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Broué's Conjecture for 2-Blocks with Elementary Abelian Defect Groups of Order 32 *

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Abstract

The first author recently classified the Morita equivalence classes of 2-blocks of finite groups with elementary abelian defect groups of order 32. In all but three cases he proved that the Morita equivalence class determines the inertial quotient of the block. We finish the remaining cases by utilizing the theory of lower defect groups. As a corollary, we verify Broué's Abelian Defect Group Conjecture in this situation.

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1 Introduction

Motivated by Donovan's Conjecture in modular representation theory, there has been some interest in determining the possible Morita equivalence classes of p-blocks B of finite groups over a complete discrete valuation ring $\mathcal{O}$ with a given defect group D. While progress in the case $p > 2$ seems out of reach at the moment, quite a few papers appeared recently addressing the situation where D is an

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abelian 2-group. For instance, in [5, 6, 7, 8, 19] a full classification was obtained whenever D is an abelian 2-group of rank at most 3 or $D \simeq C_2^4$. Building on that, the first author determined in [1] the Morita equivalence classes of blocks with defect group $D \simeq C_2^5$. Partial results on larger defect groups are given in [2, 3, 14].

A fundamental open problem in modular representation theory, Broué's Abelian Defect Group Conjecture, states that a block with an abelian defect group is derived equivalent to its Brauer correspondent in the normaliser of a defect group. There are several versions of this conjecture, which consider different equivalences between the blocks, such as splendid derived equivalences, isotypies or perfect isometries. Here we focus on derived equivalences, so in particular we do not prove splendidness.

Since every Morita equivalence is also a derived equivalence, it is reasonable to expect that Broué's Abelian Defect Group Conjecture for B follows once all Morita equivalence classes have been identified. It is however not known in general whether a Morita equivalence preserves inertial quotients. In fact, there are three cases in Theorem 1.1. of [1] where the identification of the inertial quotient was left open. We settle these cases by making use of lower defect groups.

2 Proofs

Our notation follows [16]. We consider a p-modular system $(K, \mathcal{O}, k)$ where k is algebraically closed. Unless otherwise specified, all blocks of finite groups are considered over $\mathcal{O}$. For each block B we consider the defect group D and the inertial quotient $E := N_G(D, e)/DC_G(D)$, where e is a block of $C_G(D)$ such that $e^G = B$, i.e. whose Brauer correspondent is B. We also use the theory of lower defect groups, for which we refer to [16, Section 1.8]. The elementary divisors of the Cartan matrix of B are related to the orders of the defect groups of certain p-regular G-conjugacy classes in D, which we call lower defect groups. In particular, the elementary divisors of the Cartan matrix can be computed by determining the lower defect group multiplicities $m_B^{(a)}(Q)$ for $Q \leqslant D$, which can further be computed using the numbers $m_B^{(a)}(Q, b_Q) := m_{B_Q}^{(a)}(Q)$, where $B_Q := b_Q^{N_G(Q, b_Q)}$, and [16, Proposition 1.41].

Theorem 1 *Let* B *be a 2-block of a finite group* G *with defect group* $D \simeq C_2^5$. *Then the Morita equivalence class of* B *determines the inertial quotient of* B.

Proof — By [1, Theorem 1.1 and Corollary 5.3] we may assume that B is Morita equivalent to the principal block of one of the following groups, since in every other situation the Morita equivalence class of B is known to determine the inertial quotient:

(i) $(C_2^4 \rtimes C_5) \times C_2$.

(ii) $(C_2^4 \rtimes C_{15}) \times C_2$.

(iii) $SL(2,16) \times C_2$.

In each case, we may compute the Cartan matrix directly in the group listed above with GAP [10], or equivalently refer to the data in the Block Library database [9].

Assume the first case occurs. The elementary divisors of the Cartan matrix C of B (a Morita invariant) are $2,2,2,2,32$. According to Corollary 5.3 of [1], we may assume by way of contradiction that B has inertial quotient $E \simeq C_7 \rtimes C_3$ such that $C_D(E) = 1$. There is an E-invariant decomposition $D = D_1 \times D_2$ where $|D_1| = 4$. Let (Q, b) be a B-subpair such that $|Q| = 2$ (i.e. b is a Brauer correspondent of B in $C_G(Q)$). Then b dominates a unique block $\overline{b}$ of $C_G(Q)/Q$ with defect 4. The possible Cartan matrices of such blocks have been computed in [17, Proposition 16] up to basic sets. If $Q \leqslant D_1$, then b has inertial quotient $C_E(Q) \simeq C_7$ (see [16, Lemma 1.34]) and the Cartan matrix C_b of b has elementary divisors $4,4,4,4,4,4,32$. By [16, Eq. (1.2) on p. 16], the 1-multiplicity $m_b^{(1)}(Q)$ of Q as a lower defect group of b is 0. But now also $m_B^{(1)}(Q, b) = 0$ by [16, Lemma 1.42]. Similarly, if $Q \nsubseteq D_1 \cup D_2$, then b is nilpotent and again $m_B^{(1)}(Q, b) = 0$. Finally let $Q \leqslant D_2$. Then b has inertial index 3 and C_b has elementary divisors $2,2,32$. In particular, $m_B^{(1)}(Q, b) = m_b^{(1)}(Q) \leqslant 2$. Since all subgroups of order 2 in D_2 are conjugate under E, the multiplicity of 2 as an elementary divisor of C is at most 2 by [16, Proposition 1.41]. Contradiction.

Now assume that case (2) or (2) occurs. In both cases the multiplicity of 2 as an elementary divisor of C is 14. By [1, Corollary 5.3], we may assume that $E \simeq (C_7 \rtimes C_3) \times C_3$. Again we have an E-invariant decomposition $D = D_1 \times D_2$ where $|D_1| = 4$. As above let $Q \leqslant D$ with $|Q| = 2$. If $Q \leqslant D_1$, then b has inertial quotient $C_7 \rtimes C_3$ and the elementary divisors of C_b are all divisible by 4. Hence, $m_B^{(1)}(Q,b)=0$. If $Q \nsubseteq D_1 \cup D_2$, then b has inertial index 3 and C_b has elementary divisors $8,8,32$. Again, $m_B^{(1)}(Q,b) = 0$. Now if $Q \leqslant D_2$, then b has inertial quotient $C_3 \times C_3$. Here either $l(b) = 1$ or C_b has elementary divisors $2,2,2,2,8,8,8,8,32$. As above we obtain $m_B^{(1)}(Q,b) \leqslant 4$. Thus, the multiplicity of 2 as an elementary divisor of C is at most 4. Contradiction. □

Now we are in a position to prove Broué's Abelian Defect Group Conjecture in the situation of Theorem 1: to do so, first we prove that Alperin's Weight Conjecture (in its characterization for abelian defect groups [12, Corollary 6.10.10]) holds, and then we construct the derived equivalence as a composition of equivalences.

Theorem 2 *Let B be a 2-block of a finite group G with defect group $D \simeq C_2^5$. Then B is derived equivalent to its Brauer correspondent b in $N_G(D)$.*

Proof — Let E be the inertial quotient of B (and of b). First we prove Alperin's Weight Conjecture for B, i.e. $l(B) = l(b)$ by [12, Corollary 6.10.10].

By [1, Corollary 5.3], E uniquely determines $l(B)$ (and $l(b)$) unless

$$E \in \{C_3^2, (C_7 \rtimes C_3) \times C_3\}.$$

Suppose first that $E = C_3^2$. Then

$$C_D(E) = \langle x \rangle \simeq C_2.$$

Let β be a Brauer correspondent of B in $C_G(D)$ such that $b = \beta^N$ where $N := N_G(D)$. A theorem of Watanabe [18] (see [16, Theorem 1.39]) shows that $l(B) = l(B_x)$ where $B_x := \beta^{C_G(x)}$. By Theorem 1.22 of [16] B_x dominates a block $\overline{B_x}$ of $C_G(x)/\langle x \rangle$ with defect 4 such that $l(B_x) = l(\overline{B_x})$. Since Alperin's Weight Conjecture holds for 2-blocks of defect 4 (see [16, Theorem 13.6]), we obtain $l(\overline{B_x}) = l(\overline{b_x})$, where $\overline{b_x}$ is the unique block of $C_N(x)/\langle x \rangle$ dominated by $b_x := \beta^{C_N(x)}$. Hence,

$$l(B) = l(B_x) = l(\overline{B_x}) = l(\overline{b_x}) = l(b_x) = l(b)$$

as desired. Next, we assume that

$$E = (C_7 \rtimes C_3) \times C_3.$$

Up to G-conjugacy there exist three non-trivial B-subsections

$$(x, B_x), \quad (y, B_y) \quad \text{and} \quad (xy, B_{xy}).$$

The inertial quotients are

$$E(B_x) = C_3^2,\ E(B_y) = C_7 \rtimes C_3 \ \text{ and } \ E(B_{xy}) = C_3.$$

By [1, Corollary 5.3],

$$l(B_y) = 5,\ l(B_{xy}) = 3 \ \text{ and } \ (k(B), l(B)) \in \{(32, 15), (16, 7)\}.$$

Since $k(B) - l(B) = l(B_x) + l(B_y) + l(B_{xy})$, we obtain as above

$$l(B) = 15 \iff l(B_x) = 9 \iff l(b_x) = 9 \iff l(b) = 15.$$

This proves Alperin's Weight Conjecture for B.

Now suppose that the Morita equivalence class of B is given as in [1, Theorem 1.1]. Then $k(B)$ can be computed and E is uniquely determined by Theorem 1. By [1, Corollary 5.3], also the action of E on D is uniquely determined. By a theorem of Külshammer [11] (see [16, Theorem 1.19]), b is Morita equivalent to a twisted group algebra of $D \rtimes E$. The corresponding 2-cocycle is determined by $l(b) = l(B)$ (see [1, proof of Theorem 5.1]). Hence, we have identified the Morita equivalence class of b and it suffices to check Broué's Abelian Defect Group Conjecture for the blocks listed in [1, Theorem 1.1].

For the solvable groups in that list, we have $G = N$ and $B = b$. For principal 2-blocks, Broué's Abelian Defect Group Conjecture holds in general as shown by Craven and Rouquier [4, Theorem 4.36]. Now the only remaining case in [1, Theorem 1.1] is a non-principal block B of

$$G := (\mathrm{SL}(2, 8) \times C_2^2) \rtimes 3_+^{1+2}.$$

As noted in [15, Remark 3.4], the splendid derived equivalence between the principal block of $\mathrm{SL}(2, 8)$ and its Brauer correspondent extends to a splendid derived equivalence between the principal block of $\mathrm{Aut}(\mathrm{SL}(2, 8))$ and its Brauer correspondent. An explicit proof of this fact can be found in [4, Section 6.2.1]. Let

$$M \simeq \mathrm{SL}(2, 8) \times C_3 \times A_4$$

be a normal subgroup of G such that $C_3 \simeq Z(G) \leqslant M$, and let B_M be the unique block of M covered by B. By composing the derived equivalence from [15] with a trivial Morita equivalence, we get that B_M is splendid derived equivalent to its Brauer correspondent b_M. Let

$$X := G/M \simeq N_G(D)/N_M(D).$$

Following the notation in [13, Theorem 3.4], let

$$\delta : X \to X \times X^{op}$$

be the diagonal morphism $\delta(x) = (x, x^{-1})$ and let $\Delta := (B \otimes b^{op})_{\delta(X)}$, a fully $\delta(X)$-graded subalgebra of the $(X \times X^{op})$-graded algebra $B \otimes b^{op}$. The complex that defines the splendid equivalence between B_M and b_M extends to a complex of Δ-modules: this follows from Remark 3.4 of [15] and the fact that a trivial Morita equivalence naturally extends, since G/M stabilizes each block of M. Therefore, by Theorem 3.4 of [13], B is derived equivalent to b. □

Note that we do not prove that the derived equivalences in Theorem 2 are splendid.

In an upcoming paper by Charles Eaton and Michael Livesey the 2-blocks with abelian defect groups of rank at most 4 are classified. It should then be possible to prove Broué's Abelian Defect Conjecture for all abelian defect 2-groups of order at most 32. Judging from [8] we expect that all blocks with defect group $C_4 \times C_2^3$ are Morita equivalent to principal blocks.

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