Advances in Group Theory and Applications
© 2021 AGTA - www.advgrouptheory.com/journal
12 (2021), pp. 79–89
ISSN: 2499-1287
DOI: 10.32037/agta-2021-013

Basic 3-Transpositions of the Symplectic Group $Sp(2n, 2)$ *

Jamshid Moori

(*Received Nov. 16, 2020; Accepted Feb. 15, 2021 — Communicated by F. de Giovanni*)

Abstract

In this paper we aim to study maximal pairwise commuting sets of 3-transpositions (transvections) of the simple symplectic group $Sp(2n, 2)$, and to construct designs from these sets. Any maximal set of pairwise 3-transpositions is called a *basic* set of transpositions. Let $G = Sp(2n, 2)$. It is well-known that G is a 3-transposition group with the set D, the conjugacy class consisting of its transvections, as the set of 3-transpositions. Let L be a set of basic transpositions in D. We aim to give general descriptions of L and $1-(v, k, \lambda)$ designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, with $\mathcal{P} = D$ and $\mathcal{B} = \{L^g \mid g \in G\}$. The parameters $k = |L|$, λ and further properties of $\mathcal{D}$ are determined. We also, as examples, apply the method to the symplectic simple groups $Sp(6, 2)$, $Sp(8, 2)$ and $Sp(10, 2)$.

Mathematics Subject Classification (2020): 20D05, 20D06, 05B05, 20G40

Keywords: design; simple group; symplectic group; 3-transposition group; basic transposition; commuting set; transvection

1 Introduction

Let G be a finite group generated by a class D of involutions such that any pair of non-commuting elements of D generate a dihedral group of order 6. Then D is called a class of conjugate 3-*transpositions* and G

* The author would like to thank the School of Mathematics at the University of Birmingham (UK)

a *3-transposition* group. Note that if $a, b \in D$ are such that $ab \neq ba$ then $o(ab) = 3$. Fischer ([6]) in his original studies on these groups considers the maximal commuting sets of 3-transpositions and denotes any such set by L. The set L is defined to be a *basic* set of transpositions. The *width* of G is defined to be the size of L and is denoted by $w_D(G)$. The normalizer $N_G(L)$, that is the stabilizer of L under conjugation, plays an important role in his classification of 3-transposition groups.

Let $G = Sp(2n, 2)$. It is well-known that G is a 3-transposition group, where the set D of 3-transpositions is the conjugacy class of its transvections. In this paper we aim first to study the maximal pairwise commuting sets of 3-transpositions (transvections) of G. Let L be a set of basic transpositions in D. We aim to give general descriptions of L. Secondly we aim to construct $1-(v, k, \lambda)$ designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, with $\mathcal{P} = D$ and $\mathcal{B} = \{L^g \mid g \in G\}$. The parameters $k = |L|$, λ and further properties of $\mathcal{D}$ are determined. We also, as examples, apply the method to the symplectic simple groups $Sp(6, 2)$, $Sp(8, 2)$ and $Sp(10, 2)$.

Recently in [13] we applied our method to the several 3-transposition groups, namely the Symmetric groups S_n and Fischer groups F_i for $i \in \{21, 22, 23, 24\}$. We must also add here that a good number of publications has been devoted to constructing designs and codes from finite simple groups. For example interested readers could be referred to [5],[8],[9],[10],[11],[14],[15] and [16].

2 Background, terminology and basic results

Our notation will be standard, and it is as in [2] and [12] for designs, and 𝔸𝕋𝕃𝔸𝕊 [4] for groups, finite simple groups and their maximal subgroups. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The *complement* of $\mathcal{D}$ is the structure $\widetilde{\mathcal{D}} = (\mathcal{P}, \mathcal{B}, \widetilde{\mathcal{I}})$, where $\widetilde{\mathcal{I}} = \mathcal{P} \times \mathcal{B} - \mathcal{I}$. The *dual* structure of $\mathcal{D}$ is $\mathcal{D}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I}^t)$, where $(B, P) \in \mathcal{I}^t$ if and only if $(P, B) \in \mathcal{I}$. Thus the transpose of an incidence matrix for $\mathcal{D}$ is an incidence matrix for $\mathcal{D}^t$. We will say that the design is *symmetric* if it has the same number of points and blocks, and *self dual* if it is isomorphic to its dual.

The groups G.H, G:H, and G˙H denote a general extension, a split extension and a non-split extension respectively. For a prime p, p^n denotes the elementary abelian group of order p^n.

Let G be a finite 3-transposition group generated by a class D of conjugate 3-transpositions. Fischer in [6] proved the following main theorem.

Theorem 1 *Let* G *be a finite 3-transposition group such that*

(i) $O_2(G)$ *and* $O_3(G)$ *lie in the centre of* G,

(ii) $G' = G''$.

Then $G/Z(G)$ *is isomorphic to a group in one of the following families:*

(a) S_n, *the symmetric groups,*

(b) $Sp(2n, 2)$, *the symplectic groups over* $GF(2)$,

(c) $O^{\mu}(2n, 2), \mu \in \{1, -1\}$, *the orthogonal groups over* $GF(2)$,

(d) $PSU(n, 2)$, *the projective special unitary groups over* $GF(4)$,

(e) $O^{\mu,\pi}(n, 3), \mu, \pi \in \{1, -1\}$, *the orthogonal groups over* $GF(3)$,

(f) F_{22}, F_{23} *and* F_{24}. *The first two groups are simple and the third one contains a simple subgroup of index 2.*

Let G be one of the groups in the above list, and L be a set of basic transpositions in D. Let S be a Sylow 2-subgroup of G containing L. Then we can easily show that $L = D \cap S$ and that $N_G(L)$ contains $\langle L \rangle$. It is also well-known that $N_G(L)$ contains a Sylow 2-subgroup of G and its action (by conjugation) on L is at least 2-transitive. Furthermore from the Fischer's work we have $C_G(L) = \langle L \rangle$ and that $N_G(L)/C_G(L)$ is

(i) $S_{|L|}$ or $A_{|L|}$ in cases (a) or (e),

(ii) $GL(n, 2)$ in the case (b),

(iii) $PSL([n/2], 4)$ in the case (d),

(iv) the holomorph of an elementary abelian 2-group of order 2^n in the case (c),

(v) the Mathieu groups $M_{2i}, i \in \{2, 3, 4\}$, in the case (f).

For $d \in D$ we define $D_d = C_D(d) \setminus \{d\}$ and $A_d = D \setminus C_D(d)$. Then D_d is a conjugacy class of elements of the group generated by D_d. This property allowed Fischer to use induction in order to prove his results on the classification of 3-transposition groups.

Proposition 2 *Assume that* G *is acting primitively by conjugation on* D *and* $w_D(G) \geqslant 2$*, then*

(i) G *is rank* 3 *on* D,

(ii) $C_G(d)$ *has three orbits* $\{d\}, D_d$ *and* A_d *on* D,

(iii) $\langle D_d \rangle$ *is transitive on* D_d,

(iv) $\langle C_D(d) \rangle$ *is transitive on* A_d.

Proof — See [6] and [1]. □

3 The symplectic group $Sp(2n, 2)$

Assume $G = Sp(2n, 2)$, the symplectic group acting on a $2n$-dimensional symplectic space V over $F = GF(2)$. Let D be the set of all symplectic transvections of G. There is a one-one correspondence between D and the nonzero elements of V and hence $|D| = 2^{2n} - 1$ with

$$|G| = 2^{n^2}(2^2 - 1)(2^4 - 1)\dots(2^{2n} - 1).$$

Using the above identification, we can see that for $d \in D$, $C_G(d)$ is the affine subgroup of the form $2^{2n-1}{:}Sp(2n-2, 2)$, see for example Mpono [17]. Furthermore, G acts primitively on D and $C_G(d)$ has three orbits $\{d\}$, D_d and A_d on D with

$$|D_d| = 2(2^{2n-2} - 1), \ |A_d| = 2^{2n-1},$$

and for $x \in A_d$ we have $\{x, d, d^x\}$ as a hyperbolic line (see Aschbacher [1]).

Let L be a set of basic 3-transpositions in D. We know that

$$N_G(L) = \langle L \rangle{:}GL(n, 2),$$

is a maximal parabolic subgroup of G (see for example Wilson [18]). In the following we study the structure of L and deduce that $\dim(\langle L \rangle) = n(n+1)/2$ with $|L| = 2^n - 1$.

Proposition 3 *Let* L *be a set of basic transpositions of* $G = Sp(2n, 2)$. *If* S *is a Sylow 2-subgroup of* G *containing* L, *then* $S = \langle L\rangle{:}T_n$ *where* T_n *is a Sylow 2-subgroup of* $GL(n, 2)$. *Furthermore, viewing* $\langle L\rangle$ *as a vector space over* GF(2), $\dim(\langle L\rangle) = n(n+1)/2$.

Proof — Since $S \geqslant \langle L\rangle$, by Section 2 we have that $N_G(L)$ contains S and hence S is a Sylow 2-subgroup of $N_G(L)$. Since

$$N_G(L) = \langle L\rangle{:}GL(n, 2),$$

we have $S = \langle L\rangle{:}T_n$ where T_n is a Sylow 2-subgroup of $GL(2n, 2)$. Furthermore since $|S| = 2^{n^2}$, we must have

$$2^{n^2} = |\langle L\rangle| \times |T_n| = \langle L\rangle \times 2^{n(n-1)/2}$$

and hence $|\langle L\rangle| = 2^{n^2-[n(n-1)/2]} = 2^{n(n+1)/2}$. Since $\langle L\rangle$ is an elementary abelian 2-group, we have $\dim(\langle L\rangle) = n(n+1)/2$. □

Remark 4 (i) Using [7], it can be shown that $\langle L\rangle$ consists of the following $2n \times 2n$ matrices over GF(2)

$$\left(\begin{array}{c|c} I_n & 0_n \\ \hline X & I_n \end{array}\right),$$

where X runs over all $n \times n$ symmetric matrices over GF(2).

(ii) As we have seen in Section 2, $L = D \cap S$. That is L is the set of all transvections in S. Let us denote by T_u, for any $0 \neq u \in V$, the corresponding transvection. Then, setting

$$H = u^{\perp},\ V = \langle w\rangle \oplus H,\ w \notin H,$$

we have $T_u(u) = u$ and $T_u(w) = w + u$. For $T_u \in L = D \cap S$, by part (i) we must have the following matrix form for T_u

$$\left(\begin{array}{c|c} I_n & 0_n \\ \hline X_u & I_n \end{array}\right),$$

where X_u runs over all $n \times n$ symmetric matrices over GF(2) satisfying $w_2 X_u = u_1$ with $u_2 = 0$, where $u = (u_1|u_2)$ and $w = (w_1|w_2)$ written as row vectors.

Let $B = \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ be a symplectic basis for V and let $f : V \times V \to GF(2)$ be a non-singular bilinear form on V such that all elements of B are prependicular to each other except that $f(e_i, f_i) = 1$ for $i \in \{1, 2, \ldots, n\}$. Let $W_n = \langle e_1, e_2, \ldots, e_n \rangle$.

Lemma 5 $D \cap S = L = \{T_u : u = (u_1|0),\ 0 \neq u_1 \in W_n\}$, *and hence* $|L| = 2^n - 1$.

Proof — Let $0 \neq u = (u_1|u_2) \in V$ with corresponding transvection $T_u \in D$. Then by Remark 4 all the transvections in $D \cap S$ we must have $u_2 = 0$ and the matrix form of T_u with respect to B is

$$\left(\begin{array}{c|c} I_n & 0_n \\ \hline X_u & I_n \end{array}\right).$$

Let $u_1 = \sum_{i=1}^{n} \lambda_i e_i, \lambda_i \in \{0, 1\}$. Then clearly we must have

$$T_u(f_i) = \begin{cases} f_i + u & \text{if } \lambda_i = 1 \\ f_i & \text{if } \lambda_i = 0. \end{cases}$$

Thus all the transvections in L are of the form T_u with $0 \neq u = (u_1|0)$, $u_1 \in W_n$. Therefore $|L| = |W_n| - 1 = 2^n - 1$. □

Remark 6 Consider L when $n = 3$. Then by Lemma 5 we have $|L| = 2^3 - 1 = 7$. Here we have

$$B = \{e_1, e_2, e_3, f_1, f_2, f_3\}, W = \langle e_1, e_2, e_3 \rangle.$$

Furthermore

$$L = \{T_{e_1}, T_{e_2}, T_{e_3}, T_{e_1+e_2}, T_{e_1+e_3}, T_{e_2+e_3}, T_{e_1+e_2+e_3}\}$$

with the following corresponding matrices:

$$T_{e_1} \sim \left(\begin{array}{ccc|c} & I_3 & & 0_3 \\ \hline 1 & 0 & 0 & \\ 0 & 0 & 0 & I_3 \\ 0 & 0 & 0 & \end{array}\right), \quad T_{e_2} \sim \left(\begin{array}{ccc|c} & I_3 & & 0_3 \\ \hline 0 & 0 & 0 & \\ 0 & 1 & 0 & I_3 \\ 0 & 0 & 0 & \end{array}\right),$$

$$T_{e_3} \sim \left(\begin{array}{ccc|c} & I_3 & & 0_3 \\ \hline 0 & 0 & 0 & \\ 0 & 0 & 0 & I_3 \\ 0 & 0 & 1 & \end{array}\right), \quad T_{e_1+e_2} \sim \left(\begin{array}{ccc|c} & I_3 & & 0_3 \\ \hline 1 & 1 & 0 & \\ 1 & 1 & 0 & I_3 \\ 0 & 0 & 0 & \end{array}\right),$$

$$T_{e_1+e_3} \sim \left(\begin{array}{ccc|c} & I_3 & & 0_3 \\ \hline 1 & 0 & 1 & \\ 0 & 0 & 0 & I_3 \\ 1 & 0 & 1 & \end{array}\right), \; T_{e_2+e_3} \sim \left(\begin{array}{ccc|c} & I_3 & & 0_3 \\ \hline 0 & 0 & 0 & \\ 0 & 1 & 1 & I_3 \\ 0 & 1 & 1 & \end{array}\right),$$

$$T_{e_1+e_2+e_3} \sim \left(\begin{array}{ccc|c} & I_3 & & 0_3 \\ \hline 1 & 1 & 1 & \\ 1 & 1 & 1 & I_3 \\ 1 & 1 & 1 & \end{array}\right).$$

Note that, for example,

$$T_{e_1+e_3}(f_1) = f_1 + e_1 + e_3, \; T_{e_1+e_3}(f_3) = f_3 + e_1 + e_3,$$

$$T_{e_1+e_3}(f_2) = f_2,$$

and

$$T_{e_1+e_2+e_3}(f_1) = f_1 + e_1 + e_2 + e_3,$$

$$T_{e_1+e_2+e_3}(f_2) = f_2 + e_1 + e_2 + e_3,$$

$$T_{e_1+e_2+e_3}(f_3) = f_3 + e_1 + e_2 + e_3.$$

4 Designs from basic transpositions of $Sp(2n, 2)$

Let $G = Sp(2n, 2)$. As we have seen in previous sections, G is a 3-transposition group with the set D, the conjugacy class consisting of its transvections, as the set of 3-transpositions. Let L be a set of basic transpositions in D. In Section 3 (see Proposition 3 and Lemma 5) we gave a general descriptions of L. In this section we aim to construct $1-(v,k,\lambda)$ designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, with $\mathcal{P} = D$ and $\mathcal{B} = \{L^g \mid g \in G\}$. The parameters $k = |L|$, λ and further properties of $\mathcal{D}$ will be determined. We also, as examples, apply the method to the symplectic simple groups $Sp(6, 2)$, $Sp(8, 2)$ and $Sp(10, 2)$.

Theorem 7 *Let* $G = Sp(2n, 2)$ *with* D *as its conjugacy class of transvections and* $B = L$ *a set of basic transpositions in* D. *Let* $\mathcal{B} = \{B^g | g \in G\}$, $\mathcal{P} = D$. *Then we have a* $1-(2^{2n}-1, 2^n-1, \lambda)$ *design* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ *with* $\prod_{i=1}^{n}(1+2^i)$ *blocks where* $\lambda = \prod_{i=1}^{n-1}(1+2^i)$. *Furthermore, The* *group* G *acts as an automorphism group of* $\mathcal{D}$*, primitive both on points and blocks of* $\mathcal{D}$*.*

Proof — Note for $d \in D$ we have $|D| = [G : C_G(d)]$. As seen in Section 3, $C_G(d)$ is the affine subgroup of the form

$$2^{2n-1}{:}Sp(2n-2,2)$$

and $|D| = 2^{2n} - 1$. If k is the size of each block, then since $B = L$, we have $|B| = k = |L| = 2^n - 1$ by Lemma 5. Now using Proposition 3, we have

$$G_B = \{g \in G : B^g = B\} = N_G(L) = \langle L\rangle{:}GL(n,2) \simeq 2^{n(n+1)/2}{:}GL(n,2).$$

Hence,

$$b = [G : N_G(L)] = |Sp(2n,2)|/\left[2^{n(n+1)/2} \times |GL(n,2)|\right] = \prod_{i=1}^{n}(1+2^i)$$

is the number of distinct blocks.

Suppose that there are λ blocks B_i containing d. If d' is another element of D, then $d' = d^g$ for some $g \in G$ and hence the λ blocks B_i^g contain d'. Therefore we have a $1-(v,k,\lambda)$ design $\mathcal{D}$ with $v = |D|$. Since $kb = \lambda v$, we deduce that

$$\begin{aligned}\lambda &= kb/v = |L| \times b/|D| \\ &= (2^n-1) \times \prod_{i=1}^{n}(1+2^i)/(2^{2n}-1) \\ &= \prod_{i=1}^{n}(1+2^i)/(2^n+1) = \prod_{i=1}^{n-1}(1+2^i).\end{aligned}$$

The action of G on points arises from the action of G on D. Now $\mathcal{B} = B^G$ implies that G is transitive on $\mathcal{B}$ with

$$G_B = \{g \in G : B^g = B\}$$

as the stabiliser of the action on blocks. Clearly G acts as an automorphism group on $\mathcal{D}$, transitive both on points and blocks. Since G acts primitively on D (note $C_G(d)$ is maximal in G), G acts primitively on points of $\mathcal{D}$. The action of G on $\mathcal{B}$ is equivalent to the action of G on the cosets of $G_B = N_G(L)$. Since $N_G(L) = 2^{n(n+1)/2} \times GL(n, 2)$ is maximal in G (a maximal parabolic subgroup), the action on blocks is also primitive. □

Corollary 8 *Let $\mathcal{D}_{2n}$ and $\mathcal{D}_{2n-2}$ be designs constructed from basic transpositions of* $Sp(2n, 2)$ *and* $Sp(2n-2, 2)$ *respectively. Then*

(i) $\mathcal{D}_{2n}$ *is a* $1-(2^{2n}-1, 2^n-1, \lambda_{2n})$ *design with* b_{2n} *blocks, where*

$$\lambda_{2n} = b_{2n}/(2^n + 1),$$

and

(ii) $b_{2n} = (1+2^n) \times b_{2n-2}$ *and* $\lambda_{2n} = (1+2^{n-1}) \times \lambda_{2n-2} = b_{2n-2}$.

Proof — (i) By Theorem 7, we have

$$\lambda_{2n} = \prod_{i=1}^{n}(1+2^i)/(2^n+1) = b_{2n}/(2^n+1).$$

(ii) We have

$$b_{2n} = \prod_{i=1}^{n}(1+2^i) = (1+2^n) \times \prod_{i=1}^{n-1}(1+2^i) = (1+2^n) \times b_{2n-2}.$$

Now by part (i) we have $\lambda_{2n} = b_{2n}/(2^n+1)$, and hence $\lambda_{2n} = b_{2n-2}$. Since

$$\lambda_{2n-2} = b_{2n-2}/(2^{n-1}+1) \quad \text{and} \quad \lambda_{2n} = b_{2n-2},$$

we have $\lambda_{2n-2} = \lambda_{2n}/(2^{n-1}+1)$, i.e. $\lambda_{2n} = \lambda_{2n-2} \times (2^{n-1}+1)$. □

We apply the results obtained in Sections 3 and 4 to $Sp(6, 2), Sp(8, 2)$ and $Sp(10, 2)$. These are summarized in Table 1. Computations with Magma [3] show that $|Aut(\mathcal{D})| = |G|$, and since $Aut(\mathcal{D}) \geqslant G$, we must have $Aut(\mathcal{D}) = G$.

Table 1: Results for $Sp(6,2)$, $Sp(8,2)$, $Sp(10,2)$

G	$\|D\|$	$\|L\|$	$\langle L\rangle$	$N_G(L)$	$\mathcal{D}_{2n}$	$Aut(\mathcal{D}_{2n})$
$Sp(6,2)$	63	7	2^6	$2^6{:}GL(3,2)$	$1-(63,7,15)$ $\lambda_6=15$ $b_6=135$	$Sp(6,2)$
$Sp(8,2)$	255	15	2^{10}	$2^{10}{:}GL(4,2)$	$1-(255,15,135)$ $\lambda_8=135$ $b_8=2295$	$Sp(8,2)$
$Sp(10,2)$	1023	31	2^{15}	$2^{15}{:}GL(5,2)$	$1-(1023,31,2295)$ $\lambda_{10}=2295$ $b_{10}=75735$	$Sp(10,2)$

REFERENCES

[1] M. Aschbacher: "3-Transposition Groups", *Cambridge University Press*, Cambridge (1997).

[2] E.F. Assmus, Jr. – J.D. Key: "Designs and their Codes", *Cambridge University Press*, Cambridge (1993).

[3] W. Bosma – J. Cannon: "Handbook of Magma Functions", *Department of Mathematics, University of Sydney* (November, 1994).

[4] J.H. Conway – R.T. Curtis – S.P. Norton – R.A. Parker – R.A. Wilson: "An Atlas of Finite Groups", *Oxford University Press*, Oxford (1985).

[5] D. Crnkovic – V. Mikulic Crnkovic – B. Rodrigues: "Designs, strongly regular graphs and codes from some primitive groups", in: Information Security, Coding Theory and Related Combinatorics – Information Coding and Combinatorics, *NATO Advanced Study Institute*, Croatia (2011), 231–252.

[6] B. Fischer: "Finite groups generated by 3-transpositions", *J. Combin. Theory Ser. A* 62 (1993), 225–233.

[7] R. Gow – M. Marjoram – A. Previtali: "On the irreducible characters of a Sylow 2-subgroup of the finite symplectic group in characteristic 2", *J. Algebra* 241 (2001), 393–409.

[8] W. Haemers – C. Parker – V. Pless – V.D. Tonchev: "A design and a code invariant under the simple group Co_3", *J. Combin. Theory Ser. A* 62 (1993), 225–233.

[9] J.D. Key – J. Moori: "Designs, codes and graphs from the Janko groups J_1 and J_2", *J. Combin. Math. Combin. Comput.* 40 (2002), 143–159.

[10] J.D. Key – J. Moori – B.G. Rodrigues: "On some designs and codes from primitive representations of some finite simple groups", *J. Combin. Math. Combin. Comput.* 45 (2003), 3–19.

[11] T. Le – J. Moori: "On the automorphisms of designs constructed from finite simple groups", *Des. Codes Cryptogr.* 76 (2015), 505–517.

[12] J. Moori: "Finite groups, designs and codes", in: Information Security, Coding Theory and Related Combinatorics – Information Coding and Combinatorics, *NATO Advanced Study Institute*, Croatia (2011), 202–230.

[13] J. Moori: "Designs and codes from basic 3-transpositions", to appear.

[14] J. Moori – B. G. Rodrigues, "Some designs and codes invariant under the simple group Co_2", *J. Algebra* 316 (2007), 649–661.

[15] J. Moori – A. Saeidi: "Some designs and codes invariant under the Tits group", *Adv. Math. Commun.* 11 (2017), 77–82.

[16] J. Moori – A. Saeidi: "Constructing some designs invariant under $PSL_2(q)$, q even", *Comm. Algebra* 46 (2018), 160–166.

[17] Z.E. Mpono: "Fischer-Clifford Theory and Character Tables of Group Extensions", *PhD Thesis*, University of Natal, Pietermaritzburg (1998).

[18] R.A. Wilson: "The Finite Simple Groups", *Springer*, London (2009).

Jamshid Moori
School of Mathematical and Physical Sciences
North-West University (Mafikeng)
P Bag X2046, Mmabatho 2735, South Africa
e-mail: jamshid.moori@nwu.ac.za