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On a Maximal Subgroup of the Affine General Linear Group of $\mathrm{GL}(6,2)$

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Abstract

The affine general linear group $2^5{:}\mathrm{GL}(5,2)$ of $\mathrm{GL}(6,2)$ has 6 conjugacy classes of maximal subgroups. The largest maximal subgroup is a group of the form

$$2_+^{1+8}{:}\mathrm{GL}(4,2) := \overline{G}.$$

In this paper we firstly determine the conjugacy classes of $\overline{G}$ using the coset analysis technique. The structures of inertia factor groups were determined. These are the groups $H_1 = H_6 = \mathrm{GL}(4,2) \simeq A_8$, $H_2 = H_3 = 2^3{:}\mathrm{GL}(3,2)$, $H_4 = 2_+^{1+4}{:}\mathrm{GL}(2,2)$ and $H_5 = \mathrm{GL}(3,2)$. We then determine the Fischer matrices and apply the Clifford-Fischer theory to compute the ordinary character table of $\overline{G}$. The Fischer matrices of $\overline{G}$ are all listed in this paper. These matrices satisfy some additional interesting properties (Lemmas 3 and 4) comparing to the Fischer matrices of other group extensions. Using information on conjugacy classes, Fischer matrices and ordinary and projective tables of $H_1, H_2, \ldots, H_6$, we concluded that we need to use the ordinary character tables of all the inertia factor groups to construct the character table of $\overline{G}$. The character table of $\overline{G}$ is a 69×69 complex matrix and is given here (in the format of Clifford-Fischer theory) as Table 7.

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1 Introduction

Let $GL(n,q)$ be the finite general linear group consisting of $n \times n$ invertible matrices over the Galois field $\mathbb{F}_q$. It is well-known that the affine general linear subgroup of $GL(n,q)$, is a group of the form $q^{n-1}{:}GL(n-1,q)$. Starting with the extra-special 2-group $P_n := 2_+^{1+2n}$, the authors were able to write a GAP [14] subroutine to generate a group of the form $2_+^{1+2\times 2n}{:}GL(2n,2) := \overline{G}_{2n}$, for many values of n. We are interested in studying the group $\overline{G}_{2n}$ in general and investigating its properties and internal structure, and find a solid theoretical proof for the existence of such group in general for any $n \in \mathbb{N}$. However this could be far away and hence for better understanding of the situation one needs to consider various special cases first and see if one could generalize some of the results obtained from the special cases to the generic case. In the special case, for small values of n, we used GAP to construct the group $\overline{G}_{2n}$ and we tested the existence of these groups inside some other bigger groups. In fact we found that for such special cases the group $\overline{G}_{2n}$ to be sitting inside the affine general linear group $2^{2n+1}{:}GL(2n+1,2)$ of $GL(2n+2,2)$. This gives indication that this result may be true in general for any $n \in \mathbb{N}$. Also another interesting result that we found for the special cases of n, is that the action of $\overline{G}_{2n}$ on the conjugacy classes of its kernel $N = 2_+^{1+2\times 2n}$ gives six orbits of lengths

$$1,\ 2^{2n+1} - 2 \text{ (twice)},\ 2(2^{2n} - 1)(2^{2n-1} - 1),\ 2^{4n} - 2^{2n} \text{ and } 1,$$

while the action of $\overline{G}_{2n}$ on the irreducible characters of $N = 2_+^{1+2\times 2n}$ gives six orbits of lengths

$$1,\ 2^{2n} - 1 \text{ (twice)},\ (2^{2n} - 1)(2^{2n-1} - 1),\ 2^{4n-1} - 2^{2n-1} \text{ and } 1.$$

The corresponding inertia factor groups are

$$GL(2n,2), 2^{2n-1}{:}GL(2n-1,2) \text{ (twice)},$$
$$\overline{G}_{2n-2} = 2_+^{1+2(2n-2)}{:}GL(2n-2,2),\ GL(2n-1,2) \text{ and } GL(2n,2).$$

We conjecture here that the above results are also true in general.

In this paper we pay our attention to the special case

$$\overline{G}_4 = 2_+^{1+8}{:}GL(4,2) := \overline{G},$$

which is a maximal subgroup, of index 31, of the affine general linear group 2^5:GL(5, 2) of GL(6, 2). In fact with the help of GAP [14], we were able to determine the structures of all the maximal subgroups of the affine general linear group $M = 2^5$:GL(5, 2) of GL(6, 2). Representatives M_i of these maximal subgroups can be taken as in Table 1.

Table 1: Maximal subgroups $M = 2^5$:GL(5, 2)

M_i	$\|M_i\|$	$[M : M_i]$
2_+^{1+8}:GL(4, 2)	10321920	31
2^{4+5}:GL(4, 2)	10321920	31
GL(5, 2)	9999360	32
2^{2+9}:(GL(3, 2) × S_3)	2064384	155
2^9:(GL(3, 2) × S_4)	2064384	155
(2^5:31):5	4960	64512

Remark 1 We would like to mention here that the work on the maximal subgroup $M_2 = 2^{4+5}$:GL(4, 2) of M, which has the same size as of our group $\overline{G} = 2_+^{1+8}$:GL(4, 2), has been completed and submitted (see [11]).

In this paper we are interested in determining the conjugacy classes, inertia factor groups and calculating the Fischer matrices and hence the ordinary character table of $\overline{G} = 2_+^{1+8}$:GL(4, 2) using the coset analysis technique together with the theory of Clifford-Fischer Matrices. This is a very good example for the applications of Clifford-Fischer theory since the kernel of the extension is non-abelian group. Not many examples of these type have been studied via Clifford-Fischer theory. The Fischer matrices of $\overline{G}$ have all been determined in this paper and their sizes range between 2 and 11. In addition to the common properties that the Fischer matrices of any extension satisfy, the Fischer matrices of our group $\overline{G}$ satisfy further interesting properties (Lemmas 3 and 4). The character table of $\overline{G}$ is a 69×69 complex valued matrix and it is partitioned into six blocks corresponding to the six inertia factor groups $H_1 = H_6 =$ GL(4, 2), $H_2 = H_3 = 2^3$:GL(3, 2), $H_4 = \overline{G}_2 = 2_+^{1+4}$:GL(2, 2) and $H_5 =$ GL(3, 2) (see Section 3). If one only interested in the calculation of the character table, then it could be computed by using GAP or Magma [12] and the generators $\overline{g}_1$ and $\overline{g}_2$ of $\overline{G}$, given below. But Clifford-Fischer Theory provides many other interesting information on the group and on the character table, in particular the character table produced by Clifford-Fischer theory is in a special format that could not be achieved by direct computations using GAP or Magma. Also providing various examples for the applications of Clifford-Fischer theory to both split and non-split extensions is making sense, since each group requires individual approach. The readers (particularly young researchers) will highly benefit from the theoretical background required for these computations. GAP and Magma are computational tools and would not replace good powerful and theoretical arguments.

Now with the help of GAP we were able to construct the group $\overline{G}$ as a permutation group on 32 points. In fact $\overline{G} \leqslant A_{32}$ with large index. The following two elements $\overline{g}_1$ and $\overline{g}_2$ generate $\overline{G}$

$$\overline{g}_1 = (1,14,24,12,29,27,8)(2,18,28,10,31,23,6)\\(3,32,19,21,7,26,20,4,30,15,25,5,22,16)(9,11),$$

$$\overline{g}_2 = (1,27,25,10,13,30,6,24,26,18,29,19)\\(2,23,21,12,17,32,8,28,22,14,31,15)(3,9,5,20)(4,11,7,16)$$

with $o(\overline{g}_1) = 14$, $o(\overline{g}_2) = 12$ and $o(\overline{g}_1\overline{g}_2) = 14$, where we write $(a_1, a_2, a_3, \ldots, a_k)$ for the permutation $(a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_k)$.

Let $N := 2_+^{1+8}$ and $G := GL(4,2) \simeq \overline{G}/2_+^{1+8}$. Then $\overline{G} = N{:}G$. Since $\overline{G}$ can be constructed in GAP [14], it is easy to obtain all its normal subgroups. In fact $\overline{G}$ contains 6 normal subgroups of orders $1, 2, 32, 32, 512$ and 10321920. The normal subgroup of order 512 is an extra-special 2-group isomorphic to N. The following elements $n_1, n_2, \ldots, n_8$ are permutations acting on 32 points that generate N.

$$n_1 = (1,5)(2,7)(3,6)(4,8)(9,10)(11,12)(13,22)(14,16)(15,24)\\(17,26)(18,20)(19,28)(21,29)(23,30)(25,31)(27,32),$$

$$n_2 = (1,13)(2,17)(3,21)(4,25)(5,22)(6,29)(7,26)(8,31)(9,15)\\(10,24)(11,19)(12,28)(14,23)(16,30)(18,27)(20,32),$$

$$n_3 = (1,9)(2,11)(3,14)(4,18)(5,10)(6,16)(7,12)(8,20)(13,15)\\(17,19)(21,23)(22,24)(25,27)(26,28)(29,30)(31,32),$$

$$n_4 = (1,3)(2,4)(5,6)(7,8)(9,14)(10,16)(11,18)(12,20)(13,21)\\(15,23)(17,25)(19,27)(22,29)(24,30)(26,31)(28,32),$$

$$n_5 = (13,17)(15,19)(21,25)(22,26)(23,27)(24,28)(29,31)(30,32),$$

$$n_6 = (9,11)(10,12)(13,17)(14,18)(16,20)(21,25)(22,26)(29,31),$$

$$n_7 = (5,7)(6,8)(9,11)(13,17)(14,18)(21,25)(24,28)(30,32),$$

$$n_8 = (1,2)(6,8)(10,12)(14,18)(15,19)(21,25)(22,26)(30,32).$$

Using GAP we have found that N has only one complement in $\overline{G}$ and hence can be identified with G. The following two elements g_1 and g_2 are permutations acting on 32 points that generate the complement $G = GL(4,2)$.

$$g_1 = (3,6,9,10)(4,8,11,12)(5,16)(7,20)$$
$$(15,24,21,29)(19,28,25,31)(22,30)(26,32),$$

$$g_2 = (3,14)(4,18)(5,27)(6,17)(7,23)(8,13)(10,25)(12,21)$$
$$(15,20)(16,19)(22,24)(26,28).$$

with $o(g_1) = 4$, $o(g_2) = 2$ and $o(g_1g_2) = 15$.

For the notation used in this paper and the description of Clifford-Fischer theory technique, we follow [1],[2],[3],[4],[5],[6],[7],[8], [9],[10],[11].

2 Conjugacy classes of $\overline{G} = 2^{1+8}_{+}{:}GL(4,2)$

In this section we determine the conjugacy classes of $\overline{G}$ using the coset analysis technique (see [1], or [16] and [17] for more details) as we are interested to organize the classes of $\overline{G}$ corresponding to the classes of G. Recall that N is a group of order $2^{1+8} = 512$ and has $2^8 + 1 = 257$ conjugacy classes. The action of $N = \langle n_1, n_2, \ldots, n_8\rangle$ on the identity coset $N1_G = N$ produces the 257 conjugacy classes of N, where we know that N has

- singleton conjugacy class consisting of 1_N,
- singleton conjugacy class consisting of the central involution σ of N,
- 135 conjugacy classes, each class consists of two non-central involutions,
- 120 conjugacy classes, each class consists of two elements each is of order 4.

Using GAP, the action of $\overline{G} = \langle \overline{g}_1, \overline{g}_2\rangle$ on these 257 orbits

- leaves invariant $\{1_N\} := \Omega_1$ and $\{\sigma\} := \Omega_2$,

- fuses 15 conjugacy classes (each class consists of two non-central involutions) together to form a new orbit Ω_3 of length 30,
- fuses 15 conjugacy classes (each class consists of two non-central involutions) together to form a new orbit Ω_4 of length 30,
- fuses 105 conjugacy classes (each class consists of two non-central involutions) together to form a new orbit Ω_5 of length 210,
- fuses the 120 orbits of elements of order 4 altogether into a single orbit Ω_6.

Thus in $\overline{G}$, we get six conjugacy classes of sizes $1, 1, 30, 30, 210$ and 240. Similarly one can apply this to the other 13 cosets Ng_i, where g_i is a representative of a conjugacy class of G (for the classes of G, we refer to page 22 of the Atlas). Corresponding to the 14 conjugacy classes of $G = GL(4, 2)$, we obtain 69 conjugacy classes for $\overline{G}$. We list these classes in Table 2, where the notations used in this table are as in [1],[2],[3],[4],[5],[6],[7].

Table 2: The conjugacy classes of $\overline{G} = 2_+^{1+8}{:}GL(4,2)$

$[g_i]_G$	k_i	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_1 = 1A$	$k_1 = 257$	$m_{11} = 1$	g_{11}	1	1	10321920
		$m_{12} = 1$	g_{12}	2	1	10321920
		$m_{13} = 1$	g_{13}	2	30	344064
		$m_{14} = 54$	g_{14}	2	30	344064
		$m_{15} = 72$	g_{15}	2	210	49152
		$m_{16} = 72$	g_{16}	4	240	43008
$g_2 = 2A$	$k_2 = 65$	$m_{21} = 4$	g_{21}	2	420	24576
		$m_{22} = 4$	g_{22}	2	420	24576
		$m_{23} = 24$	g_{23}	2	2520	4096
		$m_{24} = 24$	g_{24}	2	2520	4096
		$m_{25} = 24$	g_{25}	2	2520	4096
		$m_{26} = 48$	g_{26}	4	5040	2048
		$m_{27} = 32$	g_{27}	4	3360	3072
		$m_{28} = 32$	g_{28}	4	3360	3072
		$m_{29} = 96$	g_{29}	4	10080	1024
		$m_{2,10} = 96$	$g_{2,10}$	4	10080	1024
		$m_{2,11} = 96$	$g_{2,11}$	4	13440	768

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$[g_i]_G$	k_i	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\|[g_{ij}]_{\overline{G}}\|$	$\|C_{\overline{G}}(g_{ij})\|$
$g_3 = 2B$	$k_3 = 17$	$m_{31} = 16$	g_{31}	2	3360	3072
		$m_{32} = 16$	g_{32}	2	3360	3072
		$m_{33} = 96$	g_{33}	4	20160	512
		$m_{34} = 96$	g_{34}	4	20160	512
		$m_{35} = 96$	g_{35}	4	20160	512
		$m_{36} = 192$	g_{36}	4	40320	256
$g_4 = 3A$	$k_4 = 2$	$m_{41} = 256$	g_{41}	3	28672	360
		$m_{42} = 256$	g_{42}	6	28672	360
$g_5 = 3B$	$k_5 = 17$	$m_{51} = 16$	g_{51}	3	17920	576
		$m_{52} = 16$	g_{52}	6	17920	576
		$m_{53} = 96$	g_{53}	6	107520	96
		$m_{54} = 96$	g_{54}	6	107520	96
		$m_{55} = 96$	g_{55}	6	107520	96
		$m_{56} = 192$	g_{56}	12	215040	48
$g_6 = 4A$	$k_6 = 5$	$m_{61} = 16$	g_{61}	4	20160	512
		$m_{62} = 16$	g_{62}	4	20160	512
		$m_{63} = 32$	g_{63}	4	40320	256
		$m_{64} = 32$	g_{64}	4	40320	256
		$m_{65} = 32$	g_{65}	4	40320	256
		$m_{66} = 64$	g_{66}	4	80640	128
		$m_{67} = 64$	g_{67}	4	80640	128
		$m_{68} = 64$	g_{68}	4	80640	128
		$m_{69} = 64$	g_{69}	4	80640	128
		$m_{6,10} = 128$	$g_{6,10}$	8	161280	64
$g_7 = 4B$	$k_7 = 2$	$m_{71} = 64$	g_{71}	4	161280	64
		$m_{72} = 64$	g_{72}	4	161280	64
		$m_{73} = 128$	g_{73}	8	322560	32
		$m_{74} = 128$	g_{74}	8	322560	32
		$m_{75} = 128$	g_{75}	8	322560	32
$g_8 = 5A$	$k_8 = 2$	$m_{81} = 256$	g_{81}	5	344064	30
		$m_{82} = 256$	g_{82}	10	344064	30
$g_9 = 6A$	$k_9 = 2$	$m_{91} = 256$	g_{91}	6	430080	24
		$m_{92} = 256$	g_{92}	6	430080	24
$g_{10} = 6B$	$k_{10} = 5$	$m_{10,1} = 64$	$g_{10,1}$	6	215040	48
		$m_{10,2} = 64$	$g_{10,2}$	6	215040	48
		$m_{10,3} = 128$	$g_{10,3}$	12	430080	24
		$m_{10,4} = 128$	$g_{10,4}$	12	430080	24
		$m_{10,5} = 128$	$g_{10,5}$	12	430080	24
$g_{11} = 7A$	$k_{11} = 5$	$m_{11,1} = 64$	$g_{11,1}$	7	184320	56
		$m_{11,2} = 64$	$g_{11,2}$	14	184320	56
		$m_{11,3} = 128$	$g_{11,3}$	14	368640	28
		$m_{11,4} = 128$	$g_{11,4}$	14	368640	28
		$m_{11,5} = 128$	$g_{11,5}$	28	368640	28

Continued on next page

$[g_i]_G$	k_i	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_{12} = 7B$	$k_{12} = 5$	$m_{12,1} = 64$	$g_{12,1}$	7	184320	56
		$m_{12,2} = 64$	$g_{12,2}$	14	184320	56
		$m_{12,3} = 128$	$g_{12,3}$	14	368640	28
		$m_{12,4} = 128$	$g_{12,4}$	14	368640	28
		$m_{12,5} = 128$	$g_{12,5}$	28	368640	28
$g_{13} = 15A$	$k_{13} = 2$	$m_{13,1} = 256$	$g_{13,1}$	15	344064	30
		$m_{13,2} = 256$	$g_{13,2}$	30	344064	30
$g_{14} = 15B$	$k_{14} = 2$	$m_{14,1} = 256$	$g_{14,1}$	15	344064	30
		$m_{14,2} = 256$	$g_{14,2}$	30	344064	30

3 Inertia factor groups of $\overline{G} = 2_+^{1+8}{:}GL(4,2)$

We have seen in Section 2 that the action of $\overline{G}$ on N produced six orbits of lengths 1, 30, 30, 210, 240, and 1. By a theorem of Brauer (for example see Theorem 5.1.1 of [1]), it follows that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ will also produce six orbits. It is well known that any extra-special p-group of order p^{1+2m} has $p^{2m}+1$ irreducible characters (p^{2m} linear characters of the vector space p^{2m} are inflated to the full extension p^{1+2m} and $p-1$ faithful irreducible characters each of degree p^m). Thus the group $N = 2_+^{1+8}$ has 257 irreducible characters in which 256 characters are linear and one unique faithful character θ of degree 16 (the values of θ are as follows: $\theta(1_N) = 16$, $\theta(\sigma) = -16$ and $\theta(t) = 0$ for any $t \in N \setminus \{1_N, \sigma\}$, where σ is the central involution of N). Using Program C of Seretlo [19] we have found that the orbit lengths of the action of $\overline{G}$ on $\mathrm{Irr}(N)$ are 1, 15, 15, 105, 120 and 1 with corresponding inertia factor groups $H_1 = GL(4,2)$, $H_2 = H_3 = 2^3{:}GL(3,2)$, $H_4 = \overline{G}_2 = 2_+^{1+4}{:}GL(2,2)$, $H_5 = GL(3,2)$ and $H_6 = GL(4,2)$. As permutation groups acting on 32 points, the inertia factors H_2, H_3, H_4 and H_5 are generated as follows: $H_2 = H_3 = \langle\alpha_1, \alpha_2\rangle$, $H_4 = \langle\beta_1, \beta_2\rangle$, and $H_5 = \langle\gamma_1, \gamma_2\rangle$, where

$$\alpha_1 = (9,32)(10,27)(11,30)(12,23)(14,28)(15,20)(16,19)(18,24),$$

$$\alpha_2 = (3,28,27,9,5,16,31)(4,24,23,11,7,20,29)(6,25,10,14,19,26,32)(8,21,12,18,15,22,30),$$

$$\beta_1 = (3,5,6)(4,7,8)(10,16,14)(12,20,18)(21,22,29)(23,24,30)(25,26,31)(27,28,32),$$

$$\beta_2 = (3,31,25,6)(4,29,21,8)(5,26)(7,22)(9,32,10,14)(11,30,12,18)(15,20,24,23)(16,28,27,19),$$

$$\gamma_1 = (9,16)(10,14)(11,20)(12,18)(15,30)(19,32)(23,24)(27,28),$$

$$\gamma_2 = (3,6,16)(4,8,20)(5,9,10)(7,11,12)(15,24,22) \\ (19,28,26)(21,29,30)(25,31,32).$$

We assume that the first orbit on the action of $\overline{G}$ on $\mathrm{Irr}(N)$ consists of the identity character while the sixth orbit consists of θ. Note that all characters of the other orbits are linear. The identity character $\mathbf{1}_N$ is extendable to a character of $\overline{G}$ (since $\mathbf{1}_{\overline{G}}\downarrow_N^{\overline{G}} = \mathbf{1}_N$). Also since $\overline{G}$ splits over N and the characters of the second, third, fourth and fifth orbits are linear, it follows by application of Theorem 5.1.8 of [1] that these characters are extendable to ordinary characters of their respective inertia groups. Thus for the construction of the character table of $\overline{G}$, all the character tables of the inertia factors H_1, H_2, H_3, H_4 and H_5 that we will use are the ordinary ones. In Section 4 we have supplied the character tables of H_2, H_3, H_4 and H_5. At this stage we are not yet sure whether the unique faithful character θ of degree 16 of N is extendable to ordinary character of $\overline{H_6} = \overline{G}$ or not. However by the Atlas [13] we know that the Schur multiplier of $H_6 = GL(4,2) \simeq A_8$ is 2 with $|\mathrm{Irr}(A_8)| = 14$ and $|\mathrm{IrrProj}(A_8, 2)| = 9$. By Section 2 we know that the number of conjugacy classes of $\overline{G} = 69$ and hence it follows that $|\mathrm{Irr}(\overline{G})| = 69$. By Atlas and Section 4 we have

$$\begin{aligned} |\mathrm{Irr}(H_1)| &= 14, \quad |\mathrm{Irr}(H_2)| = |\mathrm{Irr}(H_3)| = 11, \quad |\mathrm{Irr}(H_4)| = 13 \\ |\mathrm{Irr}(H_5)| &= 6, \quad |\mathrm{Irr}(H_6)| = 14 \quad \text{and} \quad |\mathrm{IrrProj}(H_6, 2)| = 9. \end{aligned}$$

Thus if we will use the projective character table of H_6 with factor set $\alpha \sim [2]$, we will get

$$\begin{aligned} &|\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| + |\mathrm{Irr}(H_4)| + |\mathrm{Irr}(H_4)| + |\mathrm{Irr}(H_6)| \\ &\qquad = 14 + 11 + 11 + 13 + 6 + 9 = 64 \neq 69 = |\mathrm{Irr}(\overline{G})|. \end{aligned}$$

This simple argument shows that we have to use the ordinary character table of H_6 to construct the character table of $\overline{G}$. Note that

$$\sum_{i=1}^{6} |\mathrm{Irr}(H_i)| = 69 = |\mathrm{Irr}(\overline{G})|.$$

Also we deduce that there exists a character $\Theta \in \mathrm{Irr}(\overline{G})$ with $\deg(\Theta) = 16$ such that $\Theta\downarrow_N^{\overline{G}} = \theta$.

4 Character tables of the inertia factor groups

The character table of $H_1 = H_6 = G$ is available in the Atlas. We have used GAP to construct the character tables of $H_2 = H_3 = 2^3{:}GL(3,2)$ and $H_4 = \overline{G}_2 = 2_+^{1+4}{:}GL(2,2)$ since we know from Section 3 that $H_2 = \langle \alpha_1, \alpha_2 \rangle$ and $H_4 = \langle \beta_1, \beta_2 \rangle$. The character table of H_5 is available in the Atlas. For the sake of convenience we have listed the character tables of $H_2 = H_3$ and H_4 in Tables 3 and 4 respectively. Also to be consistent with the notations of [1] we rename the classes of H_2 and H_4 in Tables 3 and 4 according to the fusions (determined in Section 5) of these groups into $GL(4,2)$.

Table 3: The character table of $H_2 = H_3 = 2^3{:}GL(3,2)$

	1a	2a	2b	2c	3a	4a	4b	4c	6a	7a	7b
	g_{121}	g_{221}	g_{222}	g_{321}	g_{521}	g_{621}	g_{622}	g_{721}	$g_{10,21}$	$g_{11,21}$	$g_{12,21}$
$\|C_{H_2}(g)\|$	1344	192	32	32	6	16	8	8	6	7	7
ϕ_1	1	1	1	1	1	1	1	1	1	1	1
ϕ_2	3	3	−1	−1	0	−1	1	1	0	A	$\overline{A}$
ϕ_3	3	3	−1	−1	0	−1	1	1	0	$\overline{A}$	A
ϕ_4	6	6	2	2	0	2	0	0	0	−1	−1
ϕ_5	7	−1	3	−1	1	−1	1	−1	−1	0	0
ϕ_6	7	7	−1	−1	1	−1	−1	−1	1	0	0
ϕ_7	7	−1	−1	3	1	−1	−1	1	−1	0	0
ϕ_8	8	8	0	0	−1	0	0	0	−1	1	1
ϕ_9	14	−2	2	2	−1	−2	0	0	1	0	0
ϕ_{10}	21	−3	1	−3	0	1	−1	1	0	0	0
ϕ_{11}	21	−3	−3	1	0	1	1	−1	0	0	0

Here, $A = -1 - b7 = -\frac{1}{2} - i\frac{\sqrt{7}}{2}$.

Table 4: The character table of $H_4 = \overline{G}_2 = 2_+^{1+4}{:}GL(2,2)$

	1a	2a	2b	2c	2d	2e	2f	3a	4a	4b	4c	4d	6a
	g_{141}	g_{241}	g_{242}	g_{243}	g_{341}	g_{244}	g_{342}	g_{5441}	g_{641}	g_{642}	g_{643}	g_{741}	$g_{10,41}$
$\|C_{H_4}(g)\|$	192	192	32	32	32	16	16	6	16	8	8	8	6
ξ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
ξ_2	1	1	1	1	1	−1	−1	1	1	−1	−1	−1	1
ξ_3	2	2	2	2	2	0	0	−1	2	0	0	0	−1
ξ_4	3	3	−1	3	−1	1	1	0	−1	−1	1	−1	0
ξ_5	3	3	3	−1	−1	1	1	0	−1	1	−1	−1	0
ξ_6	3	3	−1	−1	3	−1	−1	0	−1	1	1	−1	0
ξ_7	3	3	−1	3	−1	−1	−1	0	−1	1	−1	1	0
ξ_8	3	3	3	−1	−1	−1	−1	0	−1	−1	1	1	0
ξ_9	3	3	−1	−1	3	1	1	0	−1	−1	−1	1	0
ξ_{10}	4	−4	0	0	0	2	−2	1	0	0	0	0	−1
ξ_{11}	4	−4	0	0	0	−2	2	1	0	0	0	0	−1
ξ_{12}	6	6	−2	−2	−2	0	0	0	2	0	0	0	0
ξ_{13}	8	−8	0	0	0	0	0	−1	0	0	0	0	1

5 Fusions of the inertia factor groups into G

In this section we determine the fusions of the conjugacy classes of the inertia factor groups $H_2 = H_3 = 2^3{:}GL(3,2)$, $H_4 = \overline{G}_2 = 2_+^{1+4}{:}GL(2,2)$ and $H_5 = GL(3,2)$ into $GL(4,2) \simeq A_8$ using the permutation characters of A_8 on these groups together with the size of centralizers. From the Atlas, the permutation character of A_8 on H_2 is of the form $\chi(A_8|H_2) = \chi_1 + \chi_3$. The group H_4 is a maximal subgroup of H_2 since $[H_2 : H_4] = 7$. From Table 3 it is clear that $\chi(H_2|H_4) = \phi_1 + \phi_4$. Also the group H_5 has index 8 in $2^3{:}GL(3,2)$, which in turns has index 15 in G. We have found the following Proposition is very useful to calculate the permutation characters $\chi(G|H_4)$ and $\chi(G|H_5)$.

Proposition 2 *Let* $K_1 \leqslant K_2 \leqslant K_3$ *and let* ψ *be a class function on* K_1*. Then*

$$(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3} = \psi\uparrow_{K_1}^{K_3}.$$

More generally if $K_1 \leqslant K_2 \leqslant \ldots \leqslant K_n$ *is a nested sequence of subgroups of* K_n *and* ψ *is a class function on* K_1*, then*

$$(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3}\cdots\uparrow_{K_{n-1}}^{K_n} = \psi\uparrow_{K_1}^{K_n}.$$

Proof — See Proposition 3.5.6 of [1]. □

From the above we have

$$\chi(A_8|H_2) = \chi_1 + \chi_3 \quad \text{and} \quad \chi(H_2|H_4) = \phi_1 + \phi_4.$$

Now by Proposition 2 the permutation character $\chi(A_8|H_4)$ can be calculated as follows

$$\begin{aligned}\chi(A_8|H_4) &= \mathbf{1}\uparrow_{H_4}^{A_8} = (\mathbf{1}\uparrow_{H_4}^{H_2})\uparrow_{H_2}^{A_8}\\ &= (\phi_1+\phi_4)\uparrow_{H_2}^{A_8} = \phi_1\uparrow_{H_2}^{A_8} + \phi_4\uparrow_{H_2}^{A_8}\\ &= \chi_1+\chi_3+\phi_4\uparrow_{H_2}^{A_8} = \chi_1+\chi_3+(\chi_3+\chi_4+\psi_{12})\\ &= \chi_1+2\chi_3+\chi_4+\chi_{12}.\end{aligned} \tag{5.1}$$

Note that in (5.1) it is easy to evaluate the values of $\phi_4\uparrow_{H_2}^{A_8}$ using the induction formula for characters, where we have found that $\phi_4\uparrow_{H_2}^{A_8} = \chi_3+\chi_4+\chi_{12}$.

Similarly the permutation character $\chi(A_8|H_5)$ has the form

$$\begin{aligned} \chi(A_8|H_5) &= \mathbf{1}\uparrow_{H_5}^{A_8} = (\mathbf{1}\uparrow_{H_5}^{H_2})\uparrow_{H_2}^{A_8} \\ &= (\phi_1+\phi_5)\uparrow_{H_2}^{A_8} = \phi_1\uparrow_{H_2}^{A_8} + \phi_5\uparrow_{H_2}^{A_8} \\ &= \chi_1+\chi_3+\chi_5\uparrow_{H_2}^{A_8} = \chi_1+\chi_3+(\chi_3+\chi_9+\chi_{12}) \\ &= \chi_1+2\chi_3+\chi_9+\chi_{12}. \end{aligned} \tag{5.2}$$

Using the Atlas, (5.1) and (5.2), we list in Table 5 the values of $\chi(A_8|H_2)$, $\chi(A_8|H_4)$ and $\chi(A_8|H_5)$ on those classes of A_8, where there is possible fusions from classes of H_2, H_4 or H_5.

Table 5: The values of the permutation characters of A_8 on H_2, H_4, H_5

$[g]_{A_8}$	1A	2A	2B	3A	3B	4A	4B	5A	6A	6B	7A	7B	15A	15B
$\chi(A_8\|H_2)$	15	7	3	0	3	3	1	0	0	1	1	1	0	0
$\chi(A_8\|H_4)$	105	25	9	0	3	5	1	0	0	1	0	0	0	0
$\chi(A_8\|H_5)$	120	24	0	0	6	4	0	0	0	0	1	1	0	0

Using the permutation characters of A_8 on H_2, H_4 and H_5 together with the size of centralizers, the fusions of H_2, H_4 and H_5 into A_8 are completely determined. We list these fusions in Table 6.

Table 6: The fusions of H_2, H_4 and H_5 into G

Class of H_2	$\hookrightarrow$	Class of G	Class of H_2	$\hookrightarrow$	Class of G
$g_{121}=1a$		1A	$g_{622}=4b$		4A
$g_{221}=2a$		2A	$g_{721}=4c$		4B
$g_{222}=2b$		2A	$g_{10,21}=6a$		6B
$g_{321}=2c$		2B	$g_{11,21}=7a$		7A
$g_{521}=3a$		3B	$g_{12,21}=7b$		7B
$g_{621}=4a$		4A			
Class of H_4	$\hookrightarrow$	Class of G	Class of H_4	$\hookrightarrow$	Class of G
$g_{141}=1a$		1A	$g_{541}=3a$		3B
$g_{241}=2a$		2A	$g_{641}=4a$		4A
$g_{242}=2b$		2A	$g_{642}=4b$		4A
$g_{243}=2c$		2A	$g_{643}=4c$		4A
$g_{341}=2d$		2B	$g_{741}=4d$		4B
$g_{244}=2e$		2A	$g_{10,41}=6a$		6B
$g_{342}=2f$		2B			
Class of H_5	$\hookrightarrow$	Class of G	Class of H_5	$\hookrightarrow$	Class of G
$g_{151}=1a$		1A	$g_{651}=4a$		4A
$g_{251}=2a$		2A	$g_{11,51}=7a$		7A
$g_{551}=3a$		3B	$g_{12,51}=7b$		7B

6 Fischer matrices of $\overline{G} = 2_+^{1+8}{:}GL(4, 2)$

In this section we calculate the Fischer matrices of $\overline{G} = 2_+^{1+8}{:}GL(4, 2)$. From Section 3 of [2] we recall that we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$, corresponding to g_i, by the sizes of the centralizers of g_{ij}, $1 \leqslant j \leqslant c(g_i)$, in $\overline{G}$ and m_{ij} respectively. Also the rows of $\mathcal{F}_i$ are partitioned into parts $\mathcal{F}_{ik}$, $1 \leqslant k \leqslant t$, corresponding to the inertia factors $H_1, H_2, \ldots, H_t$, where each $\mathcal{F}_{ik}$ consists of $c(g_{ik})$ rows correspond to the α_k^{-1}-regular classes (those are the H_k-classes that fuse to class $[g_i]_G$). Thus every row of $\mathcal{F}_i$ is labeled by the pair (k, m), where $1 \leqslant k \leqslant t$ and $1 \leqslant m \leqslant c(g_{ik})$. In Table 2 we supplied $|C_{\overline{G}}(g_{ij})|$ and m_{ij}, $1 \leqslant i \leqslant 14$, $1 \leqslant j \leqslant c(g_i)$. Also the fusions of classes of $H_2 = H_3$, H_4 and H_5 into classes of G are given in Table 6. Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 2 that the sizes of the Fischer matrices of $\overline{G} = 2_+^{1+8}{:}GL(4, 2)$ range between 2 and 11 for each $i \in \{1, 2, \ldots, 14\}$.

We have used the arithmetical properties of the Fischer matrices, given in Proposition 3.6 of [2], to calculate some of the entries of these matrices and to build a system of algebraic equations. In addition to these properties, we have the following important lemmas, which help us more in determining some entries of the Fischer matrices of $\overline{G}$.

Lemma 3 *For every Fischer matrix* $\mathcal{F}_i$*, of size* $c(g_i)$*, the sum of the* $1^{st}, 2^{nd}, 3^{rd}, \ldots, (c(g_i) - 1)^{th}$ *rows equal (componentwise) the square of the modulus of the last row.*

Proof — The proof is similar to the proof of Lemma 6 of [18]. □

Lemma 4 *For any Fischer matrix* $\mathcal{F}_i$*, we can order the* g_{ij}*,* $1 \leqslant j \leqslant c(g_i)$*, so that the last row of* $\mathcal{F}_i$ *is of the form* $[z_i \quad -z_i \quad 0 \quad \cdots \quad 0]$ *and we may choose the* $g_{i2} = \sigma g_{i1}$*, where* σ *is the central involution in* $\overline{G}$*. Furthermore*

$$a_{i1}^{(k,m)} = a_{i2}^{(k,m)} = \frac{|C_{H_k}(g_{i11})|}{|C_{H_k}(g_{ikm})|}, \tag{6.1}$$

where $k \in \{1, 2, 3, 4, 5\}$ *and* $1 \leqslant m \leqslant c(g_{ik})$*.*

Proof — The proof is similar to the proof of Lemma 7 of [18]. □

Note 5 The proof of Lemma 7 of [18] contains a very important piece of information, that is, the last row of every Fischer matrix of $2_+^{1+22}{}^{\cdot}Co_2$ is

$$[\eta(g_{i1}) \quad \eta(g_{i2}) \quad \cdots \quad \eta(g_{is_i})],$$

where s_i in his notation has the same meaning of $c(g_i)$ in our notation. In our group $\overline{G}$, the last row of every Fischer matrix $\mathcal{F}_i$ is given by $[\theta_2(g_{i1}) \quad \theta_2(g_{i2}) \quad 0 \quad \cdots \quad 0]$.

Note 6 Observe that with Lemma 3, (6.1) and Note 5 we know the first two columns and the last row of every Fischer matrix $\mathcal{F}_i$. Also from Proposition 3.6 (iii) of [2], we know the first row of every Fischer matrix $\mathcal{F}_i$. This reduces the number of unknowns in every Fischer matrix of size $c(g_i)$ to $c(g_i)^2 - 4c(g_i) + 4$.

With the help of the symbolic mathematical package Maxima [15], we were able to solve the systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, which we list below.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}
$o(g_{1j})$		1	2	2	2	2	4
$\lvert C_{\overline{G}}(g_{1j})\rvert$		10321920	10321920	344064	344064	49152	43008
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$						
$(1,1)$	20160	1	1	1	1	1	1
$(2,1)$	1344	15	15	−1	15	−1	−1
$(3,1)$	1344	15	15	15	−1	−1	−1
$(4,1)$	192	105	105	−7	−7	9	−7
$(5,1)$	168	120	120	−8	−8	−8	8
$(6,1)$	20160	16	−16	0	0	0	0
m_{1j}		1	1	30	30	210	240

$\mathcal{F}_2$

g_2		g_{21}	g_{22}	g_{23}	g_{24}	g_{25}	g_{26}	g_{27}	g_{28}	g_{29}	$g_{2,10}$	$g_{2,11}$
$o(g_{2j})$		2	2	2	2	2	4	4	4	4	4	4
$\lvert C_{\overline{G}}(g_{2j})\rvert$		24576	24576	4096	4096	4096	2048	3072	3072	1024	1024	768
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$											
$(1,1)$	192	1	1	1	1	1	1	1	1	1	1	1
$(2,1)$	192	1	1	1	1	1	1	−1	1	1	−1	−1
$(2,2)$	32	6	6	−2	−2	6	−2	0	6	−2	0	0
$(3,1)$	192	1	1	1	1	1	1	1	−1	−1	1	−1
$(3,2)$	32	6	6	6	−2	−2	−2	6	0	0	−2	0
$(4,1)$	192	1	1	1	1	1	1	−1	−1	−1	−1	1
$(4,2)$	32	6	6	0	0	0	2	3	−3	−1	1	0
$(4,3)$	32	6	6	4	−4	4	2	−3	3	1	−1	0
$(4,4)$	16	12	12	−4	12	−4	4	−6	−6	2	2	0
$(5,1)$	8	24	24	−8	−8	−8	8	0	0	0	0	0
$(6,1)$	192	8	−8	0	0	0	0	0	0	0	0	0
m_{2j}		4	4	24	24	24	48	32	32	96	96	128

$\mathcal{F}_3$

g_3		g_{31}	g_{32}	g_{33}	g_{34}	g_{35}	g_{36}
$o(g_{3j})$		2	2	4	4	4	4
$\|C_{\overline{G}}(g_{3j})\|$		3072	3072	512	512	512	256
(k,m)	$\|C_{H_k}(g_{3km})\|$						
$(1,1)$	96	1	1	1	1	1	1
$(2,1)$	32	3	3	-1	-1	3	-1
$(3,1)$	32	3	3	3	-1	-1	-1
$(4,1)$	32	3	3	-1	3	-1	-1
$(4,2)$	16	6	6	-2	-2	-2	2
$(6,1)$	96	4	-4	0	0	0	0
m_{3j}		16	16	96	96	96	192

$\mathcal{F}_4$

g_4		g_{41}	g_{42}
$o(g_{4j})$		3	6
$\|C_{\overline{G}}(g_{4j})\|$		360	360
(k,m)	$\|C_{H_k}(g_{4km})\|$		
$(1,1)$	180	1	1
$(6,1)$	180	1	-1
m_{4j}		256	256

$\mathcal{F}_5$

g_5		g_{51}	g_{52}	g_{53}	g_{54}	g_{55}	g_{56}
$o(g_{5j})$		3	6 6	6	6	12	
$\|C_{\overline{G}}(g_{5j})\|$		576	576	96	96	96	48
(k,m)	$\|C_{H_k}(g_{5km})\|$						
$(1,1)$	18	1	1	1	1	1	1
$(2,1)$	6	3	3	-1	3	-1	-1
$(3,1)$	6	3	3	3	-1	-1	-1
$(4,1)$	6	3	3	-1	-1	3	-1
$(5,1)$	3	6	6	-2	-2	-2	2
$(6,1)$	18	4	-4	0	0	0	0
m_{5j}		16	16	96	96	96	192

$\mathcal{F}_6$

g_6		g_{61}	g_{62}	g_{63}	g_{64}	g_{65}	g_{66}	g_{67}	g_{68}	g_{69}	$g_{6,10}$
$o(g_{6j})$		4	4	4	4	4	4	4	4	4	8
$\|C_{\overline{G}}(g_{6j})\|$		512	512	256	256	256	128	128	128	128	64
(k,m)	$\|C_{H_k}(g_{6km})\|$										
$(1,1)$	16	1	1	1	1	1	1	1	1	1	1
$(2,1)$	16	1	1	1	1	1	-1	1	1	-1	-1
$(2,2)$	8	2	2	-2	-2	2	0	-2	2	0	0
$(3,1)$	16	1	1	1	1	1	1	-1	-1	1	-1
$(3,2)$	8	2	2	2	-2	-2	2	0	0	0	0
$(4,1)$	16	1	1	1	1	1	-1	-1	-1	-1	1
$(4,2)$	8	2	2	-2	-2	2	0	2	-2	-2	0
$(4,3)$	8	2	2	2	-2	-2	-2	-2	-2	2	0
$(5,1)$	4	4	4	-4	4	-4	0	0	0	0	0
$(6,1)$	16	4	-4	0	0	0	0	0	0	0	0
m_{6j}		16	16	32	32	32	64	64	64	64	64

$\mathcal{F}_7$

g_7		g_{71}	g_{72}	g_{73}	g_{74}	g_{75}
$o(g_{7j})$		4	4	8	8	8
$\lvert C_{\overline{G}}(g_{7j})\rvert$		64	64	32	32	32
(k,m)	$\lvert C_{H_k}(g_{7km})\rvert$					
$(1,1)$	8	1	1	1	1	1
$(2,1)$	8	1	1	-1	-1	1
$(3,1)$	8	1	1	1	-1	-1
$(4,1)$	8	1	1	-1	1	-1
$(6,1)$	8	2	-2	0	0	0
m_{7j}		64	64	128	128	128

$\mathcal{F}_8$

g_8		g_{81}	g_{82}
$o(g_{4j})$		5	10
$\lvert C_{\overline{G}}(g_{8j})\rvert$		30	30
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$		
$(1,1)$	15	1	1
$(6,1)$	15	1	-1
m_{8j}		256	256

$\mathcal{F}_9$

g_9		g_{91}	g_{92}
$o(g_{4j})$		6	6
$\lvert C_{\overline{G}}(g_{9j})\rvert$		24	24
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$		
$(1,1)$	12	1	1
$(6,1)$	12	1	-1
m_{9j}		256	256

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$	$g_{10,2}$	$g_{10,3}$	$g_{10,4}$	$g_{10,5}$
$o(g_{10j})$		6	6	12	12	12
$\lvert C_{\overline{G}}(g_{10j})\rvert$		48	48	24	24	24
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$					
$(1,1)$	6	1	1	1	1	1
$(2,1)$	6	1	1	-1	-1	1
$(3,1)$	6	1	1	1	-1	-1
$(4,1)$	6	1	1	-1	1	-1
$(6,1)$	6	2	-2	0	0	0
m_{10j}		64	64	128	128	128

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$	$g_{11,2}$	$g_{11,3}$	$g_{11,4}$	$g_{11,5}$
$o(g_{11j})$		7	14	14	14	28
$\lvert C_{\overline{G}}(g_{11j})\rvert$		56	56	28	28	28
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$					
$(1,1)$	7	1	1	1	1	1
$(2,1)$	7	1	1	-1	1	-1
$(3,1)$	7	1	1	1	-1	-1
$(5,1)$	7	1	1	-1	-1	1
$(6,1)$	7	2	-2	0	0	0
m_{11j}		64	64	128	128	128

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$	$g_{12,2}$	$g_{12,3}$	$g_{12,4}$	$g_{12,5}$
$o(g_{12j})$		7	14	14	14	28
$\lvert C_{\overline{G}}(g_{12j})\rvert$		56	56	28	28	28
(k, m)	$\lvert C_{H_k}(g_{12km})\rvert$					
$(1, 1)$	7	1	1	1	1	1
$(2, 1)$	7	1	1	-1	1	-1
$(3, 1)$	7	1	1	1	-1	-1
$(5, 1)$	7	1	1	-1	-1	1
$(6, 1)$	7	2	-2	0	0	0
m_{12j}		64	64	128	128	128

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$	$g_{13,2}$
$o(g_{13j})$		15	30
$\lvert C_{\overline{G}}(g_{13j})\rvert$		30	30
(k, m)	$\lvert C_{H_k}(g_{13km})\rvert$		
$(1, 1)$	15	1	1
$(6, 1)$	15	1	-1
m_{13j}		256	256

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$	$g_{14,2}$
$o(g_{14j})$		15	30
$\lvert C_{\overline{G}}(g_{14j})\rvert$		30	30
(k, m)	$\lvert C_{H_k}(g_{14km})\rvert$		
$(1, 1)$	15	1	1
$(6, 1)$	15	1	-1
m_{14j}		256	256

Remark 7 In the above matrices, if we omit the first column and the last row of every Fischer matrix $\mathcal{F}_i$, we obtain the corresponding Fischer matrix $\widetilde{\mathcal{F}}_i$ of the split extension $H = 2^8{:}GL(4, 2)$. This shows that the group $H = 2^8{:}GL(4, 2)$ has

$$\sum_{i=1}^{14} c(g_i) - 14 = 69 - 14 = 55$$

conjugacy classes, which is also equal to $14 + 11 + 11 + 13 + 6 = |\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| + |\mathrm{Irr}(H_4)| + |\mathrm{Irr}(H_5)| = 55$ the number of irreducible characters of H.

7 Character table of $\overline{G} = 2_+^{1+8}{:}GL(4, 2)$

Through Sections 2, 3, 4, 5 and 6 we have determined

- the conjugacy classes of $\overline{G} = 2_+^{1+8}{:}GL(4,2)$ (Table 2),
- the inertia factors $H_1,\ H_2,\dots,H_6$,
- the character tables of all the inertia factor groups of G (Tables 3 and 4),
- the fusions of classes of the inertia factors $H_2 = H_3,\ H_4$ and H_5 into classes of G (Table 6)
- the Fischer matrices of $\overline{G}$ (see Section 6).

By Subsection 3.1 of [2], it follows that the full character table of $\overline{G}$ can be constructed easily. Let

- $g_1, g_2, \dots, g_r$ be representatives for the conjugacy classes of $G \simeq \overline{G}/N$. For each $i \in \{1, 2, \dots, r\}$, let g_{i1}, g_{i2}, …, $g_{ic(g_i)}$ be representatives for the conjugacy classes of $\overline{G}$, correspond to the class $[g_i]_G$, obtained using the coset analysis technique (see [1] for more details),
- $\mathcal{K}_{ik}$ be the fragment of the projective character table of H_k, with factor set α_k^{-1}, consisting of columns correspond to the α_k^{-1}-regular classes of H_k that fuse to $[g_i]_G$ (let such classes be represented by $g_{ik1}, g_{ik2}, \dots, g_{ikc(g_{ik})}$) and
- $\mathcal{F}_{ik}$ be the sub-matrix of the Fischer matrix $\mathcal{F}_i$ with rows correspond to the pairs (k, g_{ik1}), (k, g_{ik2}), …, $(k, g_{ikc(g_{ik})})$ or for brevity $(k,1)$, $(k,2)$, …, $(k, c(g_{ik}))$ as described by Equation (3) of [6].

For each $i \in \{1, 2, \dots, r\}$ and $k \in \{1, 2, \dots, t\}$, where t is the number of the inertia factor groups (that is the number of orbits on the action of G on Irr(N)), the part of the character table of $\overline{G}$ on the classes $[g_{ij}]_{\overline{G}}$, $1 \leqslant j \leqslant c(g_i)$, is given by $\mathcal{K}_{ik}\mathcal{F}_{ik}$. Note that the size of $\mathcal{K}_{ik}$ is $|\mathrm{IrrProj}(H_k, \alpha_k^{-1})| \times c(g_{ik})$, while the size of $\mathcal{F}_{ik}$ is $c(g_{ik}) \times c(g_i)$ and thus $\mathcal{K}_{ik}\mathcal{F}_{ik}$ is of size $|\mathrm{IrrProj}(H_k, \alpha_k^{-1})| \times c(g_i)$. If we let $\mathcal{K}_s$, $s \in \{1, 2, \dots, t\}$, be the irreducible characters of $\overline{G}$ correspond to the inertia factor group H_k, then the character table of $\overline{G}$ in the format of Clifford-Fischer theory will be composed of the $r \times t$ parts $\mathcal{K}_{ik}\mathcal{F}_{ik}$ and will have the form:

$[g_i]_G$	g_1	g_2	$\cdots$	g_r
$[g_{ij}]_{\overline{G}}$	g_{11} g_{12} $\cdots$ $g_{1c(g_1)}$	g_{21} g_{22} $\cdots$ $g_{2c(g_2)}$	$\cdots$	g_{r1} g_{r2} $\cdots$ $g_{rc(g_r)}$
$\mathcal{K}_1$	$\mathcal{K}_{11}\mathcal{F}_{11}$	$\mathcal{K}_{12}\mathcal{F}_{12}$	$\cdots$	$\mathcal{K}_{1r}\mathcal{F}_{1r}$
$\mathcal{K}_2$	$\mathcal{K}_{21}\mathcal{F}_{21}$	$\mathcal{K}_{22}\mathcal{F}_{22}$	$\cdots$	$\mathcal{K}_{2r}\mathcal{F}_{2r}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathcal{K}_t$	$\mathcal{K}_{t1}\mathcal{F}_{t1}$	$\mathcal{K}_{t2}\mathcal{F}_{t2}$	$\cdots$	$\mathcal{K}_{tr}\mathcal{F}_{tr}$

Note 8 From Note 3.4 of [6] we know that characters of $\overline{G}$ consisted in $\mathcal{K}_1$ are just Irr(G) and therefore the size of $\mathcal{K}_{1i}\mathcal{F}_{1i}$, for each $1 \leqslant i \leqslant r$, is $|\mathrm{Irr}(G)| \times c(g_i)$. In particular, columns of $\mathcal{K}_{11}\mathcal{F}_{11}$ are the degrees of irreducible characters of G repeated themselves $c(g_1)$ times, where we know that $c(g_1)$ is number of $\overline{G}$-conjugacy classes obtained from the normal subgroup N.

We illustrate the above by giving an example on how to construct the character table of $\overline{G}$, which is partitioned into 84 parts corresponding to the 14 cosets and the six inertia factor groups. As an example we construct the parts $\mathcal{K}_{31}\mathcal{F}_{31}$, $\mathcal{K}_{32}\mathcal{F}_{32}$, $\mathcal{K}_{33}\mathcal{F}_{33}$, $\mathcal{K}_{34}\mathcal{F}_{34}$, $\mathcal{K}_{35}\mathcal{F}_{35}$ and $\mathcal{K}_{36}\mathcal{F}_{36}$ of the character table of $\overline{G}$. This means that we are listing the values of all the irreducible characters of $\overline{G}$ on the classes g_{31}, g_{32}, g_{33}, g_{34}, g_{35} and g_{36} of $\overline{G}$, which correspond to the conjugacy class of G represented by $g_3 = 2B$. The six parts $\mathcal{K}_{31}\mathcal{F}_{31}$, $\mathcal{K}_{32}\mathcal{F}_{32}$, $\mathcal{K}_{33}\mathcal{F}_{33}$, $\mathcal{K}_{34}\mathcal{F}_{34}$, $\mathcal{K}_{35}\mathcal{F}_{35}$ and $\mathcal{K}_{36}\mathcal{F}_{36}$ can be derived as follows: From Table 6 we can see that there is a class of H_2, namely g_{321}, a class of H_3, namely g_{331}, and two classes of H_4, namely g_{341} and g_{342} that fuse into the class $[g_3]_G = [2B]_{GL(4,2)}$. To construct the part $\mathcal{K}_{31}\mathcal{F}_{31}$, we multiply the column of the character table of $H_1 = G = GL(4, 2)$ corresponds to the class 2B of G (see Atlas), by the first row of $\mathcal{F}_1$, namely

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and thus the part $\mathcal{K}_{31}\mathcal{F}_{31}$ of size 14×6, consists of the column of the character table of G corresponds to the class 2B repeated 6 times. To construct the part $\mathcal{K}_{34}\mathcal{F}_{34}$, we multiply again the fragment of the character table of H_4, corresponds to the class g_{341} and g_{342}, by the two rows of $\mathcal{F}_3$ labeled by the pairs $(4, 1)$ and $(4, 2)$. Thus we get a part in the character table of $\overline{G}$ of size 13×6. Similar arguments can be used to construct the other remaining parts. The above six parts will have the following forms:

$$\mathcal{K}_{31}\mathcal{F}_{31} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 1 \\ 1 \\ 1 \\ 4 \\ -5 \\ -3 \\ -3 \\ 0 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{array}{c} \begin{array}{cccccc} g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ -5 & -5 & -5 & -5 & -5 & -5 \\ -3 & -3 & -3 & -3 & -3 & -3 \\ -3 & -3 & -3 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix} \end{array},$$

$$\mathcal{K}_{32}\mathcal{F}_{32} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \\ -1 \\ -1 \\ 3 \\ 0 \\ 2 \\ -3 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 & -1 & 3 & -1 \end{pmatrix} = \begin{array}{c} \begin{array}{cccccc} g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \end{array} \\ \begin{pmatrix} 3 & 3 & -1 & -1 & 3 & -1 \\ -3 & -3 & 1 & 1 & -3 & 1 \\ -3 & -3 & 1 & 1 & -3 & 1 \\ 6 & 6 & -2 & -2 & 6 & -2 \\ -3 & -3 & 1 & 1 & -3 & 1 \\ -3 & -3 & 1 & 1 & -3 & 1 \\ 9 & 9 & -3 & -3 & 9 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & -2 & -2 & 6 & -2 \\ -9 & -9 & 3 & 3 & -9 & 3 \\ 3 & 3 & -1 & -1 & 3 & -1 \end{pmatrix} \end{array},$$

$$\mathcal{K}_{33}\mathcal{F}_{33} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \\ -1 \\ -1 \\ 3 \\ 0 \\ 2 \\ -3 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 3 & -1 & -1 & -1 \end{pmatrix} = \begin{array}{c} \begin{array}{cccccc} g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \end{array} \\ \begin{pmatrix} 3 & 3 & 3 & -1 & -1 & -1 \\ -3 & -3 & -3 & -1 & -1 & -1 \\ -3 & -3 & -3 & -1 & -1 & -1 \\ 6 & 6 & 6 & -2 & -2 & -2 \\ -3 & -3 & -3 & -1 & -1 & -1 \\ -3 & -3 & -3 & -1 & -1 & -1 \\ 9 & 9 & 9 & -3 & -39 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & -2 & -2 & -2 \\ -9 & -9 & -9 & 3 & 3 & 3 \\ 3 & 3 & 3 & -1 & -1 & -1 \end{pmatrix} \end{array},$$

$$\mathcal{K}_{34}\mathcal{F}_{34} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ -1 & -1 \\ -1 & 1 \\ -1 & -1 \\ -1 & 1 \\ 3 & -1 \\ 3 & 1 \\ 0 & -2 \\ 0 & 2 \\ -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 & 3 & -1 & -1 \\ 6 & 6 & -2 & -2 & -2 & 2 \end{pmatrix} = \begin{array}{c} \begin{array}{cccccc} g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \end{array} \\ \begin{pmatrix} 9 & 9 & -3 & 1 & -3 & 1 \\ -3 & -3 & 1 & 5 & 1 & -3 \\ 6 & 6 & -2 & 6 & -2 & -2 \\ 3 & 3 & -1 & -5 & -1 & 3 \\ 3 & 3 & -1 & -5 & -1 & 3 \\ 3 & 3 & -1 & 11 & -1 & -5 \\ 15 & 15 & -5 & 7 & -5 & -1 \\ -9 & -9 & 3 & -1 & 3 & -1 \\ -9 & -9 & 3 & -1 & 3 & -1 \\ -12 & -12 & 4 & 4 & 4 & -4 \\ 12 & 12 & -4 & -4 & -4 & 4 \\ -6 & -6 & 2 & -6 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$\mathcal{K}_{35}\mathcal{F}_{35} = \begin{array}{c} \begin{array}{cccccc} g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(since there is no fusion from classes of H_5 into classes of G),

$$\mathcal{K}_{36}\mathcal{F}_{36} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 1 \\ 1 \\ 1 \\ 4 \\ -5 \\ -3 \\ -3 \\ 0 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 4 & -4 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{array}{c} \begin{array}{cccccc} g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \end{array} \\ \begin{pmatrix} 4 & -4 & 0 & 0 & 0 & 0 \\ 12 & -12 & 0 & 0 & 0 & 0 \\ 8 & -8 & 0 & 0 & 0 & 0 \\ 16 & -16 & 0 & 0 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 \\ 16 & -16 & 0 & 0 & 0 & 0 \\ -20 & 20 & 0 & 0 & 0 & 0 \\ -12 & 12 & 0 & 0 & 0 & 0 \\ -12 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & -8 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Similarly one can obtain all the other 78 parts $\mathcal{K}_{ik}\mathcal{F}_{ik}$, $1 \leqslant i \leqslant 14$, $i \neq 3$, $1 \leqslant k \leqslant 6$ and hence the full character table of $\overline{G}$, which is 69×69 $\mathbb{C}$-valued matrix. The full character table of $\overline{G}$ in the format of Clifford-Fischer theory is given as Table 7.

Table 7: The character table of $\overline{G} = 2_+^{1+8}{:}GL(4,2)$

	1A						2A										
	1a	2a	2b	2c	2d	4a	2e	2f	2g	2h	2i	4b	4c	4d	4e	4f	4g
$\|C_{\overline{G}}(g_{ij})\|$	10321920	10321920	344064	344064	49152	43008	24576	24576	4096	4096	4096	2048	3072	3072	1024	1024	768
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	7	7	7	7	7	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
χ_3	14	14	14	14	14	14	6	6	6	6	6	6	6	6	6	6	6
χ_4	20	20	20	20	20	20	4	4	4	4	4	4	4	4	4	4	4
χ_5	21	21	21	21	21	21	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
χ_6	21	21	21	21	21	21	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
χ_7	21	21	21	21	21	21	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
χ_8	28	28	28	28	28	28	−4	−4	−4	−4	−4	−4	−4	−4	−4	−4	−4
χ_9	35	35	35	35	35	35	3	3	3	3	3	3	3	3	3	3	3
χ_{10}	45	45	45	45	45	45	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
χ_{11}	45	45	45	45	45	45	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3	−3
χ_{12}	56	56	56	56	56	56	8	8	8	8	8	8	8	8	8	8	8
χ_{13}	64	64	64	64	64	64	0	0	0	0	0	0	0	0	0	0	0
χ_{14}	70	70	70	70	70	70	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
χ_{15}	15	15	−1	15	−1	−1	7	7	−1	−1	7	−1	−1	7	−1	−1	−1
χ_{16}	45	45	−3	45	−3	−3	−3	−3	5	5	−3	5	−3	−3	5	−3	−3
χ_{17}	45	45	−3	45	−3	−3	−3	−3	5	5	−3	5	−3	−3	5	−3	−3
χ_{18}	90	90	−6	90	−6	−6	18	18	2	2	18	2	−6	18	2	−6	−6
χ_{19}	105	105	−7	105	−7	−7	17	17	−7	−7	17	−7	1	17	−7	1	1
χ_{20}	105	105	−7	105	−7	−7	1	1	9	9	1	9	−7	1	9	−7	−7
χ_{21}	105	105	−7	105	−7	−7	−7	−7	1	1	−7	1	1	−7	1	1	1
χ_{22}	120	120	−8	120	−8	−8	8	8	8	8	8	8	−8	8	8	−8	−8
χ_{23}	210	210	−14	210	−14	−14	10	10	−6	−6	10	−6	2	10	−6	2	2
χ_{24}	315	315	−21	315	−21	−21	3	3	−5	−5	3	−5	3	3	−5	3	3
χ_{25}	315	315	−21	315	−21	−21	−21	−21	3	3	−21	3	3	−21	3	3	3
χ_{26}	15	15	15	−1	−1	−1	7	7	7	−1	−1	−1	7	−1	−1	−1	−1
χ_{27}	45	45	45	−3	−3	−3	−3	−3	−3	5	5	5	−3	−3	−3	5	−3
χ_{28}	45	45	45	−3	−3	−3	−3	−3	−3	5	5	5	−3	−3	−3	5	−3
χ_{29}	90	90	90	−6	−6	−6	18	18	18	2	2	2	18	−6	−6	2	−6
χ_{30}	105	105	105	−7	−7	−7	17	17	17	−7	−7	−7	17	1	1	−7	1
χ_{31}	105	105	105	−7	−7	−7	1	1	1	9	9	9	1	−7	−7	9	−7
χ_{32}	105	105	105	−7	−7	−7	−7	−7	−7	1	1	1	−7	1	1	1	1
χ_{33}	120	120	120	−8	−8	−8	8	8	8	8	8	8	8	−8	−8	8	−8
χ_{34}	210	210	210	−14	−14	−14	10	10	10	−6	−6	−6	10	2	2	−6	2
χ_{35}	315	315	315	−21	−21	−21	3	3	3	−5	−5	−5	3	3	3	−5	3
χ_{36}	315	315	315	−21	−21	−21	−21	−21	−21	3	3	3	−21	3	3	3	3

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Table 7 (continued)

	1A						2A										
	1a	2a	2b	2c	2d	4a	2e	2f	2g	2h	2i	4b	4c	4d	4e	4f	4g
$\lvert C_{\overline{G}}(g_{ij})\rvert$	10321920	10321920	344064	344064	49152	43008	24576	24576	4096	4096	4096	2048	3072	3072	1024	1024	768
χ_{37}	105	105	−7	−7	9	−7	25	25	1	9	1	−7	−7	−7	1	1	1
χ_{38}	105	105	−7	−7	9	−7	1	1	9	−15	9	1	−7	−7	1	1	1
χ_{39}	210	210	−14	−14	18	−14	26	26	10	−6	10	−6	−14	−14	2	2	2
χ_{40}	315	315	−21	−21	27	−21	27	27	−13	11	19	−5	3	−21	3	−5	3
χ_{41}	315	315	−21	−21	27	−21	27	27	19	11	−13	−5	−21	3	−5	3	3
χ_{42}	315	315	−21	−21	27	−21	−21	−21	3	−5	3	11	3	3	−5	−5	3
χ_{43}	315	315	−21	−21	27	−21	3	3	−5	19	−5	3	3	3	−5	−5	3
χ_{44}	315	315	−21	−21	27	−21	3	3	−5	−13	27	3	3	−21	3	−5	3
χ_{45}	315	315	−21	−21	27	−21	3	3	27	−13	−5	3	−21	3	−5	3	3
χ_{46}	420	420	−28	−28	36	−28	20	20	−12	20	−12	−12	4	4	4	4	−4
χ_{47}	420	420	−28	−28	36	−28	−28	−28	4	−28	4	4	4	4	4	4	−4
χ_{48}	630	630	−42	−42	54	−42	−18	−18	−2	14	−2	14	6	6	−10	−10	6
χ_{49}	840	840	−56	−56	72	−56	−8	−8	−8	−8	−8	−8	8	8	8	8	−8
χ_{50}	120	120	−8	−8	−8	8	24	24	−8	−8	−8	8	0	0	0	0	0
χ_{51}	360	360	−24	−24	−24	24	−24	−24	8	8	8	−8	0	0	0	0	0
χ_{52}	360	360	−24	−24	−24	24	−24	−24	8	8	8	−8	0	0	0	0	0
χ_{53}	720	720	−48	−48	−48	48	48	48	−16	−16	−16	16	0	0	0	0	0
χ_{54}	840	840	−56	−56	−56	56	−24	−24	8	8	8	−8	0	0	0	0	0
χ_{55}	960	960	−64	−64	−64	64	0	0	0	0	0	0	0	0	0	0	0
χ_{56}	16	−16	0	0	0	0	8	−8	0	0	0	0	0	0	0	0	0
χ_{57}	112	−112	0	0	0	0	−8	8	0	0	0	0	0	0	0	0	0
χ_{58}	224	−224	0	0	0	0	48	−48	0	0	0	0	0	0	0	0	0
χ_{59}	320	−320	0	0	0	0	32	−32	0	0	0	0	0	0	0	0	0
χ_{60}	336	−336	0	0	0	0	−24	24	0	0	0	0	0	0	0	0	0
χ_{61}	336	−336	0	0	0	0	−24	24	0	0	0	0	0	0	0	0	0
χ_{62}	336	−336	0	0	0	0	−24	24	0	0	0	0	0	0	0	0	0
χ_{63}	448	−448	0	0	0	0	−32	32	0	0	0	0	0	0	0	0	0
χ_{64}	560	−560	0	0	0	0	24	−24	0	0	0	0	0	0	0	0	0
χ_{65}	720	−720	0	0	0	0	−24	24	0	0	0	0	0	0	0	0	0
χ_{66}	720	−720	0	0	0	0	−24	24	0	0	0	0	0	0	0	0	0
χ_{67}	896	−896	0	0	0	0	64	−64	0	0	0	0	0	0	0	0	0
χ_{68}	1024	−1024	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{69}	1120	−1120	0	0	0	0	−16	16	0	0	0	0	0	0	0	0	0

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Table 7 (continued)

	2B						3A		3B						4A									
	2j	2k	4h	4i	4j	4k	3a	6a	3b	6b	6c	6d	6e	12a	4l	4m	4n	4o	4p	4q	4r	4s	4t	8a
$\lvert C_{\overline{G}}(g_{ij})\rvert$	3072	3072	512	512	512	256	360	360	576	576	96	96	96	48	512	512	256	256	256	128	128	128	128	64
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	3	3	3	3	4	4	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
χ_3	2	2	2	2	2	2	−1	−1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
χ_4	4	4	4	4	4	4	5	5	−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
χ_5	1	1	1	1	1	1	6	6	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
χ_6	1	1	1	1	1	1	−3	−3	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
χ_7	1	1	1	1	1	1	−3	−3	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
χ_8	4	4	4	4	4	4	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
χ_9	−5	−5	−5	−5	−5	−5	5	5	2	2	2	2	2	2	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
χ_{10}	−3	−3	−3	−3	−3	−3	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
χ_{11}	−3	−3	−3	−3	−3	−3	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
χ_{12}	0	0	0	0	0	0	−4	−4	−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	0	0	0
χ_{13}	0	0	0	0	0	0	4	4	−2	−2	−2	−2	−2	−2	0	0	0	0	0	0	0	0	0	0
χ_{14}	2	2	2	2	2	2	−5	−5	1	1	1	1	1	1	−2	−2	−2	−2	−2	−2	−2	−2	−2	−2
χ_{15}	3	3	−1	−1	3	−1	0	0	3	3	−1	3	−1	−1	3	3	−1	−1	3	−1	−1	3	−1	−1
χ_{16}	−3	−3	1	1	−3	1	0	0	0	0	0	0	0	0	1	1	−3	−3	1	1	−3	1	1	1
χ_{17}	−3	−3	1	1	−3	1	0	0	0	0	0	0	0	0	1	1	−3	−3	1	1	−3	1	1	1
χ_{18}	6	6	−2	−2	6	−2	0	0	0	0	0	0	0	0	2	2	2	2	2	−2	2	2	−2	−2
χ_{19}	−3	−3	1	1	−3	1	0	0	3	3	−1	3	−1	−1	1	1	−3	−3	1	1	−3	1	1	1
χ_{20}	−3	−3	1	1	−3	1	0	0	3	3	−1	3	−1	−1	−3	−3	1	1	−3	1	1	−3	1	1
χ_{21}	9	9	−3	−3	9	−3	0	0	3	3	−1	3	−1	−1	−3	−3	1	1	−3	1	1	−3	1	1
χ_{22}	0	0	0	0	0	0	0	0	−3	−3	1	−3	1	1	0	0	0	0	0	0	0	0	0	0
χ_{23}	6	6	−2	−2	6	−2	0	0	−3	−3	1	−3	1	1	−2	−2	−2	−2	−2	2	−2	−2	2	2
χ_{24}	−9	−9	3	3	−9	3	0	0	0	0	0	0	0	0	−1	−1	3	3	−1	−1	3	−1	−1	−1
χ_{25}	3	3	−1	−1	3	−1	0	0	0	0	0	0	0	0	3	3	−1	−1	3	−1	−1	3	−1	−1
χ_{26}	3	3	3	−1	−1	−1	0	0	3	3	3	−1	−1	−1	3	3	3	−1	−1	3	−1	−1	−1	−1
χ_{27}	−3	−3	−3	1	1	1	0	0	0	0	0	0	0	0	1	1	1	−3	−3	1	1	1	−3	1
χ_{28}	−3	−3	−3	1	1	1	0	0	0	0	0	0	0	0	1	1	1	−3	−3	1	1	1	−3	1
χ_{29}	6	6	6	−2	−2	−2	0	0	0	0	0	0	0	0	2	2	2	2	2	2	−2	−2	2	−2
χ_{30}	−3	−3	−3	1	1	1	0	0	3	3	3	−1	−1	−1	1	1	1	−3	−3	1	1	1	−3	1
χ_{31}	−3	−3	−3	1	1	1	0	0	3	3	3	−1	−1	−1	−3	−3	−3	1	1	−3	1	1	1	1
χ_{32}	9	9	9	−3	−3	−3	0	0	3	3	3	−1	−1	−1	−3	−3	−3	1	1	−3	1	1	1	1
χ_{33}	0	0	0	0	0	0	0	0	−3	−3	−3	1	1	1	0	0	0	0	0	0	0	0	0	0
χ_{34}	6	6	6	−2	−2	−2	0	0	−3	−3	−3	1	1	1	−2	−2	−2	−2	−2	−2	2	2	−2	2
χ_{35}	−9	−9	−9	3	3	3	0	0	0	0	0	0	0	0	−1	−1	−1	3	3	−1	−1	−1	3	−1
χ_{36}	3	3	3	−1	−1	−1	0	0	0	0	0	0	0	0	3	3	3	−1	−1	3	−1	−1	−1	−1

Continued on next page

Table 7 (continued)

	2B						3A		3B						4A									
	2j	2k	4h	4i	4j	4k	3a	6a	3b	6b	6c	6d	6e	12a	4l	4m	4n	4o	4p	4q	4r	4s	4t	8a
$\lvert C_{\overline{G}}(g_{ij})\rvert$	3072	3072	512	512	512	256	360	360	576	576	96	96	96	48	512	512	256	256	256	128	128	128	128	64
χ_{37}	9	9	−3	1	−3	1	0	0	3	3	−1	−1	3	−1	5	5	1	−3	1	−3	1	−3	1	1
χ_{38}	−3	−3	1	5	1	−3	0	0	3	3	−1	−1	3	−1	−3	−3	1	5	1	1	−3	1	−3	1
χ_{39}	6	6	−2	6	−2	−2	0	0	−3	−3	1	1	−3	1	2	2	2	2	2	−2	−2	−2	−2	2
χ_{40}	3	3	−1	−5	−1	3	0	0	0	0	0	0	0	0	−1	−1	−5	−1	3	3	3	−1	−1	−1
χ_{41}	3	3	−1	−5	−1	3	0	0	0	0	0	0	0	0	−1	−1	3	−1	−5	−1	−1	3	3	−1
χ_{42}	3	3	−1	11	−1	−5	0	0	0	0	0	0	0	0	3	3	−1	−5	−1	−1	3	−1	3	−1
χ_{43}	15	15	−5	7	−5	−1	0	0	0	0	0	0	0	0	−5	−5	−1	3	−1	3	−1	3	−1	−1
χ_{44}	−9	−9	3	−1	3	−1	0	0	0	0	0	0	0	0	−1	−1	3	−1	−5	−1	−1	3	3	−1
χ_{45}	−9	−9	3	−1	3	−1	0	0	0	0	0	0	0	0	−1	−1	−5	−1	3	3	3	−1	−1	−1
χ_{46}	−12	−12	4	4	4	−4	0	0	3	3	−1	−1	3	−1	0	0	0	0	0	0	0	0	0	0
χ_{47}	12	12	−4	−4	−4	4	0	0	3	3	−1	−1	3	−1	0	0	0	0	0	0	0	0	0	0
χ_{48}	−6	−6	2	−6	2	2	0	0	0	0	0	0	0	0	2	2	2	2	2	−2	−2	−2	−2	2
χ_{49}	0	0	0	0	0	0	0	0	−3	−3	1	1	−3	1	0	0	0	0	0	0	0	0	0	0
χ_{50}	0	0	0	0	0	0	0	0	6	6	−2	−2	−2	2	4	4	−4	4	−4	0	0	0	0	0
χ_{51}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	−4	4	−4	0	0	0	0	0
χ_{52}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	−4	4	−4	0	0	0	0	0
χ_{53}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{54}	0	0	0	0	0	0	0	0	6	6	−2	−2	−2	2	−4	−4	4	−4	4	0	0	0	0	0
χ_{55}	0	0	0	0	0	0	0	0	−6	−6	2	2	2	−2	0	0	0	0	0	0	0	0	0	0
χ_{56}	4	−4	0	0	0	0	1	−1	4	−4	0	0	0	0	4	−4	0	0	0	0	0	0	0	0
χ_{57}	12	−12	0	0	0	0	4	−4	4	−4	0	0	0	0	−4	4	0	0	0	0	0	0	0	0
χ_{58}	8	−8	0	0	0	0	−1	1	8	−8	0	0	0	0	8	−8	0	0	0	0	0	0	0	0
χ_{59}	16	−16	0	0	0	0	5	−5	−4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{60}	4	−4	0	0	0	0	6	−6	0	0	0	0	0	0	4	−4	0	0	0	0	0	0	0	0
χ_{61}	4	−4	0	0	0	0	−3	3	0	0	0	0	0	0	4	−4	0	0	0	0	0	0	0	0
χ_{62}	4	−4	0	0	0	0	−3	3	0	0	0	0	0	0	4	−4	0	0	0	0	0	0	0	0
χ_{63}	16	−16	0	0	0	0	1	−1	4	−4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{64}	−20	20	0	0	0	0	5	−5	8	−8	0	0	0	0	−4	4	0	0	0	0	0	0	0	0
χ_{65}	−12	12	0	0	0	0	0	0	0	0	0	0	0	0	4	−4	0	0	0	0	0	0	0	0
χ_{66}	−12	12	0	0	0	0	0	0	0	0	0	0	0	0	4	−4	0	0	0	0	0	0	0	0
χ_{67}	0	0	0	0	0	0	−4	4	−4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{68}	0	0	0	0	0	0	4	−4	−8	8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{69}	8	−8	0	0	0	0	−5	5	4	−4	0	0	0	0	−8	8	0	0	0	0	0	0	0	0

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Table 7 (continued)

	4B					5A		6A		6B				
	4u	4v	8b	8c	8d	5a	10a	6f	6g	6h	6i	12b	12c	12d
	64	64	32	32	32	30	30	24	24	48	48	24	24	24
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	2	2	0	0	−1	−1	−1	−1	−1
χ_3	0	0	0	0	0	−1	−1	−1	−1	0	0	0	0	0
χ_4	0	0	0	0	0	0	0	1	1	1	1	1	1	1
χ_5	−1	−1	−1	−1	−1	1	1	−2	−2	0	0	0	0	0
χ_6	−1	−1	−1	−1	−1	1	1	1	1	0	0	0	0	0
χ_7	−1	−1	−1	−1	−1	1	1	1	1	0	0	0	0	0
χ_8	0	0	0	0	0	−2	−2	1	1	−1	−1	−1	−1	−1
χ_9	−1	−1	−1	−1	−1	0	0	1	1	0	0	0	0	0
χ_{10}	1	1	1	1	1	0	0	0	0	0	0	0	0	0
χ_{11}	1	1	1	1	1	0	0	0	0	0	0	0	0	0
χ_{12}	0	0	0	0	0	1	1	0	0	−1	−1	−1	−1	−1
χ_{13}	0	0	0	0	0	−1	−1	0	0	0	0	0	0	0
χ_{14}	0	0	0	0	0	0	0	−1	−1	1	1	1	1	1
χ_{15}	1	1	−1	−1	1	0	0	0	0	1	1	−1	−1	1
χ_{16}	1	1	−1	−1	1	0	0	0	0	0	0	0	0	0
χ_{17}	1	1	−1	−1	1	0	0	0	0	0	0	0	0	0
χ_{18}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{19}	−1	−1	1	1	−1	0	0	0	0	−1	−1	1	1	−1
χ_{20}	−1	−1	1	1	−1	0	0	0	0	1	1	−1	−1	1
χ_{21}	1	1	−1	−1	1	0	0	0	0	−1	−1	1	1	−1
χ_{22}	0	0	0	0	0	0	0	0	0	−1	−1	1	1	−1
χ_{23}	0	0	0	0	0	0	0	0	0	1	1	−1	−1	1
χ_{24}	1	1	−1	−1	1	0	0	0	0	0	0	0	0	0
χ_{25}	−1	−1	1	1	−1	0	0	0	0	0	0	0	0	0
χ_{26}	1	1	1	−1	−1	0	0	0	0	1	1	1	−1	−1
χ_{27}	1	1	1	−1	−1	0	0	0	0	0	0	0	0	0
χ_{28}	1	1	1	−1	−1	0	0	0	0	0	0	0	0	0
χ_{29}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{30}	−1	−1	−1	1	1	0	0	0	0	−1	−1	−1	1	1
χ_{31}	−1	−1	−1	1	1	0	0	0	0	1	1	1	−1	−1
χ_{32}	1	1	1	−1	−1	0	0	0	0	−1	−1	−1	1	1
χ_{33}	0	0	0	0	0	0	0	0	0	−1	−1	−1	1	1
χ_{34}	0	0	0	0	0	0	0	0	0	1	1	1	−1	−1
χ_{35}	1	1	1	−1	−1	0	0	0	0	0	0	0	0	0
χ_{36}	−1	−1	−1	1	1	0	0	0	0	0	0	0	0	0
χ_{37}	1	1	−1	1	−1	0	0	0	0	1	1	−1	1	−1
χ_{38}	−1	−1	1	−1	1	0	0	0	0	1	1	−1	1	−1
χ_{39}	0	0	0	0	0	0	0	0	0	−1	−1	1	−1	1
χ_{40}	−1	−1	1	−1	1	0	0	0	0	0	0	0	0	0
χ_{41}	−1	−1	1	−1	1	0	0	0	0	0	0	0	0	0
χ_{42}	−1	−1	1	−1	1	0	0	0	0	0	0	0	0	0
χ_{43}	1	1	−1	1	−1	0	0	0	0	0	0	0	0	0
χ_{44}	1	1	−1	1	−1	0	0	0	0	0	0	0	0	0
χ_{45}	1	1	−1	1	−1	0	0	0	0	0	0	0	0	0
χ_{46}	0	0	0	0	0	0	0	0	0	−1	−1	1	−1	1
χ_{47}	0	0	0	0	0	0	0	0	0	−1	−1	1	−1	1
χ_{48}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{49}	0	0	0	0	0	0	0	0	0	1	1	−1	1	−1
χ_{50}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{51}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{52}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{53}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{54}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{55}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{56}	2	−2	0	0	0	1	−1	1	−1	2	−2	0	0	0
χ_{57}	2	−2	0	0	0	2	−2	0	0	−2	2	0	0	0
χ_{58}	0	0	0	0	0	−1	1	−1	1	0	0	0	0	0
χ_{59}	0	0	0	0	0	0	0	1	−1	2	−2	0	0	0
χ_{60}	−2	2	0	0	0	1	−1	−2	2	0	0	0	0	0
χ_{61}	−2	2	0	0	0	1	−1	1	−1	0	0	0	0	0
χ_{62}	−2	2	0	0	0	1	−1	1	−1	0	0	0	0	0
χ_{63}	0	0	0	0	0	−2	2	1	−1	−2	2	0	0	0
χ_{64}	−2	2	0	0	0	0	0	1	−1	0	0	0	0	0
χ_{65}	2	−2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{66}	2	−2	0	0	0	0	0	0	0	0	0	0	0	0
χ_{67}	0	0	0	0	0	1	−1	0	0	−2	2	0	0	0
χ_{68}	0	0	0	0	0	−1	1	0	0	0	0	0	0	0
χ_{69}	0	0	0	0	0	0	0	−1	1	2	−2	0	0	0

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Table 7 (continued)

	7A					7B					15A		15B	
	7a	14a	14b	14c	28a	7b	14d	14e	14f	28b	15a	30a	15b	30b
	56	56	28	28	28	56	56	28	28	28	30	30	30	30
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	0	0	0	0	0	0	0	0	0	0	−1	−1	−1	−1
χ_3	0	0	0	0	0	0	0	0	0	0	−1	−1	−1	−1
χ_4	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	0	0	0
χ_5	0	0	0	0	0	0	0	0	0	0	1	1	1	1
χ_6	0	0	0	0	0	0	0	0	0	0	C	C	$\overline{C}$	$\overline{C}$
χ_7	0	0	0	0	0	0	0	0	0	0	$\overline{C}$	$\overline{C}$	C	C
χ_8	0	0	0	0	0	0	0	0	0	0	1	1	1	1
χ_9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{10}	A	A	A	A	A	$\overline{A}$	$\overline{A}$	$\overline{A}$	$\overline{A}$	$\overline{A}$	0	0	0	0
χ_{11}	$\overline{A}$	$\overline{A}$	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	A	A	A	0	0	0	0
χ_{12}	0	0	0	0	0	0	0	0	0	0	1	1	1	1
χ_{13}	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1
χ_{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{15}	1	1	−1	1	−1	1	1	−1	1	−1	0	0	0	0
χ_{16}	A	A	$-A$	A	$-A$	$\overline{A}$	$\overline{A}$	$-\overline{A}$	$\overline{A}$	$-\overline{A}$	0	0	0	0
χ_{17}	$\overline{A}$	$\overline{A}$	$-\overline{A}$	$\overline{A}$	$-\overline{A}$	A	A	$-A$	A	$-A$	0	0	0	0
χ_{18}	−1	−1	1	−1	1	−1	−1	1	−1	1	0	0	0	0
χ_{19}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{20}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{21}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{22}	1	1	−1	1	−1	1	1	−1	1	−1	0	0	0	0
χ_{23}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{24}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{25}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{26}	1	1	1	−1	−1	1	1	1	−1	−1	0	0	0	0
χ_{27}	A	A	A	$-A$	$-A$	$\overline{A}$	$\overline{A}$	$\overline{A}$	$-\overline{A}$	$-\overline{A}$	0	0	0	0
χ_{28}	$\overline{A}$	$\overline{A}$	$\overline{A}$	-$\overline{A}$	-$\overline{A}$	A	A	A	$-A$	$-A$	0	0	0	0
χ_{29}	−1	−1	−1	1	1	−1	−1	−1	1	1	0	0	0	0
χ_{30}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{31}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{32}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{33}	1	1	1	−1	−1	1	1	1	−1	−1	0	0	0	0
χ_{34}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{35}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{36}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{37}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{38}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{39}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{40}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{41}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{42}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{44}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{45}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{46}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{47}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{48}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{49}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{50}	1	1	−1	−1	1	1	1	−1	−1	1	0	0	0	0
χ_{51}	A	A	$-A$	$-A$	A	$\overline{A}$	$\overline{A}$	$-\overline{A}$	$-\overline{A}$	$\overline{A}$	0	0	0	0
χ_{52}	$\overline{A}$	$\overline{A}$	$-\overline{A}$	$-\overline{A}$	$\overline{A}$	A	A	$-A$	$-A$	A	0	0	0	0
χ_{53}	−1	−1	1	1	−1	−1	−1	1	1	−1	0	0	0	0
χ_{54}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{55}	1	1	−1	−1	1	1	1	−1	−1	1	0	0	0	0
χ_{56}	2	−2	0	0	0	2	−2	0	0	0	1	−1	1	−1
χ_{57}	0	0	0	0	0	0	0	0	0	0	−1	1	−1	1
χ_{58}	0	0	0	0	0	0	0	0	0	0	−1	1	−1	1
χ_{59}	−2	2	0	0	0	−2	2	0	0	0	0	0	0	0
χ_{60}	0	0	0	0	0	0	0	0	0	0	1	−1	1	−1
χ_{61}	0	0	0	0	0	0	0	0	0	0	C	$-C$	$\overline{C}$	$-\overline{C}$
χ_{62}	0	0	0	0	0	0	0	0	0	0	$\overline{C}$	$-\overline{C}$	C	$-C$
χ_{63}	0	0	0	0	0	0	0	0	0	0	1	−1	1	−1
χ_{64}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{65}	B	$-B$	0	0	0	$\overline{B}$	$-\overline{B}$	0	0	0	0	0	0	0
χ_{66}	$\overline{B}$	$-\overline{B}$	0	0	0	B	$-B$	0	0	0	0	0	0	0
χ_{67}	0	0	0	0	0	0	0	0	0	0	1	−1	1	−1
χ_{68}	2	−2	0	0	0	2	−2	0	0	0	−1	1	−1	1
χ_{69}	0	0	0	0	0	0	0	0	0	0	0	0	0	0

where in Table 7, $A = -\frac{1}{2} - \frac{\sqrt{7}}{2}i = -1 - b7$, $B = -1 - \sqrt{7}i$ and $C = -\frac{1}{2} - \frac{\sqrt{15}}{2}i = -1 - b15$.

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