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# Profinite Groups with Many Elements of Bounded Order *

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**Abstract**

Lévai and Pyber [5] proposed the following as a conjecture (see also Problem 14.53 of [9]): if $G$ is a profinite group such that the set of solutions of the equation $x^n = 1$ has positive Haar measure, then $G$ has an open subgroup $H$ and an element $t$ such that all elements of the coset $tH$ have order dividing $n$.

We define a constant $c_n$ for all finite groups and prove that the latter conjecture is equivalent with a conjecture saying $c_n < 1$. Using the latter equivalence we observe that correctness of Lévai and Pyber conjecture implies the existence of the universal upper bound $1/1 - c_n$ on the index of generalized Hughes-Thompson subgroup $H_n$ of finite groups whenever it is non-trivial. It is known that the latter is widely open even for all primes $n = p \geqslant 5$. For odd $n$ we also prove that Lévai and Pyber conjecture is equivalent to show that $c_n$ is less than 1 whenever $c_n$ is only computed on finite solvable groups.

The validity of the conjecture has been proved in [5] for $n = 2$. Here we confirm the conjecture for $n = 3$.

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## 1 Introduction and results

Let $G$ be a Hausdorff compact group. Then $G$ has a unique normalized Haar measure denoted by $m_G$. In general, the question of

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weather the interior of every non-empty measurable subset of G with positive Haar measure is non-empty has negative answer even if G is profinite (see e.g. [5]). However the same question for subsets defined by words is still open. In [5] the following conjecture is proposed.

**Conjecture 1.1** (Conjecture 3 of [5], Problem 14.53 of [9]) *Let* $G$ *be a profinite group such that the set* $X_n(G)$ *of solutions of the equation* $x^n = 1$ *in* $G$ *has positive Haar measure. Then* $G$ *has an open subgroup* $H$ *and an element* $t$ *such that all elements of the coset* $tH$ *have order dividing* $n$.

The validity of Conjecture 1.1 has been proved in [5] for $n = 2$. In [7] it is shown that the conjecture is valid for $n = 2$ even if G is Hausdorff compact. It is also proved in [7] that if $X_3(G)$ has positive Haar measure in a compact group G, then G contains an open normal subgroup which is 2-Engel. Here we confirm Conjecture 1.1 for $n = 3$. To do so, we first show that Conjecture 1.1 is equivalent to the following one. We need the following notation in the statement of the conjecture. For an arbitrary group K and an automorphism $\phi$ of K of order dividing a positive integer n, define

$$X_{n,\phi}(K) := \left\{x \in K \mid xx^{\phi}x^{\phi^2}\dots x^{\phi^{n-1}} = 1\right\}.$$

The automorphism group of K will be denoted by Aut(K). In the following we denote by $\mathcal{F}$ the class of all finite groups.

**Conjecture 1.2**

$$c_n := \sup\left(\left\{\frac{|X_{n,\phi}(H)|}{|H|} : H \in \mathcal{F},\ \phi \in \mathrm{Aut}(H),\ \phi^n = \mathrm{id}\right\} \setminus \{1\}\right) < 1.$$

It is known that Conjecture 1.2 is valid for $n = 2$ and the supremum $c_2$ is 3/4 (see [8]). We shall prove that $c_3 < 1$.

If n is odd, using Theorem 1.10 of [4], we prove that Conjecture 1.1 is equivalent to the following. Here we denote by $\mathcal{S}$ the class of finite solvable groups.

**Conjecture 1.3**

$$c^{\mathcal{S}}_{2n+1} := \sup\left(\left\{\frac{|X_{n,\phi}(H)|}{|H|} : H \in \mathcal{S},\ \phi \in \mathrm{Aut}(H),\ \phi^{2n+1} = \mathrm{id}\right\} \setminus \{1\}\right) < 1.$$

The following easy proposition and the next remark show how difficult the above conjectures may be. Recall that for any finite group G and any positive integer $n$ the generalized Hughes-Thompson subgroup $H_n(G)$ is defined as the subgroup $\langle x \in G \mid x^n \neq 1 \rangle$ of $G$, see Chapter 7 of [3].

**Proposition 1.4** *If* Conjecture 1.2 *is true for* $n$ *i.e.* $c_n < 1$, *then for any finite group* $G$ *with* $H_n(G) \neq 1$, $|G : H_n(G)| < 1/1 - c_n$.

Proof — Since $H_n(G) \neq 1$, $X_n(G) \neq G$. Now the proof follows from the fact that $G \setminus H_n(G) \subset X_n(G)$ and the definition of $c_n$. □

The problem of finding a universal bound from above depending only on $p$ for the index $|G : H_p(G)|$ of the Hughes-Thompson subgroup $H_p(G)$ of finite $p$-groups $G$ with $H_p(G) \neq 1$ is widely open for all $p \geqslant 5$ (see Chapter 7 of [3] for the history and results on the problem). Indeed to get that universal bound, one would have to prove, inter alia, that $L(B(\infty, p))$ has a finite basis of multilinear identities, where $B := B(\infty, p)$ is the free Burnside group of exponent $p$ with infinite countable rank and $L(B(\infty, p))$ is the associated Lie ring on $\bigoplus_{i=1}^{\infty} \gamma_i(B)/\gamma_{i+1}(B)$.

We finish this section with the following two questions.

**Question 1.5** *Suppose that* $n$ *is a positive integer for which there exists a positive integer* $k_n$ *depending only on* $n$ *such that* $|G : H_n(G)| \leqslant k_n$ *for all finite groups* $G$ *with* $1 \neq H_n(G)$. *Is it true that* $c_n < 1$? *The same question is open even when* $n \geqslant 5$ *is prime.*

**Question 1.6** *Let* $n > 1$ *be a positive integer such that* $c_d < 1$ *for all prime power divisors* $d$ *of* $n$. *Is it true that* $c_n < 1$?

## 2 Profinite groups

In the following, we denote the normalized Haar measure of a compact group $G$ by $m_G$, and we will simply write $m$ if there is no ambiguity. Our first easy lemma will be used in the sequel without any further reference.

**Lemma 2.1** (cf. [2], Lemma 2.5) *Let* $G$ *be a compact group and* $A \subseteq G$ *be a measurable subset. If* $m(A) \geqslant 1 - \epsilon$, *then* $m\left(\bigcap_{k=1}^{n} g_k A\right) \geqslant 1 - n\epsilon$ *for all* $g_1, \ldots, g_n \in G$. *The similar result with strict inequalities holds.*

Proof — By induction on $n$, we prove the result. For $n = 1$, it holds as the measure is left-invariant. Assume that the result is true for $n$; therefore

$$m\left(\bigcap_{k=1}^{n+1} g_k A\right) = m\left(\bigcap_{k=1}^{n} g_k A\right) + m(g_{n+1}A) - m\left(\left(\bigcap_{k=1}^{n} g_k A\right) \cup (g_{n+1}A)\right)$$
$$\geqslant (1 - n\epsilon) + (1 - \epsilon) - 1 = 1 - (n+1)\epsilon$$

by the induction hypothesis and the statement is proved. □

**Lemma 2.2** *Let* $G$ *be a compact group and* $\phi$ *be a continuous automorphism of* $G$ *of order dividing* $n$*. Denote by* $G \rtimes \langle\phi\rangle$ *the semidirect product of* $G$ *by* $\langle\phi\rangle$*. Then:*

(i) $X_{n,\phi}(G)$ *has nonempty interior if and only if* $X_n(G \rtimes \langle\phi\rangle)$ *has nonempty interior.*

(ii) *If* $X_{n,\phi}(G)$ *has positive Haar measure then* $X_n(G \rtimes \langle\phi\rangle)$ *has positive Haar measure.*

Proof — Since

$$X_n(G \rtimes \langle\phi\rangle) \cap G\phi^{-1} = X_{n,\phi}(G)\phi^{-1},$$

both points follow. □

**Proposition 2.3** Conjecture 1.1 *implies* Conjecture 1.2.

Proof — If $n$ is such that Conjecture 1.2 is not valid, then there exist sequences $(G_k)$ of finite groups and $(\phi_k) \in \prod_{k=1}^{\infty} \mathrm{Aut}(G_k)$ such that $\phi_k^n = 1$ and

$$0 < \prod_{k=1}^{\infty} \frac{|X_{n,\phi_k}(G_k)|}{|G_k|} < 1.$$

Consider the cartesian product $G = \prod_k G_k$ which is clearly profinite. Then $\phi = (\phi_k)$ is an automorphism of $G$ of order dividing $n$. It is clear that the measure of $X_{n,\phi}(G)$ is equal to $\prod_k \frac{|X_{n,\phi_k}(G_k)|}{|G_k|}$ and its interior is empty, so by Lemma 2.2, $X_n(G \rtimes \langle\phi\rangle)$ has positive Haar measure and empty interior showing that Conjecture 1.1 is not valid. □

The following lemma will be used in the proof that "Conjecture 1.2 implies Conjecture 1.1". We write "$N \trianglelefteq_o G$" whenever $N$ is a normal and open subgroup of $G$.

**Lemma 2.4** *Let* $A$ *be a closed subset of a profinite group with positive Haar measure and* $M$ *be any normal open subgroup of* $G$. *If* $\mathcal{X}$ *is the set of all normal open subgroups of* $G$ *contained in* $M$, *then*

$$\sup\left\{\frac{m(Ng\cap A)}{m(N)} : g \in G, N \in \mathcal{X}\right\} = 1$$

Proof — Let $N \in \mathcal{X}$ be such that $r := |G : N|$. If $s$ is the number of cosets of $N$ which intersect $A$, then

$$(r-s)m(N) \leqslant 1 - m(A). \tag{2.1}$$

On the other hand, assume that

$$m(Nx\cap A) = \max\left\{m(Ng\cap A) : g \in G\right\},$$

so

$$m(A) \leqslant sm(Nx\cap A) \tag{2.2}$$

It follows from inequalities (2.1) and (2.2) that

$$\frac{m(A)}{1-m(A)}\frac{r-s}{s} \leqslant \frac{m(Nx\cap A)}{m(N)}.$$

Since $A$ is closed,

$$m(A) = \inf\left\{\frac{|AK/K|}{|G/K|} : K \trianglelefteq_o G\right\}.$$

Now since $|AK/K|/|G/K| \leqslant |AN/N|/|G/N|$, whenever $K \leqslant N$ are normal subgroups of $G$ of finite index, it follows that

$$m(A) = \inf\left\{\frac{|AN/N|}{|G/N|} : N \in \mathcal{X}\right\}.$$

Since $r/s = |G/N|/|AN/N|$, the result now follows from the last inequality. □

For any positive integer $n$ and any class $\mathcal{X}$ of finite groups, we denote by $c_n^{\mathcal{X}}$ the following positive real number which is at most 1:

$$\sup\left(\left\{\frac{|X_{n,\phi}(H)|}{|H|} : H \in \mathcal{X} \text{ and } \ \phi \in \mathrm{Aut}(H),\ \phi^n = \mathrm{id}\right\} \setminus \{1\}\right).$$

**Proposition 2.5** *Assume* $c_n^{\mathfrak{X}} < 1$ *and suppose that* $G$ *is a profinite group. Let* $M$ *be a normal open subgroup of* $G$ *and* $\phi$ *be a continuous automorphism of* $M$ *of order dividing* $n$ *such that* $N^{\phi} \subset N$ *for all normal open subgroups* $N$ *of* $G$ *contained in* $M$ *and* $M/N \in \mathfrak{X}$*. Then* $m_M(X_{n,\phi}(M)) \leqslant c_n^{\mathfrak{X}}$ *if* $X_{n,\phi}(M) \neq M$.

Proof — Seeking a contradiction, suppose $m_M(X_{n,\phi}(M)) > c_n^{\mathfrak{X}}$. Let $N$ be a normal open subgroup of $G$. Consider the following automorphism of $M/(M \cap N)$,

$$\overline{\phi} : \frac{M}{M \cap N} \to \frac{M}{M \cap N}, \quad \overline{x}^{\overline{\phi}} := \overline{x^{\phi}}$$

We have

$$m_M\left(X_{n,\phi}(M)\right) \leqslant \frac{|X_{n,\overline{\phi}}(M/(M \cap N))|}{|M/(M \cap N)|}$$

which holds because if $x \in X_{n,\phi}(M)$, then $\overline{x} \in X_{n,\overline{\phi}}(M/(M \cap N))$. Since $c_n^{\mathfrak{X}} < 1$, it follows that

$$X_{n,\overline{\phi}}(M/(M \cap N)) = M/(M \cap N),$$

whence $\prod_{k=0}^{n-1} x^{\phi^k} \in N$ for all $x \in M$. Since

$$\bigcap\{N \trianglelefteq_o G : N \leqslant M\} = \{1\},$$

we have $\prod_{k=0}^{n-1} x^{\phi^k} = 1$ for all $x \in M$, i.e. $X_{n,\phi}(M) = M$. □

We are now in a position to prove the following result.

**Proposition 2.6** Conjecture 1.2 *implies* Conjecture 1.1.

Proof — Let $G$ be a profinite group such that $m_G(X_n(G)) > 0$. By Lemma 2.4, there exist a normal open subgroup $M$ and $g \in X_n(G)$, such that

$$c_n < \frac{m_G\left(Mg \cap X_n(G)\right)}{m_G(M)} = \frac{m_G\left(M \cap X_n(G)g^{-1}\right)}{m_G(M)}. \tag{2.3}$$

Put $\phi : M \to M$, $x \mapsto gxg^{-1}$. Since $g^n = 1$, then $\phi^n = 1$. Moreover

$$M \cap X_n(G)g = X_{n,\phi}(M)$$

and the inequality (2.3) means that $m_M(X_{n,\phi}(M)) > c_n$, so by Proposition 2.5, $X_{n,\phi}(M) = M$, whence $Mg \subseteq X_n(G)$. □

We use the following result in the proof of Proposition 2.8.

**Theorem 2.7** (Theorem 1.10 (ii) of [4]) *Let $G$ be a profinite group and suppose that the set of elements of $G$ of finite odd order has positive Haar measure. Then $G$ has a prosolvable open normal subgroup.*

**Proposition 2.8** *For odd $n$,* Conjecture 1.3 *implies* Conjecture 1.1.

Proof — Let $G$ be a profinite group with $m_G(X_n(G)) > 0$. By Theorem 2.7 and Lemma 2.4, there exist a normal open subgroup $M$ and $g \in X_n(G)$ such that such that $M/N$ is finite solvable for all open normal subgroups $N$ of $G$ contained in $M$, and

$$c_n^{\mathcal{S}} < \frac{m_G\left(Mg \cap X_n(G)\right)}{m_G(M)} = \frac{m_G\left(M \cap X_n(G)g^{-1}\right)}{m_G(M)}. \qquad (2.4)$$

Put $\phi : M \to M$, $x \mapsto gxg^{-1}$. Since $g^n = 1$, then $\phi^n = 1$. Moreover

$$M \cap X_n(G)g = X_{n,\phi}(M)$$

and the inequality (2.4) means that $m_M(X_{n,\phi}(M)) > c_n^{\mathcal{S}}$, so by Proposition 2.5, $X_{n,\phi}(M) = M$, whence $Mg \subseteq X_n(G)$. □

# 3 Compact groups with splitting automorphisms of order 3

In this section we prove that $c_3 < 1$.

**Theorem 3.1** *Let $G$ be a compact group and $\alpha$ be an automorphism of $G$ such that $\alpha^3 = \mathrm{id}$ and the set $X = \{x \in G \mid xx^{\alpha}x^{\alpha^2} = 1\}$ is measurable with $m(X) > 15/16$. Then $X = G$.*

Proof — First we prove that $G$ is 2-Engel. The proof is similar to an argument used in the proof of Theorem 4.4 of [7]. We give the proof for the reader's convenience. For any $a, b \in G$ we must prove that $[a, b, b] = 1$. Consider the set

$$M := X \cap b^{-1}X \cap aX \cap a^{-1}X \cap ab^{-1}X \cap ba^{-1}X \cap abX \cap b^{-1}a^{-1}X.$$

Since $m(M) > 1/2$, there exists $x \in X$ such that

$$1 = (x\alpha^{-1})^3 = (bx\alpha^{-1})^3 = (ax\alpha^{-1})^3 = (a^{-1}x\alpha^{-1})^3 = (ab^{-1}x\alpha^{-1})^3$$
$$= (ba^{-1}x\alpha^{-1})^3 = (abx\alpha^{-1})^3 = (b^{-1}a^{-1}x\alpha^{-1})^3,$$

where all the elements written above belong to the semidirect product $G \rtimes \langle\alpha\rangle$ by noting that $g \in X$ if and only if $(g\alpha^{-1})^3 = 1$ in $G \rtimes \langle\alpha\rangle$. Now Lemma 4.1 of [7] implies that $[a, b, b] = 1$.

In the sequel we prove that for an arbitrary element $g$ of $G$, we have $gg^{\alpha}g^{\alpha^2} = 1$.

Let $X^{-1} = \{x^{-1} \mid x \in X\}$ and $Y = X \cap X^{-1}$. Then $m(Y) > 7/8$. Consider the set $Z = Y \cap g^{-1}Y$. Since $m(Z) > 3/4$, $Z \neq \varnothing$ and for all $x \in Z$ we have

$$xx^{\alpha}x^{\alpha^2} = gx(gx)^{\alpha}(gx)^{\alpha^2} = 1. \tag{3.1}$$

It follows from $gx(gx)^{\alpha}(gx)^{\alpha^2} = 1$ that

$$1 = gxg^{\alpha}x^{\alpha}g^{\alpha^2}x^{\alpha^2}$$
$$= gg^{\alpha}x[x, g^{\alpha}]x^{\alpha}x^{\alpha^2}g^{\alpha^2}[g^{\alpha^2}, x^{\alpha^2}] \text{ (using the identity } XY = YX[X, Y])$$
$$= gg^{\alpha}[x, g^{\alpha}]xx^{\alpha}x^{\alpha^2}g^{\alpha^2}[g^{\alpha^2}, x^{\alpha^2}] \text{ (since } G \text{ is 2-Engel and} [x, [x, g^{\alpha}]] = 1)$$
$$= gg^{\alpha}[x, g^{\alpha}]g^{\alpha^2}[g^{\alpha^2}, x^{\alpha^2}] \text{ (since by (3.1), } xx^{\alpha}x^{\alpha^2} = 1)$$
$$= gg^{\alpha}g^{\alpha^2}[x, g^{\alpha}][x, g^{\alpha}, g^{\alpha^2}][g^{\alpha^2}, x^{\alpha^2}] \text{ (using the identity } XY = YX[X, Y]).$$

Thus

$$(gg^{\alpha}g^{\alpha^2})^{-1} = [x, g^{\alpha}][x, g^{\alpha}, g^{\alpha^2}][g^{\alpha^2}, x^{\alpha^2}] \text{ for all } x \in Z.$$

Now consider $W = Z \cap x_0^{-1}Z$ for some $x_0 \in Z$. Since $m(W) > 1/2$, $W$ is nonempty and for all $y \in W$ we have

$$\begin{aligned}(gg^{\alpha}g^{\alpha^2})^{-1} &= [x_0y, g^{\alpha}][x_0y, g^{\alpha}, g^{\alpha^2}][g^{\alpha^2}, (x_0y)^{\alpha^2}] \\ &= [y, g^{\alpha}][y, g^{\alpha}, g^{\alpha^2}][g^{\alpha^2}, y^{\alpha^2}]\end{aligned} \tag{3.2}$$

Note that $G$ is nilpotent of class at most 3 (see e.g. [10], Corollary 3 on page 45). By expanding the commutators in the middle equality of (3.2) and using the fact that commutators of weight 3 are central, we have

$$(gg^{\alpha}g^{\alpha^2})^{-1} = [x_0y, g^{\alpha}][x_0y, g^{\alpha}, g^{\alpha^2}][g^{\alpha^2}, (x_0y)^{\alpha^2}]$$
$$= [x_0, g^{\alpha}][x_0, g^{\alpha}, y][y, g^{\alpha}][x_0, g^{\alpha}, g^{\alpha^2}][y, g^{\alpha}, g^{\alpha^2}][g^{\alpha^2}, y^{\alpha^2}][g^{\alpha^2}, x_0^{\alpha^2}][g^{\alpha^2}, x_0^{\alpha^2}, y^{\alpha^2}]$$
$$= [x_0, g^{\alpha}][x_0, g^{\alpha}, g^{\alpha^2}][g^{\alpha^2}, x_0^{\alpha^2}][y, g^{\alpha}][y, g^{\alpha}, g^{\alpha^2}][g^{\alpha^2}, y^{\alpha^2}][x_0, g^{\alpha}, y][g^{\alpha^2}, x_0^{\alpha^2}, y^{\alpha^2}]$$
$$= (gg^{\alpha}g^{\alpha^2})^{-1}(gg^{\alpha}g^{\alpha^2})^{-1}[x_0, g^{\alpha}, y][g^{\alpha^2}, x_0^{\alpha^2}, y^{\alpha^2}].$$

Therefore

$$gg^{\alpha}g^{\alpha^2} = [x_0, g^{\alpha}, y][g^{\alpha^2}, x_0^{\alpha^2}, y^{\alpha^2}] \text{ for all } y \in W. \tag{3.3}$$

Now consider $T = W \cap y_0^{-1}W$ for some $y_0 \in W$. Since $m(T) > 0$, there exists $z \in W$ such that $y_0z \in W$. It follows from (3.3) that

$$gg^{\alpha}g^{\alpha^2} = [x_0, g^{\alpha}, y_0z][g^{\alpha^2}, x_0^{\alpha^2}, (y_0z)^{\alpha^2}] = [x_0, g^{\alpha}, y_0][g^{\alpha^2}, x_0^{\alpha^2}, y_0^{\alpha^2}].$$

Since commutators of weight 3 are central in G,

$$[x_0, g^{\alpha}, y_0z][g^{\alpha^2}, x_0^{\alpha^2}, (y_0z)^{\alpha^2}]$$
$$= [x_0, g^{\alpha}, y_0][g^{\alpha^2}, x_0^{\alpha^2}, y_0^{\alpha^2}][x_0, g^{\alpha}, z][g^{\alpha^2}, x_0^{\alpha^2}, z^{\alpha^2}].$$

Therefore,

$$[x_0, g^{\alpha}, z][g^{\alpha^2}, x_0^{\alpha^2}, z^{\alpha^2}] = 1$$

and so, as $z \in W$, it follows from (4) that $gg^{\alpha}g^{\alpha^2} = 1$. This completes the proof. □

Using the first part of our proof of Theorem 3.1 and applying Theorem 6.5 of [2] (cf. [1], Théorème 5) the number 15/16 can be reduced to 7/8.

In [6] groups G having an automorphism $\phi \in \mathrm{Aut}(G)$ with $\phi^n = 1$ such that $X_{n,\phi}(G)$ is a large set in the sense of [1] for $n = 3, 4$ are studied. For the case $n = 3$ it is proved in [6] that $G = X_{3,\phi}(G)$. For the case $n = 4$, it is proved that a normal solvable subgroup H of G of derived length $d \geqslant 3$ is nilpotent of class at most $\frac{9^{d-2}+1}{2}$. It is interesting to know if the same latter result is valid by replacing "largness" of $X_{n,\phi}(G)$ by weaker condition "k-largness" for some k in the sense of [2]. If the latter is valid, then for compact groups G with $m_G(X_{4,\phi}(G)) > 1 - \frac{1}{k}$, where $\phi$ is continuous, by Lemma 2.1 we have that $G = X_{4,\phi}(G)$.

**Theorem 3.2** $c_3 < 1$.

Proof — It follows from Theorem 3.1. □

We now see that Conjecture 1.1 of Lévai and Pyber is true for $n = 3$.

**Theorem 3.3** *Let* $G$ *be a profinite group such that the equation* $x^3 = 1$ *holds on a set with positive Haar measure. Then the solutions set of the equation* $x^3 = 1$ *has non-empty interior.*

Proof — It follows from Theorem 3.2 and Proposition 2.6. □

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