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About an Extension of the Mirsky-Newman, Davenport-Rado Result to the Herzog-Schönheim Conjecture for Free Groups

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Abstract

Let G be a group and $H_1, \dots, H_s$ be subgroups of G of indices $d_1, \dots, d_s$ respectively. In 1974, M. Herzog and J. Schönheim conjectured that if $\{H_i\alpha_i\}_{i=1}^{i=s}$, $\alpha_i \in G$, is a coset partition of G, then $d_1, \dots, d_s$ cannot be distinct. We consider the Herzog-Schönheim conjecture for free groups of finite rank and propose a new approach, based on an extension of the Mirsky-Newman, Davenport-Rado result for $G = \mathbb{Z}$.

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1 Introduction

Let G be a group, $s > 1$ a natural number, and $H_1, \dots, H_s$ be subgroups of G. If there exist $\alpha_i \in G$ such that $G = \bigcup_{i=1}^{i=s} H_i\alpha_i$, and the sets $H_i\alpha_i$, $1 \leqslant i \leqslant s$, are pairwise disjoint, then $\{H_i\alpha_i\}_{i=1}^{i=s}$ is a *coset partition of* G (or a *disjoint cover of* G). In this case, all the subgroups $H_1, \dots, H_s$ can be assumed to be of finite index in G (see [28, 29, 25]). We denote by $d_1, \dots, d_s$ the indices of $H_1, \dots, H_s$ respectively. The coset partition $\{H_i\alpha_i\}_{i=1}^{i=s}$ has *multiplicity* if $d_i = d_j$ for some $i \neq j$.

In 1974, M. Herzog and J. Schönheim conjectured that any coset partition of any group G has multiplicity. In the 1980's, in a series of papers, M.A. Berger, A. Felzenbaum and A.S. Fraenkel studied the Herzog-Schönheim conjecture (see [2, 3, 4]) and in [5] they proved the conjecture is true for the pyramidal groups, a subclass of the finite solvable groups. Coset partitions of finite groups with additional assumptions on the subgroups of the partition have been extensively studied. We refer to [7, 44, 45, 43]. In [27], the authors very recently proved that the conjecture is true for all groups of order less than 1440.

The common approach to the Herzog-Schönheim (HS) conjecture is to study it in finite groups. Indeed, given any group G, every coset partition of G induces a coset partition of a finite quotient group of G with the same indices (see [25]). In [8], we initiated a completely different approach to the Herzog-Schönheim conjecture. The idea is to study it in free groups of finite rank and from there to provide answers for every group. This is possible since any finite or finitely generated group is a quotient group of a free group of finite rank and any coset partition of a quotient group F/N induces a coset partition of F with the same indices (see [8]). In [8], we consider free groups of finite rank and develop a new combinatorial approach to the problem, based on the machinery of covering spaces. The fundamental group of the bouquet with $n \geqslant 1$ leaves (or the wedge sum of n circles), X, is F_n, the free group of finite rank n. As X is a "good" space (connected, locally path connected and semilocally 1-connected), X has a universal covering which can be identified with the Cayley graph of F_n, an infinite simplicial tree. Furthermore, there exists a one-to-one correspondence between the subgroups of F_n and the covering spaces (together with a chosen point) of X.

For any subgroup H of F_n of finite index d, there exists a d-sheeted covering space $(\widetilde{X}_H, p)$ with a fixed basepoint. The underlying graph of $(\widetilde{X}_H, p)$ is a directed labelled graph with d vertices. We call it the *Schreier graph of* H and denote it by $\widetilde{X}_H$. It can be seen also as a finite complete bi-deterministic automaton; fixing the start and the end state at the basepoint, it recognises the set of elements in H. It is called the *Schreier coset diagram for* F_n *relative to the subgroup* H (see [40, p.107]) or the *Schreier automaton for* F_n *relative to the subgroup* H (see [38, p.102]). The d vertices (or states) correspond to the d right cosets of H, each edge (or transition) $Ht \xrightarrow{a} Hta$, $t \in F_n$, a a generator of F_n, describes the right action of a on Ht. If we fix the start state at H, the basepoint, and the end state at another vertex Ht, where t denotes the label of some path from the start state to the end state, then this automaton recognises the set of elements in Ht and we call it the *Schreier automaton of* Ht and denote it by $\widetilde{X}_{Ht}$.

In general, for any automaton M, with alphabet Σ, and d states, there exists a square matrix A of order $d \times d$, with a_{ij} equal to the number of directed edges from vertex i to vertex j, $1 \leqslant i, j \leqslant d$. This matrix is non-negative and it is called the *transition matrix* [13]. If for every $1 \leqslant i, j \leqslant d$, there exists $m \in \mathbb{Z}^+$ such that $(A^m)_{ij} > 0$, the matrix is *irreducible*. For an irreducible non-negative matrix A, the *period of* A is the gcd of all $m \in \mathbb{Z}^+$ such that $(A^m)_{ii} > 0$ (for any i). If i and j denote respectively the start and end states of M, then the number of words of length k (in the alphabet Σ) accepted by M is $a_k = (A^k)_{ij}$. The *generating function of* M is defined by $p(z) = \sum_{k=0}^{k=\infty} a_k z^k$. It is a rational function: the fraction of two polynomials in z with integer coefficients (see [13] and [39, p.575]).

The intuitive idea behind our approach in this paper is as follows. Let $F_n = \langle \Sigma \rangle$, and Σ^* the free monoid generated by Σ. Let $\{H_i \alpha_i\}_{i=1}^{i=s}$ be a coset partition of F_n with $H_i < F_n$ of index $d_i > 1$, $\alpha_i \in F_n$, $1 \leqslant i \leqslant s$. Let $\widetilde{X}_i$ denote the Schreier automaton of $H_i \alpha_i$, with transition matrix A_i and generating function $p_i(z)$, $1 \leqslant i \leqslant s$. For each $\widetilde{X}_i$, A_i is a non-negative irreducible matrix and $a_{i,k}$, $k \geqslant 0$, counts the number of words of length k that belong to $H_i \alpha_i \cap \Sigma^*$. Since F_n is the disjoint union of the sets $\{H_i \alpha_i\}_{i=1}^{i=s}$, each element in Σ^* belongs to one and exactly one such set, so n^k, the number of words of length k in Σ^*, satisfies $n^k = \sum_{i=1}^{i=s} a_{i,k}$, for every $k \geqslant 0$. So,

$$\sum_{k=0}^{k=\infty} n^k z^k = \sum_{i=1}^{i=s} p_i(z).$$

Using this kind of counting argument, we prove that there is a repetition of the maximal period $h > 1$ and in some cases we could prove there is a repetition of the index also.

Theorem 1 *Let F_n be the free group on $n \geqslant 1$ generators. Let $\{H_i \alpha_i\}_{i=1}^{i=s}$, $s > 1$, be a coset partition of F_n with $H_i < F_n$ of index d_i, $\alpha_i \in F_n$, $1 \leqslant i \leqslant s$, and $1 < d_1 \leqslant \ldots \leqslant d_s$. Let $\widetilde{X}_i$ denote the Schreier graph of H_i, with transition matrix A_i, and period $h_i \geqslant 1$, $1 \leqslant i \leqslant s$. Then, for every $1 \leqslant i \leqslant s$, there exists $j \neq i$ such that $h_i \mid h_j$. In particular, any period h_i which does not properly divide any other period has multiplicity* *at least two.*

If $n = 1$ in Theorem 1, $\{H_i\alpha_i\}_{i=1}^{i=s}$ is a coset partition of $\mathbb{Z}$. A coset partition of $\mathbb{Z}$ is $\{d_i\mathbb{Z}+r_i\}_{i=1}^{i=s}$, $r_i \in \mathbb{Z}$, with each $d_i\mathbb{Z}+r_i$ the residue class of r_i modulo d_i. These coset partitions of $\mathbb{Z}$ were first introduced by P. Erdős [14] and he conjectured that if $\{d_i\mathbb{Z}+r_i\}_{i=1}^{i=s}$, $r_i \in \mathbb{Z}$, is a coset partition of $\mathbb{Z}$, then the largest index d_s appears at least twice. Erdős' conjecture was proved by L. Mirsky and D. Newman and independently by H. Davenport and R. Rado, using analysis of complex function. Their proof appears in P. Erdős' paper [16, p.126]. So, with Theorem 1, we recover the Mirsky-Newman, Davenport-Rado result for the Erdős' conjecture. Indeed, for every index d, the Schreier graph of $d\mathbb{Z}$ has a transition matrix with period equal to d, so a repetition of the period is equivalent to a repetition of the index: for the unique subgroup H of $\mathbb{Z}$ of index d, its Schreier graph $\widetilde{X}_H$ is a closed directed path of length d (with each edge labelled 1), and its transition matrix A is the permutation matrix corresponding to the d-cycle $(1,2,\ldots,d)$, with period d.

Some consequences of the Mirsky-Newman, Davenport-Rado result for the Erdős' conjecture were proved: the largest index d_s appears at least p times, where p is the smallest prime dividing d_s (see [29, 30, 41]), each index d_i divides another index d_j, $j \neq i$, and each index d_k that does not properly divide any other index appears at least twice (see [46]). With Theorem 1, we recover these consequences with index replaced by period, as for the free groups in general, the repetition of the period is not equivalent to the repetition of the index. In Theorem 2, we prove that in some cases, the repetition of the period implies the repetition of the index. More precisely, we have the following theorem.

Theorem 2 *Let F_n be the free group on $n \geqslant 1$ generators. Let $\{H_i\alpha_i\}_{i=1}^{i=s}$, $s > 1$, be a coset partition of F_n with $H_i < F_n$ of index d_i, $\alpha_i \in F_n$, $1 \leqslant i \leqslant s$, and $1 < d_1 \leqslant \ldots \leqslant d_s$. Let $\widetilde{X}_i$ denote the Schreier graph of H_i, with transition matrix A_i, and period $h_i \geqslant 1$, $1 \leqslant i \leqslant s$. Let h_i be a period that does not properly divide any other period. Let $J = \{1 \leqslant j \leqslant s \mid h_j = h_i\}$, and let $k \in J$ such that $d_k = \max\{d_j\}_{j\in J}$. If the period of A_k is d_k, then d_k has multiplicity at least two.*

Note that Theorems 1 and 2 can be extended to coset partitions of finitely generated groups. Indeed, given a finitely generated group $G \simeq F_n/N$, any coset partition of G gives rise to a coset partition of F_n with subgroups of the same indices (see [8]). To each subgroup H of finite index d in G, there corresponds a subgroup K of finite index d in F_n, $N \subseteq K$, and $H \simeq K/N$. So, each coset of H gives rise to a bi-deterministic automaton, and in turn to an irreducible transition matrix and all the arguments applied in the case of F_n can be applied also for G.

In [10, 11], we continue our study of the Schreier automaton and its generating function, and our results there are based on Theorems 1 and 2. The paper is organized as follows. In Section 2, we give some preliminaries on automata and their growth functions. In Section 3, we prove the main result. We also refer to [8] for more preliminaries and examples, Section 2 for free groups and covering spaces, and Section 3.1, for graphs. We refer to [22] for a recent proof of the Erdős' conjecture based on group representations.

2 Premilinaries on automata

2.1 Automata and generating function of their language

We refer the reader to [38, p.96], [12, p.7], [32, 33], [13]. A *finite state automaton* is a quintuple (S, Σ, μ, Y, s_0), where S is a finite set, called the *state set*, Σ is a finite set, called the *alphabet*, $\mu : S \times \Sigma \to S$ is a function, called the *transition function*, Y is a (possibly empty) subset of S called the *accept* (or *end*) *states*, and s_0 is called the *start state*. It is a directed graph with vertices the states and each transition $s \xrightarrow{a} s'$ between states s and s' is an edge with label $a \in \Sigma$. The *label of a path* p of length n is the product $a_1 a_2 \dots a_n$ of the labels of the edges of p. The finite state automaton $M = (S, \Sigma, \mu, Y, s_0)$ is *deterministic* if there is only one initial state and each state is the source of exactly one arrow with any given label from Σ. In a deterministic automaton, a path is determined by its starting point and its label [38, p.105]. It is *co-deterministic* if there is only one final state and each state is the target of exactly one arrow with any given label from Σ. The automaton $M = (S, \Sigma, \mu, Y, s_0)$ is *bi-deterministic* if it is both deterministic and co-deterministic. An automaton M is *complete* if for each state $s \in S$ and for each $a \in \Sigma$, there is exactly one edge from s labelled a.

Definition 3 Let $M = (S, \Sigma, \mu, Y, s_0)$ be a finite state automaton. Let Σ^* be the free monoid generated by Σ. Let $\mathrm{Map}(S, S)$ be the monoid consisting of all maps from S to S. The map $\phi : \Sigma \to \mathrm{Map}(S, S)$

given by μ can be extended in a unique way to a monoid homomorphism $\phi : \Sigma^* \to \mathrm{Map}(S, S)$. The range of this map is a monoid called the *transition monoid of* M, which is generated by $\{\phi(a) \mid a \in \Sigma\}$. An element $w \in \Sigma^*$ is *accepted* by M if the corresponding element of $\mathrm{Map}(S, S)$, $\phi(w)$, takes s_0 to an element of the accept states set Y. The set $L \subseteq \Sigma^*$ recognized by M is called the *language accepted by* M, denoted by $L(M)$.

For any directed graph with d vertices or any finite state automaton M, with alphabet Σ, and d states, there exists a square matrix A of order $d \times d$, with a_{ij} equal to the number of directed edges from vertex i to vertex j, $1 \leqslant i, j \leqslant d$. This matrix is non-negative and it is called the *transition matrix* (as in [13]) or the *adjacency matrix* (as in [39, p.575]). For any $k \geqslant 1$, $(A^k)_{ij}$ is equal to the number of directed paths of length k from vertex i to vertex j. If for every $1 \leqslant i, j \leqslant d$, there exists $m \in \mathbb{Z}^+$ such that $(A^m)_{ij} > 0$, the matrix is *irreducible*. For A an irreducible non-negative matrix, the *period of* A is the gcd of all $m \in \mathbb{Z}^+$ such that $(A^m)_{ii} > 0$ (for any i), or equivalently the gcd of the lengths of closed (directed) loops in the underlying graph or automaton. The period is bounded from above by d, the number of vertices. If the period is 1, A is called *aperiodic*.

Let M be a bi-deterministic automaton with alphabet Σ, d states, start state i, accept state j and transition matrix A. Let $a_k = (A^k)_{ij}$, the number of words of length k in the free monoid Σ^*, accepted by M. The function $p_{ij}(z) = \sum_{k=0}^{k=\infty} a_k z^k$ is called the *generating function of* M.

Theorem 4 (see [39], p.574) *The generating function* $p_{ij}(z)$ *is given by*

$$p_{ij}(z) = \frac{(-1)^{i+j} \det(I - zA : j, i)}{\det(I - zA)},$$

where $(B : j, i)$ *denotes the matrix obtained by removing the* j*th row and* i*th column of B,* $\det(I - zA)$ *is the reciprocal polynomial of the characteristic polynomial of* A.

2.2 The Schreier automaton of a coset of a subgroup of F_n

Let's introduce the special automata we are interested in, i.e. the *Schreier coset diagram for* F_n *relative to the subgroup* H (see [40, p.107]) or the *Schreier automaton for* F_n *relative to the subgroup* H (see [38, p.102]).

Definition 5 Let $F_n = \langle \Sigma \rangle$ and Σ^* the free monoid generated by Σ. Let $H < F_n$ of index d. Let $(\widetilde{X}_H, p)$ be the covering of the n-leaves bouquet with basepoint $\widetilde{x}_1$ and vertices $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_d$. Let $t_i \in \Sigma^*$ denote the label of a path from $\widetilde{x}_1$ to $\widetilde{x}_i$. Let $\mathcal{T} = \{1, t_i \mid 1 \leqslant i \leqslant d\}$. Let $\widetilde{X}_H$ be the Schreier coset diagram for F_n relative to the subgroup H, with $\widetilde{x}_1$ representing the subgroup H and the other vertices $\widetilde{x}_2, \ldots, \widetilde{x}_d$ representing the cosets Ht_i accordingly. We call $\widetilde{X}_H$ *the Schreier graph of* H, with this correspondence between the vertices $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_d$ and the cosets Ht_i accordingly.

From its definition, $\widetilde{X}_H$ is a strongly-connected graph (i.e any two vertices are connected by a directed path), so its transition matrix A is non-negative and irreducible. As $\widetilde{X}_H$ is a directed n-regular graph, the sum of the elements at each row and at each column of A is equal to n. So, from the Perron-Frobenius result for non-negative irreducible matrices, n is the *Perron-Frobenius eigenvalue* of A, that is the positive real eigenvalue with maximal absolute value [20, 18, 19, 31]. If A has period $h \geqslant 1$, then A is similar to the matrix $Ae^{\frac{2\pi i}{h}}$, that is the set

$$\left\{\lambda e^{\frac{2\pi i k}{h}} \mid 0 \leqslant k \leqslant h-1\right\}$$

is a set of eigenvalues of A, for each eigenvalue λ of A. In particular,

$$\left\{n e^{\frac{2\pi i k}{h}} \mid 0 \leqslant k \leqslant h-1\right\}$$

is a set of simple eigenvalues of A (see [1, Ch.16] and [6]). This implies that the period is bounded from above by the order of A.

Theorem 6 (see [17], p.343) *Let* A *be a non-negative and irreducible matrix of order* $d \times d$. *Let* $P(z) = \sum_{k=0}^{k=\infty} z^k A^k$. *Then*

(1) $P(z) = (I - zA)^{-1}$, *since* $(I + zA + z^2A^2 + \ldots)(I - zA) = I$;

(2) *All the entries of* $P(z)$ *have the same radius of convergence* $\frac{1}{\lambda_{PF}}$, *where* λ_{PF} *is the Perron-Frobenius eigenvalue of* A;

(3) $P(z) = I + z(AP(z)) = I + z(P(z)A)$.

Definition 7 Let $F_n = \langle \Sigma \rangle$ and Σ^* the free monoid generated by Σ. Let $H < F_n$ of index d. Let $\widetilde{X}_H$ be the Schreier graph of H. Using the notation from Definition 5, let $\widetilde{x}_1$ be the start state and $\widetilde{x}_j$ be the end state for some $1 \leqslant j \leqslant d$. We call the automaton obtained *the Schreier automaton of* Ht_j and denote it by $\widehat{X}_{Ht_j}$. The language accepted by $\widehat{X}_{Ht_j}$ is the set of elements in Σ^* that belong to Ht_j. We call the elements in $\Sigma^* \cap Ht_j$, *the positive words in* Ht_j. The identity may belong to this set.

In contrast with our approach in [8], as we are interested here in counting positive words of a given length, we do not add the inverses of the generators from Σ to the alphabet.

Example 8 Let $\Sigma = \{a, b\}$ be an alphabet. Let Σ^* be the free monoid generated by Σ. Let $F_2 = \langle a, b\rangle$. Let $H = \langle a^4, b^4, ab^{-1}, a^2b^{-2}, a^3b^{-3}\rangle$ be a subgroup of index 4 in F_2. Let $\widetilde{X}_H$ be the Schreier graph of H:

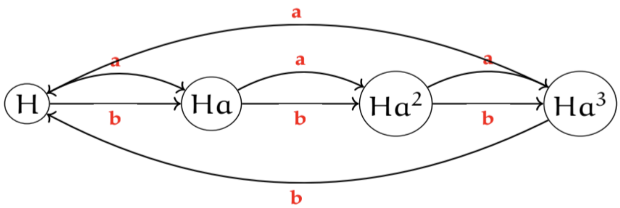

Figure 1: The Schreier graph $\widetilde{X}_H$ of $H = \langle a^4, b^4, ab^{-1}, a^2b^{-2}, a^3b^{-3}\rangle$.

The transition matrix of $\widetilde{X}_H$ is

$$\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

and its period is 4. If H is both the start and accept state, then the language accepted, L, is the set of elements in Σ^* that belong to H, that is the set of positive words in H. The generating function is then $p(z) = \frac{1}{1-16z^4}$, with $p(0) = 1$, since it contains 1. If H is the start state and Ha the accept state, then L is the set of positive words in the coset Ha, and $p(z) = \frac{2z}{1-16z^4}$.

3 The generating functions of Schreier automata

3.1 Properties of the Schreier automaton

For $H < F_n$ of index d, we prove some properties of its Schreier automaton, its transition matrix and its generating function. We use the following notation: $\widetilde{X}_H$ is the Schreier graph of H, A its transition matrix with period $h \geqslant 1$, and $p_{ij}(z)$ denotes the generating function of the Schreier automaton, with i and j the start and end states respectively. First, we show the following correlation between the generating functions of the Schreier automaton, with start state i fixed and and j running over all the possible end states.

Lemma 9 *For $|z| < \frac{1}{n}$, and every $1 \leqslant i \leqslant d$,*

$$\sum_{j=1}^{j=d} p_{ij}(z) = \sum_{j=1}^{j=d} \frac{(-1)^{i+j} \det(I - zA : j, i)}{\det(I - zA)} = \frac{1}{1 - nz}$$

Proof — The number of positive words of length $k \geqslant 0$ in F_n is n^k, so the generating function of F_n is $\sum_{k=0}^{k=\infty} n^k z^k = \frac{1}{1-nz}$, for z with $|z| < \frac{1}{n}$. As F_n is the disjoint union of the d cosets of H, the generating function of F_n is equal to the sum of the generating functions corresponding to each coset of H. $\square$

Lemma 10 *Let λ be a non-zero eigenvalue of A of algebraic multiplicity n_λ. Then $\frac{1}{\lambda}$ is a pole of $\frac{1}{\det(I-zA)}$ of order n_λ. Moreover,*

$$\left\{ \frac{1}{n} e^{\frac{2\pi i m}{h}} \;\middle|\; 0 \leqslant m \leqslant h-1 \right\}$$

is a set of simple poles of $\frac{1}{\det(I-zA)}$ of minimal absolute value.

Proof — For any eigenvalue λ of A with algebraic multiplicity n_λ, $1 - \lambda z$ is an eigenvalue of $I - zA$ with same algebraic multiplicity n_λ. And if $\lambda \neq 0$, $\frac{1}{\lambda}$ is a pole of $\frac{1}{\det(I-zA)}$ of order n_λ. From the Perron-Frobenius result for non-negative irreducible matrices, n is the Perron-Frobenius eigenvalue of A. So, $\frac{1}{n}$ is a simple pole of $\frac{1}{\det(I-zA)}$. As n is the eigenvalue of A of maximal absolute value, $\frac{1}{n}$ is the pole of $\frac{1}{\det(I-zA)}$ of minimal absolute value. The same holds for $\frac{1}{n} e^{\frac{2\pi i m}{h}}$. $\square$

Note that if $\frac{1}{\lambda}$, $\lambda \neq 0$, is a pole of $\frac{1}{\det(I-zA)}$, then it may occur that for some generating function $p_{ij}(z)$, it may not be a pole anymore. Indeed, if $\frac{1}{\lambda}$ is not in the domain of convergence of the power series $\sum_{k=0}^{k=\infty}(z^k A^k)_{ij}$, there is no correlation anymore between the power series and the function $p_{ij}(z)$, and a simplification may occur between the numerator and the denumerator of $p_{ij}(z)$.

Lemma 11 *For every $1 \leqslant i,j \leqslant d$, $p_{ij}(z)$ have the same radius of convergence $\frac{1}{n}$. Furthermore, for every $1 \leqslant i,j \leqslant d$, $\left\{\frac{1}{n}e^{\frac{2\pi i m}{h}} \mid 0 \leqslant m \leqslant h-1\right\}$ is a set of simple poles of $p_{ij}(z)$ of minimal absolute value.*

Proof — It results directly from Theorem 6 (2), that $p_{ij}(z)$ have the same radius of convergence $\frac{1}{n}$, for every $1 \leqslant i,j \leqslant d$. So,

$$p_{ij}(z) = \sum_{k=0}^{k=\infty} (z^k A^k)_{ij},$$

for z with $|z| < \frac{1}{n}$. Let $v \in \mathbb{C}^d$ be an eigenvector of A with eigenvalue $ne^{\frac{2\pi i(-m)}{h}}$, for some $0 \leqslant m \leqslant h-1$. So, on one hand,

$$(I + zA + z^2A^2 + \ldots)(I - zA)\,v = Iv = v.$$

On the second hand,

$$(I + zA + z^2A^2 + \ldots)(I - zA)\,v = (I + zA + z^2A^2 + \ldots)\big((I - zA)\,v\big),$$

and if $z \to \frac{1}{n}e^{\frac{2\pi i m}{h}}$, then

$$(I - zA)\,v \to \left(I - \frac{1}{n}e^{\frac{2\pi i m}{h}}A\right) v \to 0.$$

By definition, $v \neq \vec{0}$, so if, whenever $z \to \frac{1}{n}e^{\frac{2\pi i m}{h}}$, all the elements in the matrix $(I - zA)^{-1}$ are finite, we get a contradiction. So, there exists $1 \leqslant i,j \leqslant d$, such that $(I - zA)^{-1}_{ij} \to \infty$, whenever $z \to \frac{1}{n}e^{\frac{2\pi i m}{h}}$. From Theorem 6 (3), each entry of $P(z)$ is positively linearly related to any other entry, that is the $p_{ij}(z)$ must all become infinite as soon as one of them does. So, for every $1 \leqslant i,j \leqslant d$,

$$\left\{\frac{1}{n}e^{\frac{2\pi i m}{h}} \;\middle|\; 0 \leqslant m \leqslant h-1\right\}$$

is a set of poles of $p_{ij}(z)$, and from Lemma 10, these are simple poles of minimal absolute value. □

3.2 Proof of Theorems 1 and 2

Let F_n be the free group on $n \geqslant 1$ generators. Let $\{H_i\alpha_i\}_{i=1}^{i=s}$, $s > 1$, be a coset partition of F_n with $H_i < F_n$ of index d_i, $\alpha_i \in F_n$, $1 \leqslant i \leqslant s$, and $1 < d_1 \leqslant \ldots \leqslant d_s$. Let $\widetilde{X}_i$ denote the Schreier graph of H_i, with transition matrix A_i of period $h_i \geqslant 1$, $1 \leqslant i \leqslant s$. Let $\widehat{X}_{H_i\alpha_i}$ denote the Schreier automaton of $H_i\alpha_i$, with generating function $p_i(z)$, $1 \leqslant i \leqslant s$. We prove some properties of the generating functions.

Lemma 12 *Let* $|z| < \frac{1}{n}$.

(1)
$$\sum_{k=1}^{k=s} p_k(z) = \frac{1}{1-nz} \tag{3.1}$$

(2) *For every* $1 \leqslant k \leqslant s$, $\left\{\frac{1}{n}e^{\frac{2\pi i m}{h_k}} \mid 0 \leqslant m \leqslant h_k - 1\right\}$ *is a set of simple poles of* $p_k(z)$ *of minimal absolute value.*

Proof — (1) The generating function of F_n is $\frac{1}{1-nz}$, for $|z| < \frac{1}{n}$. As $\{H_i\alpha_i\}_{i=1}^{i=s}$ is a coset partition of F_n, the generating function of F_n is equal to the sum of the corresponding generating functions.

(2) This follows from Lemma 11. □

Lemma 13 *Let* $h > 1$, *where* $h = \max\{h_i \mid 1 \leqslant i \leqslant s\}$. *Assume* $h_k = h$. *Let* $J = \{j \mid 1 \leqslant j \leqslant s,\ h_j = h\}$.

(1) *There exists (at least one)* $j \neq k$ *such that* $\frac{1}{n}e^{\frac{2\pi i}{h}}$ *is also a pole of* $p_j(z)$ *and* $h_j = h$.

(2) $\sum_{j\in J} \mathrm{Res}\big(p_j(z), \frac{1}{n}e^{\frac{2\pi i}{h}}\big) = 0$ *and moreover* $\sum_{j\in J} \mathrm{Res}\big(p_j(z), \frac{1}{n}e^{\frac{2\pi i m}{h}}\big) = 0$, *for every* m *with* $\gcd(m,h) = 1$.

Proof — (1) From Lemma 11, $\left\{\frac{1}{n}e^{\frac{2\pi i m}{h}} \mid 0 \leqslant m \leqslant h-1\right\}$ is a set of simple poles of $p_k(z)$. Let $z \to \frac{1}{n}e^{\frac{2\pi i}{h}}$ in (3.1). Then $p_k(z) \to \infty$ and the left-hand side of (3.1) also, while the right-hand side of (3.1) is a finite number, a contradiction. So, there exists $j \neq k$ such that $\frac{1}{n}e^{\frac{2\pi i}{h}}$ is a simple pole of $p_j(z)$ which implies that $h_j = h$.

(2) From Lemma 12(i),

$$\sum_{i=1}^{i=s} \mathrm{Res}(p_i(z), \frac{1}{n}e^{\frac{2\pi i}{h}}) = \mathrm{Res}(\frac{1}{1-nz}, \frac{1}{n}e^{\frac{2\pi i}{h}}) = 0.$$

For every $l \notin J$, as $\frac{1}{n}e^{\frac{2\pi i}{h}}$ is not a pole of $p_l(z)$, $\mathrm{Res}\big(p_l(z), \frac{1}{n}e^{\frac{2\pi i}{h}}\big) = 0$. So, $\sum\limits_{j\in J} \mathrm{Res}\big(p_j(z), \frac{1}{n}e^{\frac{2\pi i}{h}}\big) = 0$. Clearly, $\sum\limits_{j\in J} \mathrm{Res}(p_j(z), \frac{1}{n}e^{\frac{2\pi i m}{h}}) = 0$, for every m with $\gcd(m, h) = 1$. $\square$

Proof of Theorem 1 — We prove that for every h_k, there exists $j \neq k$ such that either $h_j = h_k$ or $h_k \mid h_j$. If $h_k = 1$, then clearly h_k divides every period h_j, $1 \leqslant j \leqslant s$. Assume $h_k > 1$ and assume by contradiction that h_k does not divide any period h_l, $1 \leqslant l \leqslant s$, $l \neq k$. That is, for every $m \geqslant 1$,

$$e^{\frac{2\pi i}{\left(\frac{h_l}{m}\right)}} \neq e^{\frac{2\pi i m}{h_k}}$$

for every h_l, $1 \leqslant l \leqslant s$, $l \neq k$. Using the same argument as in the proof of Lemma 13, with $z \to \frac{1}{n}e^{\frac{2\pi i}{h_k}}$ in (3.1), we get a contradiction. So, there exists $j \neq k$ such that $h_k \mid h_j$, and if h_k does not properly divide any other period, h_k has multiplicity at least two. $\square$

Proof of Theorem 2 — If the period of A_k is d_k, then from Theorem 1, there exists $j \in J$ such that $h_j = d_k$. As $h_j \leqslant d_j \leqslant d_k$, since the period is bounded from above by the index, we have $d_j = d_k$. $\square$

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