that

$$\mathcal{E}\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{g_i \in G})\big) \simeq G$$

where $\mathcal{E}\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leqslant j \leqslant n})\big)$ denotes the group of self homotopy cochain equivalences of $\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{g_i \in G})$ (see [3] and [1] for more details).

Now let $V^{119} = \mathbb{Q}\{z_1, \dots, z_n\}$ be the vector space spanned by the set $\{z_1, \dots, z_n\}$. Recall that $|z_i| = 119$ for every $1 \leqslant i \leqslant n$. In [1], Proposition 3.9, it is shown that

$$\mathcal{E}\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{g_i \in G})\big) \simeq \mathcal{D}_{40}^{119},$$

where $\mathcal{D}_{40}^{119}$ is the subgroup of

$$\mathrm{aut}(V^{119}) \times \mathcal{E}\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{w_j\}_{g_i \in G})\big)$$

consisting of the couples $(\xi, [\alpha])$ making the following diagram commutes:

$$\begin{array}{ccc} V^{119} & \xrightarrow{\ \ \xi\ \ } & V^{119} \\ \Big\downarrow b & & \Big\downarrow b \\ \Gamma_G^{120} & \xrightarrow{\ H^{120}(\alpha)\ } & \Gamma_G^{120} \end{array} \tag{2.2}$$

where $\Gamma_G^{120} = H^{120}\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{w_j\}_{g_i \in G})\big)$ and where b is defined by

$$b(z_i) = \widehat{\partial(z_i)}, \quad 1 \leqslant j \leqslant n \tag{2.3}$$

Here $\widehat{\partial(z_i)}$ is the cohomology class of $\partial(z_i)$ in

$$H^{120}\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{w_j\}_{g_i \in G})\big).$$

Moreover, it is shown that if $(\xi, [\alpha]) \in \mathcal{D}_{40}^{119}$, then there exists a unique permutation

$$\sigma_s = \begin{pmatrix} 1 & 2 & \dots & n \\ s & \sigma_s(2) & \dots & \sigma_s(n) \end{pmatrix} \tag{2.4}$$

such that

$$\begin{aligned} &\xi(z_j) = z_{\sigma_s(j)}, \quad \alpha(w_j) = w_{\sigma_s(j)}, \\ &\alpha = id, \quad \text{on} \ \ x_1, x_2, y_1, y_2, y_3. \end{aligned} \tag{2.5}$$

Thus, there is an isomorphism

$$\Psi : \mathcal{D}_{40}^{119} \to G$$

defined by $\Psi\big((\xi, [\alpha])\big) = g_s$, where the element g_s corresponds to the permutation σ_s, given in (2.4), via Cayley's theorem.

Set

$$u = y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6.$$

As the following set of generators

$$\Sigma = \Big\{ w_1^3 \ ; \ \dots \ ; \ w_n^3 \ ; \ w_j w_{\sigma_{j+1}(1)} x_2^4 \ ; \\ w_j w_{\sigma_{j+1}(i_\tau)} x_2^4 \ ; \ u \ ; \ x_1^{15} \Big\} \tag{2.6}$$

where $1 \leqslant j \leqslant n$ and $1 \leqslant \tau \leqslant k$, is linearly independent in the vector space

$$\Gamma_G^{120} = H^{120}\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{w_j\}_{i\in G})\big)$$

it follows that Σ can be chosen, according the formulas (2.1) and (2.3), as a basis for the vector space $b(V^{119}) \subseteq \Gamma_G^{120}$. Notice that

$$\dim\ b(V^{119}) = \text{cardinal}(\Sigma) = (1+k)n + 2 \tag{2.7}$$

Thus, if B_G denotes the matrix of order $((1+k)n+2) \times n$ which is associated to the linear map b defined in (2.3) with respects to the basis Σ, then we can write

$$B_G = \begin{bmatrix} I_n \\ M \\ D \end{bmatrix} \qquad \text{where} \qquad D = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{bmatrix},$$

where the matrix $M = \big[m_{ij}\big]$ is defined by

$$m_{ij} = \begin{cases} 1, & \text{if } i \in \{\sigma_{j+1}(1), \sigma_{j+1}(i_1), \dots, \sigma_{j+1}(i_k)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, taking into construction (2.5), the matrices associated to the linear maps ξ and the restriction of the linear map $H^{120}(\alpha)$ to $b(V^{119})$, given in the diagram (2.2) and corresponding to the element $(\xi, [\alpha]) \in \mathcal{D}_{40}^{119}$, can be written, respectively, as

$$C_{g_s} = \sigma_s I_n\,, \qquad\qquad A_{g_s} = \begin{bmatrix} \sigma_s I_n & 0 & 0 \\ 0 & \widetilde{A}_{g_s} & 0 \\ 0 & 0 & I_2 \end{bmatrix}, \tag{2.8}$$

where

$$\sigma_s I_n = \left[c_{i,j}\right]_{1\leqslant i,j\leqslant (k+1)n}, \quad c_{i,j} = \begin{cases} 1, & \text{if } i = \sigma_s(j) \\ 0, & \text{otherwise} \end{cases},$$

and where

$$\widetilde{A}_{g_s} = \left[a_{n+i,n+j}\right]_{1\leqslant i,j\leqslant (k+1)n}, \quad a_{n+i,n+j} = \begin{cases} 1, & \text{if } i = \sigma_s(j) \\ 0, & \text{otherwise} \end{cases}.$$

Here σ_s is the permutation corresponding to g_s via Cayley's theorem.

From (2.8), it is clear to see that A_{g_s} is a permutation matrix. Recall that the commutativity of the diagram (2.2) implies that

$$A_{g_s} B_G = B_G C_{g_s}, \qquad \forall g_s \in G. \tag{2.9}$$

Let $G = \{g_1, \dots, g_n\}$ be a group, we define the following two sets

$$\mathcal{A}_G = \{A_{g_s},\ g_s \in G\}, \qquad \Omega_G = \left\{(A_{g_s}, C_{g_s}) \in \mathcal{A}_G \times \mathcal{C}_G,\ \ g_s \in G\right\}.$$

Theorem 3 *The sets $\mathcal{A}_G$ and Ω_G are groups isomorphic to* G.

Proof — First let us prove that $\mathcal{A}_G$ is a group. Let $A_{g_s}, A_{g_r} \in \mathcal{A}_G$. By (2.9) there exist two matrices C_{g_s}, C_{g_r} such that

$$A_{g_s} B_G = B_G C_{g_s} \qquad \text{and} \qquad A_{g_r} B_G = B_G C_{g_r}$$

therefore

$$A_{g_s} A_{g_r} B_G = A_{g_s} B_G C_{g_r} = B_G C_{g_s} C_{g_r},$$

it follows that $A_{g_s} A_{g_r} \in \mathcal{A}_G$. Here we use the fact that

$$A_{g_s} A_{g_r} = A_{g_s g_r} \qquad \text{and} \qquad C_{g_s} C_{g_r} = C_{g_s g_r} \tag{2.10}$$

Next let $A_{g_s} \in \mathcal{A}_G$. Since A_{g_s} and C_{g_s} are invertible, we deduce that $B_G C_{g_s}^{-1} = (A_{g_s})^{-1} B_G$ implying that $(A_{g_s})^{-1} \in \mathcal{A}_G$. Notice also that $A_{g_s}^{-1} = A_{g_s^{-1}}$.

Then, using the same arguments, it is easy to check that the set Ω_G is a group. Finally, it is clear that the two maps

$$\chi : G \to \mathcal{A}_G \quad \text{and} \quad \varphi : G \to \Omega_G,$$

defined by $\chi(g_s) = A_{g_s}$ and $\varphi(g_s) = (A_{g_s}, C_{g_s})$ respectively, are isomorphisms of groups. □

2.2 Realisable subgroups

A subgroup H of $\mathcal{P}(n)$ is called *realisable* if each element $M \in H$ is obtained by permuting the rows of the identity matrix I_n using a permutation $\tau \in S_n$ satisfying $\tau(i) \neq i$ for all $1 \leqslant i \leqslant n$.

Let $G = \{g_1, \dots, g_n\}$ be a group. Based on the formula (2.8) , let us define the following matrix

$$M_{g_s} = \begin{bmatrix} \sigma_s I_n & 0 \\ 0 & \widetilde{A}_{g_s} \end{bmatrix} \quad , \quad g_s \in G \tag{2.11}$$

Theorem 4 *If* $H_G = \{M_{g_s}\ ,\ g_s \in G\}$*, then* H_G *is a realisable subgroup of* $\mathcal{P}\big((1+k)n\big)$ *isomorphic to* G

Proof — According to the formula (2.8), the matrix M_{g_s} is defined in terms of the permutation σ_s corresponding to the element $g_s \in G$, so it follows that $\sigma_s(i) \neq i$ for every $1 \leqslant i \leqslant n$ implying that M_{g_s} belongs to $\mathcal{P}\big((1+k)n\big)$.

Taking into consideration the relation (2.10), the map

$$G = \{g_1, \dots, g_n\} \to H_G$$

which assign $g_s \mapsto M_{g_s}$ is obviously an isomorphism of groups. □

2.3 Examples

In the following examples we illustrate our study by determining all the groups introduced in this paper for the cyclic group $\mathbb{Z}_4$ and the Klein group $\mathbb{V}$.

Example 5 If $G = \mathbb{Z}_4$, then the monomorphism

$$\mathbb{Z}_4 = \{g_1, g_2, g_3, g_4\} \to S_4$$

is given by

$$g_1 \to \mathrm{id},\ \ g_2 \leftrightarrow \sigma_2 = (1234),\ \ g_3 \leftrightarrow \sigma_3 = (13)(24),\ \ g_4 \leftrightarrow \sigma_4 = (1432)$$

therefore according to (2.1) the model associated with $\mathbb{Z}_4$ is

$$\big(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4), \partial\big)$$

where $|x_1| = 8, |x_2| = 10, |w_j| = 40$, and the differential is given by

$$\partial(x_1) = \partial(x_2) = \partial(w_j) = 0, \;\; \partial(y_1) = x_1^3 x_2 \;, \;\; \partial(y_2) = x_1^2 x_2^2 \;, \;\; \partial(y_3) = x_1 x_2^3,$$

$$\partial(z_1) = w_1^3 + w_1 w_2 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15},$$

$$\partial(z_2) = w_2^3 + w_2 w_3 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15},$$

$$\partial(z_3) = w_3^3 + w_3 w_4 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15},$$

$$\partial(z_4) = w_2^4 + w_4 w_1 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15}.$$

For the above construction it is clear that $V^{119} = \mathbb{Q}\{z_1, z_2, z_3, z_4\}$ and by (2.6) the base Σ of the vector space $b(V^{119})$ is given

$$\Sigma = \Big\{ w_1^3, w_2^3, w_3^3, w_4^3, w_1 w_2 x_2^4, w_2 w_3 x_2^4, w_3 w_4 x_2^4, \\ w_4 w_1 x_2^4, y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6, x_1^{15} \Big\} \tag{2.12}$$

implying that the matrix $B_{\mathbb{Z}_4}$ associated with the linear map b, given in (2.3), is

$$B_{\mathbb{Z}_4} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix},$$

and we have

$$\mathcal{C}_{\mathbb{Z}_4} = \Big\{ I_4, C_{(1234)}, C_{(13)(24)}, C_{(1432)} \Big\},$$

where

$$C_{(1234)} = \begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix}, \quad C_{(13)(24)} = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix},$$

$$C_{(1432)} = \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix}$$

For instance, $C_{(1234)}$ is simply the permutation matrix obtained by permuting the rows of I_4 using the permutation (1234) and likewise $C_{(13)(24)}$ and $C_{(1432)}$.

Next we have $\mathcal{A}_{\mathbb{Z}_4} = \Big\{I_{10}, A_{(1234)}, A_{(13)(24)}, A_{(1432)}\Big\}$, where

$$A_{(1234)} = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad A_{(13)(24)} = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$A_{(1432)} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

and $H_{\mathbb{Z}_4} = \Big\{I_8, M_{(1234)}, M_{(13)(24)}, M_{(1432)}\Big\}$, where

$$M_{(1234)} = \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \end{pmatrix}, \quad M_{(13)(24)} = \begin{pmatrix} 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{pmatrix}$$

$$M_{(1432)} = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \end{pmatrix}.$$

Recall that $A_{(1234)}$ is the matrix associated to the restriction of the linear map $H^{120}(\alpha)$ to the vector space $b(V^{119})$, where the cochain map α is given by (2.5), with respects to the basis Σ in (2.12). Thus, $A_{(1234)}$ is obtained by using the permutation $\sigma_2 = (1234)$ as follows:

$$w_1^3 \mapsto w_2^3\ ,\ w_2^3 \mapsto w_3^3\ ,\ w_3^3 \mapsto w_4^3\ ,\ w_4^3 \mapsto w_1^3\ ,\ w_1w_2x_2^4 \mapsto w_2w_3x_2^4\ ,$$
$$w_2w_3x_2^4 \mapsto w_3w_4x_2^4\ ,\ w_3w_4x_2^4 \mapsto w_4w_1x_2^4\ ,\ w_4w_1x_2^4 \mapsto w_1w_4x_2^4\ ,$$
$$y_1y_2x_1^4x_2^2 - y_1y_3x_1^5x_2 + y_2y_3x_1^6 \mapsto y_1y_2x_1^4x_2^2 - y_1y_3x_1^5x_2 + y_2y_3x_1^6,$$
$$x_1^{15} \mapsto x_1^{15}.$$

and likewise we obtain the matrices $A_{(13)(24)}$ and $A_{(1432)}$. Notice that matrices $M_{(1234)}, M_{(13)(24)}, M_{(1432)}$ are constructed from the matrices $A_{(1234)}, A_{(13)(24)}, A_{(1432)}$ using (2.10) and finally we have

$$\Omega_{\mathbb{Z}_4} = \Big\{(I_4, I_{12}), (A_{(1234)}, C_{(1234)}), \\ (A_{(13)(24)}, C_{(13)(24)}), (A_{(1432)}, C_{(1432)})\Big\}$$

It is also worth noting to point out that the group $M_{\mathbb{Z}_4}$, which is isomorphic to $\mathbb{Z}_4$ is a realisable subgroup of the group of permutation matrices $\mathcal{P}(8)$.

Example 6 In this example we use the same analysis and computation as in the example (5), but we omit all the details, to determine the groups $\mathcal{A}_{\mathbb{V}}, \mathcal{C}_{\mathbb{V}}, H(\mathbb{V})$ and $\Omega_{\mathbb{V}}$ for the Klein group $\mathbb{V}$. Indeed, the monomorphism

$$\mathbb{V} = \{g_1, g_2, g_3, g_4\} \leftrightarrow S_4$$

is given by

$$g_2 \leftrightarrow (12)(34)\ ,\ g_3 \leftrightarrow (13)(24)\ ,\ g_4 \to (14)(23),$$

so the model associated to $\mathbb{V}$ is

$$\big(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4), \partial\big)$$

where $|x_1| = 8, |x_2| = 10, |w_j| = 40$, and where the differential is given by

$$\partial(x_1) = \partial(x_2) = \partial(w_j) = 0\ ,\ \partial(y_1) = x_1^3 x_2\ ,\ \partial(y_2) = x_1^2 x_2^2\ ,\ \partial(y_3) = x_1 x_2^3,$$

$$\partial(z_1) = w_1^3 + w_1 w_2 x_2^4 + w_1 w_4 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15},$$

$$\partial(z_2) = w_2^3 + w_2 w_3 x_2^4 + w_2 w_1 x_1^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15},$$

$$\partial(z_3) = w_3^3 + w_3 w_4 x_2^4 + w_3 w_2 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15},$$

$$\partial(z_4) = w_4^3 + w_4 w_1 x_2^4 + w_4 w_3 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15}$$

If $u = y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6$, then the basis Σ is given by

$$\Sigma = \Big\{w_1^3, w_2^3, w_3^3, w_4^3, w_1 w_2 x_2^4, w_2 w_3 x_2^4, w_3 w_4 x_2^4, \\ w_4 w_1 x_2^4, w_1 w_4 x_2^4, w_2 w_1 x_2^4, w_3 w_2 x_2^4, w_4 w_3 x_2^4, u, x_1^{15}\Big\}$$

implying that dim $b(V^{119}) = 14$ and the matrix $B_{\mathbb{V}}$ is

$$B_{\mathbb{V}} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&1&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&1\\ 0&0&1&0\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix}$$

We have $\mathcal{C}_{\mathbb{V}} = \left\{I_4, C_{(12)(34)}, C_{(13)(24)}, C_{(14)(23)}\right\}$, where

$$C_{(12)(34)} = \begin{pmatrix} 0&1&0&0\\ 1&0&0&1\\ 0&0&0&0\\ 0&0&1&0 \end{pmatrix}\ ,\ C_{(13)(24)} = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}$$

$$C_{(14)(23)} = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&1 \end{pmatrix}$$

and $\mathcal{A}_{\mathbb{V}} = \left\{I_{14}, A_{(12)(34)}, A_{(13)(24)}, A_{14)(23)}\right\}$, where

$$A_{(12)(34)} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$A_{(13)(24)} = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$A_{(14)(23)} = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$$

Next we have $H_{\mathbb{V}} = \Big\{ I_{12}, M_{(12)(34)}, M_{(13)(24)}, M_{14)(23)} \Big\}$, where

$$M_{(12)(34)} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \end{pmatrix},$$

$$M_{(13)(24)} = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \end{pmatrix},$$

$$M_{(14)(23)} = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \end{pmatrix}.$$

Notice that $H(\mathbb{V})$ is a realisable subgroup of $\mathcal{P}(12)$. Finally, we have

$$\Omega_{\mathbb{V}} = \Big\{ (I_4, I_{14}), (A_{(12)(34)}, C_{(12)(34)}), (A_{(13)(24)}, C_{(13)(24)}), (A_{(14)(32)}, C_{(14)(32)}) \Big\}.$$

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