



On Groups whose Non-Normal Subgroups are either Contranormal or Core-Free *

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Abstract

A subgroup H of a group G is called contranormal in G if $H^G = G$. A subgroup H of a group G is called core-free in G if $\text{Core}_G(H) = \langle 1 \rangle$. Obviously, these two types of subgroups are the complete opposite of normal subgroups. In this paper, we will obtain the structure of soluble and non-soluble groups whose non-normal subgroups are contranormal. Moreover, we will obtain the structure of some periodic groups whose non-normal subgroups are either contranormal or core-free.

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1 Introduction

Let G be a group. The following two normal subgroups are associated with any subgroup H of G : H^G , the *normal closure* of H in G , the least normal subgroup of G that contains H , and $\text{Core}_G(H)$, the *normal core* of H in G , the greatest normal subgroup of G which is contained in a subgroup H .

We have

$$H^G = \langle H^x \mid x \in G \rangle$$

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and

$$\text{Core}_G(H) = \bigcap_{x \in G} H^x.$$

A subgroup H is normal in G if and only if $\text{Core}_G(H) = H$. In this sense, the subgroups H , for which $\text{Core}_G(H) = \langle 1 \rangle$, are the complete opposite to the normal subgroups. A subgroup H of a group G is called *core-free* in G if $\text{Core}_G(H) = \langle 1 \rangle$.

A subgroup H is normal in G if and only if $H^G = H$. In this sense, the subgroups H , for which $H^G = G$, are the complete opposites to the normal subgroups. A subgroup H of a group G is called *contranormal* in G if $H^G = G$. J.S. Rose has introduced the term “contranormal subgroup” in the paper [20].

For each subgroup H of a group G we have the following two extreme and opposite situations:

$$H^G = H \quad \text{or} \quad H^G = G,$$

and, respectively,

$$\text{Core}_G(H) = H \quad \text{or} \quad \text{Core}_G(H) = \langle 1 \rangle.$$

The following extreme cases immediately appear. The first case: every proper subgroup of G is normal. Such group is called a *Dedekind group*. A description of Dedekind groups has been obtained by R. Baer [1]. The second case: every proper subgroup of G is core-free. In this case, G does not contain proper non-trivial normal subgroups, that is, G is a simple group. The third case: every proper non-trivial subgroup of G is contranormal. In this case, G does not contain proper non-trivial normal subgroups, so that again G is a simple group.

In the last two cases we came only to simple groups. Note again that a simple group has the only three following types of subgroups: normal, core-free and contranormal.

Therefore, the following question natural appears: *what we can say about the groups whose subgroups are either normal, core-free or contranormal?*

We study groups whose subgroups are either normal or core-free, and the groups whose subgroups are either normal or contranormal. Note that groups having only two types of subgroups, which are also antagonistic in some sense to each other, have been considered by many authors. Here we provide a list of papers whose subject is to some extent related to our topic: [5, 6, 7, 8, 11, 14, 16, 17, 18, 22].

Groups whose subgroups are either normal or core-free have been studied in the paper [15]. The study of groups whose subgroups are either normal or contranormal was initiated in the paper [22]. In Theorem 2 of [22], basis structural features of such groups were shown. However, this theorem was not provided with a proof. Therefore, in this paper we provide a more detailed description of such groups.

A group G is called *quasisimple*, if the central factor-group $G/\zeta(G)$ is simple and $G = [G, G]$.

It is not hard to show that every subgroup of quasisimple group is either normal or contranormal.

Theorem A *Let G be a group whose non-normal subgroups are contranormal. If G is not soluble, then G is simple or quasisimple.*

Recall that an *infinite generalized quaternion group* is a group

$$Q_\infty = A\langle b \rangle$$

where A is a normal Prüfer 2-group, $|b| = 4$, $b^2 \in A$ and $a^b = a^{-1}$ for all $a \in A$.

Theorem B *Let G be a soluble group whose non-normal subgroups are contranormal. Suppose that G is not a Dedekind group.*

If G is a p -group for some prime p , then $p = 2$ and G is a group of one of the following types:

- (i) $G = D \rtimes \langle g \rangle$ where D is a normal divisible abelian 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$;
- (ii) $G = D\langle g \rangle$ where D is a normal divisible abelian 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$, $D = A \times B$, where A is a Prüfer 2-subgroup, $g^2 \in \Omega_1(A)$ and $\langle A, g \rangle$ is an infinite generalized quaternion group.

If G is a periodic group and $|\Pi(G)| \geq 2$, then G is a group of one of the following types:

- (iii) $G = S \rtimes \langle g \rangle$ where g is a p -element for some prime p , S is an abelian Sylow p' -subgroup of G , $C_{\langle g \rangle}(S) = \langle g^p \rangle$ and every subgroup of S is G -invariant;
- (iv) $G = D \rtimes \langle g \rangle$ where D is a normal abelian subgroup, $D = S \times K$ where S is a Sylow 2'-subgroup of G , K is a divisible 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$;

- (v) $G = D\langle g \rangle$ where D is a normal abelian subgroup, $D = S \times A \times B$ where S is a Sylow 2'-subgroup of G , $A \times B$ is a divisible 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$, $g^2 \in \Omega_1(A)$, and $\langle A, g \rangle$ is an infinite generalized quaternion group.

If G is a non-periodic group, then G is a group of one of the following types:

- (vi) $G = D \rtimes \langle g \rangle$, g has order 2 or 4, and $x^g = x^{-1}$ for each $x \in D$, $C_{\langle g \rangle}(D) = \langle g^2 \rangle$, $2 \notin \Pi(D)$, $D^2 = D$, and every subgroup of D is G -invariant;
- (vii) $G = D \rtimes \langle g \rangle$, g has order 2 or 4, and $x^g = x^{-1}$ for each $x \in D$, $C_{\langle g \rangle}(D) = \langle g^2 \rangle$, $D^2 = D$, $D = S \times B$, $2 \notin \Pi(B)$, S is a divisible Sylow 2-subgroup of D ;
- (viii) $G = D\langle g \rangle$, g has order 2 or 4, and $x^g = x^{-1}$ for each $x \in D$, $C_{\langle g \rangle}(D) = \langle g^2 \rangle$, $D^2 = D$, $D = S \times B$, where $2 \notin \Pi(B)$, $S = A \times C$ is a divisible Sylow 2-subgroup of D , $\langle g \rangle \cap (A \times B) = \langle 1 \rangle$, $g^2 \in \Omega_1(C)$ and $\langle C, g \rangle$ is an infinite generalized quaternion group.

Further, for the study of groups whose subgroups are either normal, core-free or contranormal, it is natural to assume that they contain proper contranormal and proper non-trivial core-free subgroups. Here we consider periodic locally soluble such groups. Their description decomposes to few natural parts.

Theorem C *Let G be a group whose non-normal subgroups are either contranormal or core-free. If G is locally soluble, then G is a soluble group.*

Let G be a group and A be a normal subgroup of G . The intersection $\text{Mon}_G(A)$ of all non-trivial G -invariant subgroups of A is called the G -monolith of A . If $\text{Mon}_G(A)$ is not trivial, then A is called G -monolithic. If $A = G$, then we will say that $\text{Mon}_G(G)$ is the monolith of G and denote it by $\text{Mon}(G)$.

Recall that a p -group G is called *extraspecial*, if $[G, G] = \zeta(G)$ is a subgroup of order p and $G/\zeta(G)$ is an elementary abelian p -group.

The following theorem is dedicated to the monolithic case.

Theorem D *Let G be a soluble periodic monolithic group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains proper contranormal and non-trivial core-free subgroups. Then G is a group of one of the following types:*

- (i) $G = D \rtimes \langle v \rangle$ where D is a normal Prüfer 2-subgroup, $v^2 = 1$, $d^v = d^{-1}$ for all $d \in D$;
- (ii) $G = M \rtimes S$ where M is an elementary abelian p -subgroup, p is a prime, S is a locally cyclic p' -subgroup, $C_G(M) = M$, and every complement to M in G is conjugate with S ; in particular, if M is finite, then G is finite and $G = M \rtimes S$ where S is a cyclic Sylow p' -subgroup of G ;
- (iii) $G = D \rtimes \langle g \rangle$ where D is a normal cyclic p -subgroup, $|g| = q$ is a prime, $q < p$, $C_G(D) = D$;
- (iv) $G = D \rtimes \langle g \rangle$ where D is a normal Prüfer p -subgroup, $|g| = q$ is a prime, $q < p$, $C_G(D) = D$;
- (v) $G = D \rtimes \langle g \rangle$ where D is an extraspecial p -subgroup, p is a prime, $|g| = q$ is a prime, $q < p$, $q \neq 2$, moreover, $M = [D, D] = \zeta(D)$ is a monolith of G , and every subgroup of D/M is G -invariant;
- (vi) $G = M \rtimes K$ where M is a finite elementary abelian p -subgroup, p is an odd prime, K is a quaternion group of order 8, M is a minimal normal subgroup of G , $C_G(M) = M$;
- (vii) $G = M \rtimes B$ where M is a minimal normal elementary abelian p -subgroup, p is an odd prime, $B = K \rtimes \langle u \rangle$ where K is a normal Prüfer 2-subgroup, $u^2 = 1$, $a^u = a^{-1}$ for each $a \in K$;
- (viii) $G = M \rtimes B$ where M is a minimal normal elementary abelian p -subgroup, p is an odd prime, B is an infinite generalized quaternion group;
- (ix) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = D_1 \rtimes \langle g \rangle$ where D_1 is a locally cyclic p' -subgroup, $|g| = p$, every subgroup of D_1 is $\langle g \rangle$ -invariant, $C_V(D_1) = D_1$;
- (x) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = D_1 \rtimes \langle g \rangle$ where D_1 is a locally cyclic subgroup, g is a q -element, q is an odd prime, $p, q \notin \Pi(D_1)$, $C_{\langle g \rangle}(D_1) = \langle g^q \rangle$, and every subgroup of D_1 is $\langle g \rangle$ -invariant;
- (xi) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = D_1 \rtimes \langle g \rangle$ where D_1 is a locally cyclic subgroup, g is a 2-element, $2, p \notin \Pi(D_1)$, $C_{\langle g \rangle}(D_1) = \langle g^2 \rangle$, $x^g = x^{-1}$ for each $x \in D_1$;

- (xii) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = (S \times K) \rtimes \langle g \rangle$ where $S \times K$ is a locally cyclic subgroup, moreover, K is a Prüfer 2-subgroup, S is a $2'$ -subgroup, $|g| = 2$, $x^g = x^{-1}$ for each $x \in S \times K$;
- (xiii) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = S \rtimes (K \langle g \rangle)$ where S is a locally cyclic $2'$ -subgroup, K is a Prüfer 2-subgroup, $|g| = 4$, $g^2 \in \Omega_1(K)$, $\langle K, g \rangle$ is an infinite generalized quaternion group, $C_V(S) = S \times K$, $x^g = x^{-1}$ for each $x \in S \times K$.

Finally, our last theorem considers the non-monolithic case.

Theorem E *Let G be a soluble periodic non-monolithic group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains proper contranormal and non-trivial core-free subgroups. Then G is a group of one of the following types:*

- (i) $G = A \rtimes \langle v \rangle$ where A is a normal divisible 2-subgroup, $v^2 = 1$, $a^v = a^{-1}$ for all $a \in A$;
- (ii) $G = M \rtimes (\langle c \rangle \times \langle g \rangle)$ where M is a normal subgroup of prime order $p \neq 2$, $|c| = s$ is a prime, $|g| = q$ is a prime, $q \neq s$, q divides $p - 1$, $C_G(M) = M \times \langle c \rangle$;
- (iii) $G = M \rtimes (\langle c \rangle \times \langle g \rangle)$ where M is a normal subgroup of prime order $p \neq 2$, $|c| = |g| = q$ is a prime, q divides $p - 1$, $C_G(M) = M \times \langle c \rangle$;
- (iv) $G = M \rtimes \langle g \rangle$ where M is a normal subgroup of prime order $p \neq 2$, g is an element of order q , q is a prime, q divides $p - 1$, $C_G(M) = M$;
- (v) $G = [G, G] \rtimes \langle g \rangle$ where $[G, G]$ is a normal cyclic p -subgroup, where p is an odd prime, $\langle g \rangle$ is a cyclic q -subgroup, q is a prime, $C_G([G, G]) = [G, G]$;
- (vi) $G = [G, G] \rtimes \langle g \rangle$ where $[G, G]$ is a normal Prüfer p -subgroup, where p is an odd prime, $\langle g \rangle$ is a cyclic q -subgroup, q is a prime, $C_G([G, G]) = [G, G]$;
- (vii) $G = (\langle a_1 \rangle \times \langle a_2 \rangle) \rtimes \langle g \rangle$ where $|a_1| = |a_2| = q$, $|g| = p$, p is a prime, $p < q$, $C_G([G, G]) = [G, G]$, $a_1^g = a_1^m$, $a_2^g = a_2^s$, $1 \leq m, s < q$, $m \neq s$;

- (viii) $G = [G, G] \rtimes \langle g \rangle$ where g is an element of order p , p is a prime, $p < q$, $[G, G]$ is an abelian Sylow q -subgroup of G , $C_G([G, G]) = [G, G]$, and every subgroup of $[G, G]$ is G -invariant;
- (ix) $G = S \rtimes \langle g \rangle$ where g is an element of order 2, S is an abelian $2'$ -subgroup, $C_G(S) = S$, and $x^g = x^{-1}$ for every $x \in S$;
- (x) $G = S \rtimes P$ where P is a Sylow 2-subgroup of G and S is an abelian $2'$ -subgroup, $P = P_1 \rtimes \langle g \rangle$ where D is a normal divisible abelian 2-subgroup, $[S, P_1] = \langle 1 \rangle$, $|g| = 2$, and $x^g = x^{-1}$ for every $x \in S \times P_1$;
- (xi) $G = S \rtimes \langle g \rangle$ where $|g| = p$, where p is the least prime of the set $\Pi(G)$, S is an abelian Sylow p' -subgroup of G , $C_G(S) = S$, and every subgroup of S is G -invariant.

2 Groups whose non-normal subgroups are contranormal

Lemma 2.1 *Assume that G is a group and H is a non-trivial normal subgroup of G . If every non-normal subgroup of G is either contranormal or core-free, then every non-normal subgroup of G/H is contranormal. More precisely, if K/H is a proper normal subgroup of G/H , then every subgroup of K/H is G -invariant, in particular, K/H is a Dedekind group.*

PROOF — Indeed, let X be an arbitrary subgroup containing H . Since X contains a non-trivial normal subgroup of G , $\text{Core}_G(X) \neq \langle 1 \rangle$. Hence X must be normal or contranormal in G . It follows that X/H is a normal or contranormal subgroup of G/H .

Let L/H be a subgroup of K/H . Then $L^G \leq K \neq G$, which shows that L cannot be contranormal. The inclusion $\langle 1 \rangle \neq H \leq \text{Core}_G(L)$ shows that L cannot be core-free. Hence L is normal in G . In other words, L/H is a G -invariant subgroup of G/H . □

Lemma 2.1 shows that if G is a group whose non-normal subgroups are either core-free or contranormal, then in every proper factor-group of G each subgroup is normal or contranormal. Therefore we need some properties of such groups.

Lemma 2.2 *Assume that G is a soluble group whose non-normal subgroups are contranormal. Suppose that G is not a Dedekind group.*

Then $G/[G, G]$ is a cyclic p -group for some prime p . Moreover, $G = \langle g \rangle [G, G]$ where $\langle g \rangle$ is contranormal, and every subgroup of $G^p [G, G]$ is G -invariant.

PROOF — Since G is not a Dedekind group, it contains a proper contranormal subgroup C . Since G is soluble, $D = [G, G] \neq G$. The subgroup CD is normal in G , therefore if we suppose that CD is a proper subgroup of G , then from $C^G \leq CD \neq G$ we obtain that a subgroup C cannot be contranormal, and we come to a contradiction. This contradiction proves that $G = CD$.

Suppose that C contains two proper subgroups U, V such that

$$C \cap D \leq U, \quad C \cap D \leq V \quad \text{and} \quad C = UV.$$

Then DU and DV are proper normal subgroups of G . Then every subgroup of DU (respectively, DV) cannot be contranormal, and hence, it is G -invariant. In particular, the subgroups U, V are G -invariant. Then equality $C = UV$ implies that C is normal in G , and we obtain a contradiction. This contradiction shows that $C/(C \cap D)$ is not a product of two proper subgroups. In this case, $C/(C \cap D)$ is either cyclic or quasicyclic p -group for some prime p (it follows, for example, from [9, Corollary 27.4]).

Assume that $C/(C \cap D)$ is quasicyclic. Then C has an ascending series

$$C \cap D = C_1 \leq C_2 \leq \dots \leq C_n \leq C_{n+1} \leq \dots \bigcup_{n \in \mathbb{N}} C_n = C,$$

whose factors C_{n+1}/C_n of order p , $n \in \mathbb{N}$. Isomorphism

$$G/D \simeq C/(C \cap D)$$

shows that $C_n D$ is a proper normal subgroup of G for each $n \in \mathbb{N}$. Using the above arguments we obtain that every subgroup of $C_n D$ is G -invariant, in particular, C_n is a normal subgroup of G for each $n \in \mathbb{N}$.

The equality

$$\bigcup_{n \in \mathbb{N}} C_n = C$$

shows that C is normal in G , and we obtain a contradiction. Thus $C/(C \cap D)$ is a cyclic p -group, so that G/D is likewise cyclic.

Let g be an element of C such that $C = \langle g \rangle (C \cap D)$. Then $G = \langle g \rangle D$.

If we suppose that $\langle g \rangle$ is not contranormal, then $\langle g \rangle$ is normal. The fact that $C \cap D$ is normal in G together with equality $C = \langle g \rangle(C \cap D)$ imply that C is normal in G , and we obtain a contradiction. This contradiction shows that $\langle g \rangle$ must be contranormal.

Finally, $G^p D$ is a proper normal subgroup of G , and, as above, we obtain that every subgroup of $G^p D$ is G -invariant. □

Lemma 2.3 *Let G be an abelian p -group for some prime p . If the factor-group G/G^p is finite, then $G = B \times D$ where B is a finite subgroup and D is a divisible subgroup.*

PROOF — Let B be a basic subgroup of G , that is B is a pure subgroup of G , in particular, $B^p = B \cap G^p$, B is a direct product of cyclic subgroups, and G/B is divisible. The existence of this subgroup follows, for example, from the results of Sections 32 and 33 of the book [9]. The fact that G/G^p is finite together with the equality

$$B^p = B \cap G^p$$

imply that B/B^p is finite. It implies that B is finite. Then $G = B \times D$ (see, for example, [9, Theorem 27.5]). The isomorphism $D \simeq G/B$ shows that D is divisible. □

Lemma 2.4 *Let G be a soluble p -group for some prime p whose non-normal subgroups are contranormal. Suppose that G is not a Dedekind group. Then $p = 2$ and G is a group of one of the following types:*

- (i) $G = D \rtimes \langle g \rangle$ where D is a normal divisible abelian 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$;
- (ii) $G = D \langle g \rangle$ where D is a normal divisible abelian 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$, $D = A \times B$, where A is a Prüfer 2-subgroup, $g^2 \in \Omega_1(A)$ and $\langle A, g \rangle$ is an infinite generalized quaternion group.

PROOF — Since G is not a Dedekind group, G contains a proper contranormal subgroup. Let $D = [G, G]$. Then Lemma 2.2 implies that $G = \langle g \rangle D$ where a subgroup $\langle g \rangle$ is contranormal in G . If we suppose that G is finite, then, being nilpotent, G does not contain proper contranormal subgroups. It follows that $G = \langle g \rangle$. But in this case, G is abelian, and we obtain a contradiction. This contradiction shows that G must be infinite.

If H is a subgroup of $L = D\langle g^p \rangle$, then $H^G \leq D \neq G$, so that a subgroup H cannot be contranormal. Thus every subgroup of L is G -invariant. In particular, it follows that L is a Dedekind group.

Suppose that D is not abelian. Then $D = Q \times E$ where Q is a quaternion group of order 8 and E is an elementary abelian 2-group. The fact that every cyclic subgroup of E is G -invariant implies that $E \leq \zeta(G)$. Being central-by-finite, G is nilpotent. On the other hand, a nilpotent group does not contain proper contranormal subgroups. It follows that G must be Dedekind, and we obtain a contradiction. This contradiction shows that D is abelian.

Suppose that $D^p \neq D$. Consider first the case when D/D^p is infinite. In this case, G/D^p contains a normal elementary abelian p -subgroup of finite index. Then G/D^p is nilpotent (see, for example, [2]). Since G/D^p is infinite, $\langle gD^p \rangle \neq G/D^p$.

Being a proper subgroup of a nilpotent group, $\langle gD^p \rangle$ cannot be contranormal, and we obtain a contradiction. This contradiction proves that D/D^p must be finite. Lemma 2.3 shows that $D = K \times P$ where K is a finite subgroup and P is a divisible subgroup. Since G/P is finite, by the above arguments, $G/P = \langle gP \rangle$. In particular, G/P is abelian, which implies that $D = P$. Thus, D is divisible.

Since L is a proper normal subgroup of G , every subgroup of L is G -invariant, in particular, L is a Dedekind group. Since L contains divisible subgroups, L is abelian. Since $|G : L| = p$, $L = C_G(L)$. Then G/L is isomorphic to a subgroup of the multiplicative group of ring \mathbb{Z}_p^∞ of p -adic integers (see, for example, [21, Theorem 1.5.6]). We recall that

$$U(\mathbb{Z}_p^\infty) = C \times J$$

where $|C| = 2$ and J is an additive group of 2-adic integers, whenever $p = 2$, or $U(\mathbb{Z}_p^\infty) = C \times J$ where C is a cyclic group of order $p - 1$ and J is an additive group of p -adic integers, whenever p is an odd prime (see, for example, [10, Chapter 4, Theorem 6.5]). Since G is a p -group, $p = 2$. In other words, G is a 2-group. Moreover, $G/C_G(L)$ is of order 2. We note that a subgroup C of $U(\mathbb{Z}_2^\infty)$ coincides with $\{1, -1\}$. It follows that $x^g = x^{-1}$ for each $x \in L$. Put $y = g^2$ and suppose that $y \neq 1$. Then $y^g = y^{-1}$. However $y^g = y$, so that $y = y^{-1}$ and $|y| = 2$, so $|g| = 4$.

Suppose that $\langle g \rangle \cap D = \langle 1 \rangle$. Then $G = D \rtimes \langle g \rangle$, g has order 2 or 4 and $x^g = x^{-1}$ for each $x \in D$. Thus G is a group of type (i).

Suppose that $\langle g \rangle \cap D = \langle a \rangle \neq \langle 1 \rangle$. Since D is divisible, there exists

a Prüfer 2-subgroup A such that $a \in A$. Let

$$\mathfrak{M} = \{H \mid H \leq D \text{ and } H \cap A = \langle 1 \rangle\}$$

and let B be a maximal element of family \mathfrak{M} . Then $D = A \times B$ (see, for example, [9, Theorem 21.2]). By this choice $\langle g \rangle \cap B = \langle 1 \rangle$. The subgroup $C = \langle g^2, A \rangle$ is abelian. By its choice,

$$\Omega_1(C) = \Omega_1(\langle g^2 \rangle) = \Omega_1(\langle g \rangle) = \Omega_1(A),$$

so that $\Omega_1(C)$ is cyclic. Since C is abelian and infinite, it follows that C must be a Prüfer 2-subgroup. Its choice shows that $C = A$. We have $\langle g \rangle \cap A = \Omega_1(A)$ and $a^g = a^{-1}$ for each $a \in A$. This means that $\langle g, A \rangle$ is an infinite generalized quaternion group. Thus G is a group of type (ii). □

Lemma 2.5 *Let G be a soluble periodic group whose non-normal subgroups are contranormal. Suppose that G is not a Dedekind group. If $|\Pi(G)| \geq 2$, then G is a group of one of the following types:*

- (i) $G = S \rtimes \langle g \rangle$ where g is a p -element for some prime p , S is an abelian Sylow p' -subgroup of G , $C_{\langle g \rangle}(S) = \langle g^p \rangle$ and every subgroup of S is G -invariant;
- (ii) $G = D \rtimes \langle g \rangle$ where D is a normal abelian subgroup, $D = S \times K$ where S is a Sylow 2'-subgroup of G , K is a divisible 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$;
- (iii) $G = D \langle g \rangle$ where D is a normal abelian subgroup, $D = S \times A \times B$ where S is a Sylow 2'-subgroup of G , $A \times B$ is a divisible 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$, $g^2 \in \Omega_1(A)$, and $\langle A, g \rangle$ is an infinite generalized quaternion group.

PROOF — Since G is not a Dedekind group, G contains a proper contranormal subgroup. Let $D = [G, G]$. Then Lemma 2.2 implies that $G = \langle g \rangle D$ where $\langle g \rangle$ is contranormal. Moreover, Lemma 2.2 shows that G/D is a p -group for some prime p . We have $g = g_1 g_2$ where g_1 is a p -element, g_2 is a p' -element and $[g_1, g_2] = 1$. The fact that G/D is a p -group implies that $g_2 \in D$. Using again Lemma 2.2 we obtain that $\langle g_2 \rangle$ is normal in G . It follows that $\langle g_1 \rangle$ is contranormal in G . Therefore without loss of generality we may assume that g is a p -element.

By Lemma 2.2 every subgroup of D is G -invariant. It follows that G is hypercyclic. Let q be the smallest prime from $\Pi(G)$. Then $G = S \rtimes P$ where P is a Sylow q -subgroup of G and S is a Sylow p' -subgroup of G . The inclusion $D \leq S[P, P]$ together with the fact that G/D is a p -group imply that $q = p$.

Suppose first that P is finite. Then G/S is finite and nilpotent. Using the above arguments we can obtain that $P = \langle g \rangle$. The choice of p shows that $2 \notin \Pi(S)$. Being a Dedekind $2'$ -group, S is abelian. Thus

$$G = S \rtimes \langle g \rangle$$

where S is an abelian Sylow p' -subgroup of G whose subgroups are G -invariant. Thus G is a group of type (i).

Suppose now that P is infinite. The isomorphism $P \simeq G/S$ shows that P is a p -group whose non-normal subgroups are contranormal. Lemma 2.4 shows that $p = 2$ and $P = \langle g \rangle K$ where K is a divisible 2-subgroup. By Lemma 2.2, every subgroup of SP^2 is G -invariant. Since a non-abelian Dedekind group does not contain divisible 2-subgroup, SP^2 is abelian. For every $x \in S$ we have $x^g = x^k$ for some positive integer k . Then

$$x = g^{-2}xg^2 = g^{-1}(g^{-1}xg)g = g^{-1}x^k g = (g^{-1}xg)^k = (x^g)^k.$$

It follows that

$$k^2 \equiv 1 \pmod{|x|}.$$

The last congruence has $\gcd(2, \varphi(|x|)) = 2$ solutions where φ is an Euler function. Thus only 1 and -1 are the solutions of this congruence. It follows that $x^g = x^{-1}$ for each $x \in S$.

If P is a group of type (i) from Lemma 2.4, then $P = K \rtimes \langle g \rangle$ where K is a normal divisible abelian 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in K$.

Then we obtain that $G = D \rtimes \langle g \rangle$ where D is a normal abelian subgroup, $D = S \times K$ where S is a Sylow $2'$ -subgroup of G , K is a divisible 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$. Thus G is a group of type (ii).

If P is a group of type (ii) of Lemma 2.4, then $P = K \rtimes \langle g \rangle$ where K is a normal divisible abelian 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $x \in K$, $K = A \times B$, where A is a Prüfer $2'$ -subgroup, $g^2 \in \Omega_1(A)$ and $\langle A, g \rangle$ is an infinite generalized quaternion group. Then we obtain that $G = D \langle g \rangle$ where D is a normal abelian

subgroup,

$$D = S \times A \times B,$$

S is a Sylow $2'$ -subgroup of G , K is a divisible 2-subgroup, g has order 2 or 4, $d^g = d^{-1}$ for every $d \in D$, $g^2 \in \Omega_1(A)$, and $\langle A, g \rangle$ is an infinite generalized quaternion group. Thus G is a group of type (iii). □

Lemma 2.6 *Let G be a soluble non-periodic group whose non-normal subgroups are contranormal. Suppose that G is not a Dedekind group. Then G is a group of one of the following types:*

- (i) $G = D \rtimes \langle g \rangle$, g has order 2 or 4, and $x^g = x^{-1}$ for each $x \in D$, $C_{\langle g \rangle}(D) = \langle g^2 \rangle$, $2 \notin \Pi(D)$, $D^2 = D$, and every subgroup of D is G -invariant;
- (ii) $G = D \rtimes \langle g \rangle$, g has order 2 or 4, and $x^g = x^{-1}$ for each $x \in D$, $C_{\langle g \rangle}(D) = \langle g^2 \rangle$, $D^2 = D$, $D = S \times B$, $2 \notin \Pi(B)$, S is a divisible Sylow 2-subgroup of D ;
- (iii) $G = D \langle g \rangle$, g has order 2 or 4, and $x^g = x^{-1}$ for each $x \in D$, $C_{\langle g \rangle}(D) = \langle g^2 \rangle$, $D^2 = D$, $D = S \times B$, where $2 \notin \Pi(B)$, $S = A \times C$ is a divisible Sylow 2-subgroup of D , $\langle g \rangle \cap (A \times B) = \langle 1 \rangle$, $g^2 \in \Omega_1(C)$ and $\langle C, g \rangle$ is an infinite generalized quaternion group.

PROOF — Let $D = [G, G]$. Then Lemma 2.2 implies that G/D is a cyclic p -group for some prime p , so that $G = \langle g \rangle D$ where $\langle g \rangle$ is contranormal, and every subgroup of $\langle g^p \rangle D$ is G -invariant. Since G is not periodic, $\langle g^p \rangle D$ is not periodic. Being a non-periodic Dedekind group, $\langle g^p \rangle D$ is abelian. Then $p = 2$ and $x^g = x^{-1}$ for each $x \in \langle g^2 \rangle D$ (see, for example, [21, Theorem 1.5.7]). Put $y = g^2$, and suppose that $y \neq 1$. Then $y^g = y^{-1}$. But $y^g = y$, so that $y = y^{-1}$ and $|y| = 2$.

Let S be a Sylow 2-subgroup of D . Suppose that S is finite. Then S is pure in D , $D = S \times B$ (see, for example, [9, Theorem 27.5]).

Thus G/B is finite. Being a 2-group, it is nilpotent. As above, G/B is cyclic. It follows that $D = B$. In particular, $2 \notin \Pi(D)$. Thus we obtain that $G = D \rtimes \langle g \rangle$ where g has order 2 or 4, and $x^g = x^{-1}$ for each $x \in D$. Thus G is a group of type (i).

Suppose now that a subgroup S is infinite. If $D^2 = D$, then being a pure subgroup of D , S is divisible. Then $D = S \times B$ (see, for example, [9, Theorems 21.2 and 27.5]). By this choice, $2 \notin \Pi(B)$. The factor-group G/S has a finite Sylow 2-subgroup, and using the above arguments we obtain that $G/S = D/S \rtimes \langle gS \rangle$ where gS has order 2

or 4. It follows that $\langle g, S \rangle \cap B = \langle 1 \rangle$. Clearly, $P = \langle g, S \rangle$ is a Sylow 2-subgroup of G . The isomorphism $P \simeq G/B$ proves that P is a group from Lemma 2.4. Then either $G = D \rtimes \langle g \rangle$ where g has order 2 or 4 and $x^g = x^{-1}$ for each $x \in D$, or $G = D \langle g \rangle$ where g has order 2 or 4 and $x^g = x^{-1}$ for each $x \in D$, $D = A \times C$ where $g^2 \in C$ and $\langle g, C \rangle$ is an infinite generalized quaternion group. Thus G is a group of type (ii) or (iii).

Suppose that $D^2 \neq D$. Then G/D^2 must be non-abelian. As in Lemma 2.4 we can prove that D/D^2 must be finite. In turn out it follows that D/D^2 must be cyclic. In particular, it is abelian, and we obtain a contradiction. This contradiction proves the equality $D^2 = D$. \square

PROOF OF THEOREM A — Suppose that G contains a non-trivial proper normal subgroup H . Each subgroup of H cannot be contranormal and therefore is G -invariant. Then $G/C_G(H)$ is abelian (see, for example, [21, Theorem 1.5.1]). If we suppose that $G \neq C_G(H)$, then, as we done above, we can prove that every subgroup of $C_G(H)$ is G -invariant, in particular, $C_G(H)$ is a Dedekind group. Being Dedekind, $C_G(H)$ is nilpotent, so that G must be soluble, and we obtain a contradiction. This contradiction proves the equality $G = C_G(H)$. In other words, the center of G contains every proper normal subgroup of G . In particular, it follows that $G/\zeta(G)$ is a simple group. If we suppose that $G \neq [G, G]$, then by above $[G, G] \leq \zeta(G)$, so that G is soluble, which is impossible. This contradiction proves that $G = [G, G]$. Thus G is a quasisimple group.

PROOF OF THEOREM B — This theorem directly follows from Lemmas 2.4, 2.5 and 2.6.

PROOF OF THEOREM C — Being locally soluble, G has a family \mathfrak{S} of normal subgroups containing $\langle 1 \rangle$ and G , which is linearly ordered by inclusion and closed with the respect of taking intersections and unions, and whose factors are G -chief and abelian (see, for example, [19, §58]). If $\mathfrak{S} = \{\langle 1 \rangle, G\}$, then G is abelian, and all is proved.

Suppose that $\mathfrak{S} \neq \{\langle 1 \rangle, G\}$. Denote by D the intersection of all non-trivial members of \mathfrak{S} . If H is a proper non-trivial normal subgroup of G , then Lemma 2.1 shows that every subgroup of G/H is normal or contranormal. Lemmas 2.4, 2.5 and 2.6 shows that G/H is metabelian. It follows that

$$[[G, G], [G, G]] \leq H.$$

Since it is true for each non-trivial subgroup $H \in \mathfrak{S}$,

$$[[G, G], [G, G]] \leq D.$$

If $D = \langle 1 \rangle$, then G is metabelian. If $D \neq \langle 1 \rangle$, then D is a minimal normal subgroup of G . In this case, G is soluble of solubility class at most 3.

3 Primary groups whose non-normal subgroups are either contranormal or core-free

Lemma 3.1 *Let G be a group whose non-normal subgroups are either contranormal or core-free and let U be a proper non-trivial normal subgroup of G . If G contains a normal non-trivial subgroup V such that $U \cap V = \langle 1 \rangle$, then every subgroup of U is G -invariant.*

PROOF — Let u be an arbitrary element of U . Since U is normal in G , then $\langle u \rangle^G \leq U$. The fact that U is proper implies that every subgroup of U cannot be contranormal. On the other hand, every subgroup W of G such that $V \leq W$ has a non-trivial core, and hence cannot be core-free. Thus, every subgroup of $(UV)/V$ is G -invariant. Hence if g is an arbitrary element of G , then

$$(uV)^{gV} = u^gV \in \langle uV \rangle.$$

In other words, $u^g = u^k v$ for some $v \in V$ and some integer k . Thus, we have $u^k v \in U$, which implies that $v \in U$. Then equality $U \cap V = \langle 1 \rangle$ implies that $v = 1$. We obtain that $u^g \in \langle u \rangle$ for each $g \in G$. It follows that $\langle u \rangle$ is G -invariant. The fact that every cyclic subgroup of U is G -invariant implies that every subgroup of U is G -invariant. \square

Lemma 3.2 *Let G be a soluble group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains a proper contranormal subgroup and $[G, G]$ contains a proper non-trivial G -invariant abelian subgroup A . Then $G/[G, G]$ is a cyclic p -group for some prime p , and every subgroup of $G^p[G, G]/A$ is G -invariant.*

PROOF — Indeed, Lemma 2.1 shows that every subgroup of G/A is contranormal or normal in G , hence we can apply Lemma 2.2. \square

Lemma 3.3 *Let G be a group whose non-normal subgroups are either contranormal or core-free, H be a proper normal elementary abelian p -subgroup of G , p is a prime. Suppose also that $|H| > p^2$. If H contains two G -invariant non-trivial subgroups U, V such that $U \cap V = \langle 1 \rangle$, then every subgroup of H is G -invariant.*

PROOF — Let h be an arbitrary element of H . The equality $U \cap V = \langle 1 \rangle$ implies that either $h \notin U$ or $h \notin V$. Suppose that $h \notin U$. If $h \in V$, then Lemma 3.1 implies that $\langle h \rangle$ is G -invariant. Therefore we will assume that $h \notin V$. By Lemma 2.1 $\langle hU \rangle$ is G -invariant. If g is an arbitrary element of G , then we obtain $h^g = h^k u_0$ for some $u_0 \in U$ and some integer k . Since H is elementary abelian, $\langle h \rangle \cap U = \langle 1 \rangle$.

Suppose that $|U| > p$. Then U contains a non-trivial subgroup U_1 such that $U = \langle u_0 \rangle \times U_1$. Using Lemma 2.1 we obtain that $\langle hU_1 \rangle$ is G -invariant. Hence

$$h^g = h^k u_0 \in \langle h \rangle U_1.$$

It follows that $u_0 \in \langle h \rangle U_1$, which implies that

$$u_0 \in \langle h \rangle U_1 \cap \langle u_0 \rangle = \langle 1 \rangle.$$

So $h^g \in \langle h \rangle$.

Suppose that $|U| = p$. If $u_0 \neq 1$, then $U = \langle u_0 \rangle$. Since $h \notin V$, $\langle hV \rangle$ is non-trivial. By Lemma 2.1, this subgroup is G -invariant, so that

$$h^g = h^k u_0 \in \langle h \rangle V.$$

It follows that $h^k u_0 = h^t v$ for some $v \in V$ and some integer t . Then

$$h^{t-k} = u_0 v^{-1}.$$

The equality $U \cap V = \langle 1 \rangle$ implies that $u_0 v^{-1} \neq 1$, so that p does not divide $t - k$, which implies that $\langle h \rangle \leq UV$, i.e. $h = u_0^s v_0$ for some $v_0 \in V$ and some integer s such that $(p, s) = 1$.

Suppose that $|V| > p$. Then V contains a non-trivial subgroup V_1 such that $V = \langle v_0 \rangle \times V_1$. Using Lemma 3.1 we obtain that both subgroups $\langle v_0 \rangle$ and V_1 are G -invariant. Since $\langle u_0 \rangle = U$ is G -invariant, $U \langle v_0 \rangle$ is G -invariant. Its choice yields that $\langle 1 \rangle = V_1 \cap (U \langle v_0 \rangle)$. Using Lemma 3.1 one more time, we obtain that every subgroup of $U \langle v_0 \rangle$ is G -invariant. In particular, $\langle h \rangle$ is G -invariant.

Finally, suppose that $|U| = |V| = p$. By our condition $|H| > p^2$. It

follows that there exists an element $h \notin UV$. By Lemma 2.1, a subgroup $\langle h \rangle U$ (respectively, $\langle h \rangle V$) of H/U (respectively, H/V) is G -invariant. If g is an arbitrary element of G , then we obtain that $h^g = h^k u_0$ for some $u_0 \in U$ and some integer k . If we assume that $u_0 \neq 1$, then repeating the above arguments, we obtain that $h \in UV$, which contradicts to the choice of h . This contradiction again proves that $\langle h \rangle$ is G -invariant. If x is an element of UV , then $x \notin \langle h \rangle U$ or $x \notin \langle h \rangle V$. Repeating the above arguments, we obtain that $\langle x \rangle$ is G -invariant.

Thus, every cyclic subgroup of H is G -invariant. It implies that every subgroup of H is G -invariant. □

Lemma 3.4 *Let G be a group whose non-normal subgroups are either contranormal or core-free, H be a proper normal elementary abelian p -subgroup of G , p is a prime. Suppose that $|H| > p^2$. If H is not G -monolithic, then every subgroup of H is G -invariant.*

PROOF — Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the family of all non-trivial G -invariant subgroups of H . Since H is not G -monolithic,

$$\bigcap_{\lambda \in \Lambda} V_\lambda = \langle 1 \rangle.$$

This equality together with Remak's theorem imply that H is embedded in the cartesian product $\text{Cr}_{\lambda \in \Lambda} H/V_\lambda$. Lemma 2.1 shows that the factors H/V_λ are Dedekind for each index $\lambda \in \Lambda$. If $\bigcap_{\lambda \in \Xi} V_\lambda = \langle 1 \rangle$ for some finite subset Ξ of Λ , then we can apply Lemma 3.3. Therefore, further we will suppose that the intersection $\bigcap_{\lambda \in \Phi} V_\lambda$ is not trivial for every finite subset Φ of Λ .

Suppose that H has an element h such that $\langle h \rangle$ is not G -invariant. In this case, G contains an element g such that $h^g \notin \langle h \rangle$. The equality

$$\bigcap_{\lambda \in \Lambda} V_\lambda = \langle 1 \rangle$$

implies that there exists an index $\mu \in \Lambda$ such that V_μ does not contains h . Since $V_\mu \neq \langle 1 \rangle$, then Lemma 2.1 implies that $\langle h V_\mu \rangle$ is G -invariant in the factor-group G/V_μ . It follows that

$$h^g = h^k v_\mu$$

for some $v_\mu \in V_\mu$ and some integer k . Moreover, our assumption about h implies that $\langle h \rangle$ does not contains v_μ . There exists an in-

dex $\nu \in \Lambda$ such that the subgroup V_ν does not contains v_μ . Since

$$\langle h \rangle \cap V_\mu = \langle 1 \rangle \quad \text{and} \quad \langle v_\mu \rangle \cap V_\nu = \langle 1 \rangle,$$

we have

$$(\langle h \rangle \langle v_\mu \rangle) \cap (V_\mu \cap V_\nu) = \langle 1 \rangle.$$

As we have noted above, $D = V_\mu \cap V_\nu$ is non-trivial, and therefore, Lemma 2.1 implies that every subgroup of H/D is G -invariant. On the other hand,

$$(hD)^9 = h^9D = h^k v_\mu D = (h^k D)(v_\mu D) \notin \langle h \rangle D,$$

and we obtain a contradiction, which proves that every subgroup of H is G -invariant. \square

Lemma 3.5 *Let G be a group whose non-normal subgroups are either contranormal or core-free, A be a proper normal p -subgroup of G , p is a prime. Let B be a G -invariant subgroup of A containing all elements of A of order p . If every cyclic subgroup of B is G -invariant, then every subgroup of A is G -invariant. In particular, if A is abelian and every cyclic subgroup of $\Omega_1(A)$ is G -invariant, then every subgroup of A is G -invariant.*

PROOF — Let b be an arbitrary element of A . Since the subgroup

$$B_1 = \Omega_1(\langle b \rangle)$$

is G -invariant, Lemma 2.1 implies that every subgroup of A/B_1 is G -invariant. In particular, $\langle b \rangle/B_1$ is G -invariant. It follows that $\langle b \rangle$ is G -invariant. In its turn, the fact that every cyclic subgroup of A is G -invariant implies that every subgroup of A is G -invariant. \square

Lemma 3.6 *Let G be a group whose non-normal subgroups are either contranormal or core-free, A be a proper normal abelian p -subgroup of G , p is a prime. If $(\Omega_2(A))^p = \Omega_1(A)$, then every subgroup of A is G -invariant.*

PROOF — Let $c \neq 1$ be an arbitrary element of $\Omega_1(A)$. The equality

$$(\Omega_2(A))^p = \Omega_1(A)$$

implies that there exists an element $d \in \Omega_2(A)$ such that $d^p = c$. Lemma 2.1 implies that every subgroup of $A/\Omega_1(A)$ is G -invariant. Hence for each x of G we have $d^x = d^k c_1$ where $1 \leq k \leq p$ and

$c_1 \in \Omega_1(A)$. Then

$$c^x = (d^p)^x = (d^x)^p = (d^k c_1)^p = d^{pk} c_1^p = c^k.$$

Since this is true for each $x \in G$, $\langle c \rangle$ is normal in G . Thus, every cyclic subgroup of $\Omega_1(A)$ is G -invariant. Using Lemma 3.5 we obtain that every subgroup of A is G -invariant. \square

Corollary 3.7 *Let G be a group whose non-normal subgroups are either contranormal or core-free, A be a proper normal abelian p -subgroup of G , p is a prime. If A is divisible, then every subgroup of A is G -invariant.*

PROOF — Indeed, the fact that a subgroup A is divisible implies the equality $(\Omega_2(A))^p = \Omega_1(A)$, and we can apply Lemma 3.6. \square

Let p be a prime. If every elementary abelian p -section of a group G is finite of order at most p^r and there is an elementary abelian p -section of order precisely p^r , then G is said to have *finite section p -rank r* , denoted by $sr_p(G) = r$.

Lemma 3.8 *Let G be a group whose non-normal subgroups are either contranormal or core-free, H be a proper non-trivial normal p -subgroup of G , p is a prime. Suppose that $sr_p(H) > 2$. If H is not G -monolithic, then every subgroup of H is G -invariant.*

PROOF — Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the family of all non-trivial G -invariant subgroups of H . Since H is not G -monolithic, $\bigcap_{\lambda \in \Lambda} V_\lambda = \langle 1 \rangle$. This equality together with Remak's theorem imply that H is embedded in the cartesian product $\text{Cr}_{\lambda \in \Lambda} H/V_\lambda$. Lemma 2.1 shows that the factors H/V_λ are Dedekind groups for each index $\lambda \in \Lambda$.

Suppose that p is odd. Then every factor H/V_λ are abelian for each index $\lambda \in \Lambda$. It follows that H is likewise abelian. Using Lemma 3.4 we obtain that every subgroup of $\Omega_1(H)$ is G -invariant. Then Lemma 3.5 implies that every subgroup of H is G -invariant.

Suppose now that $p = 2$. Lemma 2.1 shows that the factors H/V_λ are Dedekind groups for each index $\lambda \in \Lambda$. Taking into account the description of the structure of Dedekind groups, we can conclude that if a factor H/V_λ is not abelian, then its center contains all its elements of order 2. The equality $\bigcap_{\lambda \in \Lambda} V_\lambda = \langle 1 \rangle$ together with Remak's theorem imply that H is embedded in the cartesian product $\text{Cr}_{\lambda \in \Lambda} H/V_\lambda$. Thus, we can see that if H is not abelian, then its center contains all its elements of order 2. It follows that the set A of all elements of H of order 2, is a subgroup of H . Using Lemma 3.4 we

obtain that every subgroup of A is G -invariant. Lemma 3.5 implies that every subgroup of H is G -invariant. \square

Lemma 3.9 *Let G be a periodic group whose non-normal subgroups are either contranormal or core-free. Suppose that $[G, G]$ is abelian and contains a proper non-trivial G -invariant subgroup. Then*

- (i) $G/[G, G]$ is a cyclic p -group for some prime p ;
- (ii) if a Sylow p -subgroup of $[G, G]$ is trivial, then either every subgroup of $G^p/[G, G]$ is G -invariant, or G is a group of one of the following types:
 - (ii.a) $G = D \rtimes \langle g \rangle$ where $D = \langle d_1 \rangle \times \langle d_2 \rangle$, $|d_1| = |d_2| = q$, q is a prime, $|g| = p$, $p \neq q$, $d_1^g = d_1^k$, $d_2^g = d_2^m$, $1 \leq k, m < q$, $k \neq m$;
 - (ii.b) if $p \neq 2$, then a Sylow p -subgroup of $[G, G]$ is trivial;
 - (ii.c) if $p = 2$ and a Sylow 2-subgroup V of $[G, G]$ is non-trivial, then V is divisible and every subgroup of $G^2/[G, G]$ is G -invariant.

PROOF — Let $D = [G, G]$. Then Lemma 3.2 implies that G/D is a cyclic p -group for some prime p , so that $G = \langle g \rangle D$, and we can choose the element g in such way that g will be a p -element.

Suppose that a Sylow p -subgroup of D is trivial. Consider the case when the set $\Pi(D)$ contains at least two primes. Let $q \in \Pi(D)$ and Q be a Sylow q -subgroup of D . Then $D = Q \times R$ where R is a non-trivial Sylow q' -subgroup of D . Using Lemma 3.1 we obtain that every subgroup of Q and every subgroup of R are G -invariant. It follows that every subgroup of D is G -invariant. Suppose that $g^p \neq 1$. Using Lemma 2.1 we obtain that the subgroups $Q\langle g^p \rangle$ and $R\langle g^p \rangle$ are G -invariant. It follows that

$$\langle g^p \rangle = Q\langle g^p \rangle \cap R\langle g^p \rangle$$

is G -invariant. Since $p \notin \Pi(D)$ we obtain that every subgroup of $D\langle g^p \rangle$ is G -invariant.

Assume now that D is a q -subgroup for some prime q . Suppose also that D is G -monolithic. If we assume that

$$\Omega_1(D) \neq \text{Mon}_G(D),$$

then $\Omega_1(D)$ contains a G -invariant subgroup W such that

$$\Omega_1(D) = \text{Mon}_G(D) \times W$$

(see, for example, [13, Corollary 5.14]), and we come to a contradiction. This contradiction proves the equality $\Omega_1(D) = \text{Mon}_G(D)$.

Since D contains a proper non-trivial G -invariant subgroup, we have $D \neq \Omega_1(D)$. Then $(\Omega_2(D))^q$ is a non-trivial G -invariant subgroup of $\Omega_1(D)$. Using the fact that $\Omega_1(D)$ is a minimal normal subgroup of G , we obtain the equality

$$(\Omega_2(D))^q = \Omega_1(D).$$

Then Lemma 3.6 implies that every subgroup of D is G -invariant. Suppose that $g^p \neq 1$. Lemma 3.2 shows that every subgroup of $D\langle g^p \rangle / \Omega_1(D)$ is G -invariant. It follows that

$$[\Omega_2(D), g^p] \leq \Omega_1(D).$$

Let d be an arbitrary element of $\Omega_1(D)$. The equality

$$(\Omega_2(D))^q = \Omega_1(D)$$

shows that there is an element $b \in \Omega_2(D)$ such that $b^q = d$. It follows that $g^{-p}bg^p = bc$ for some $c \in \Omega_1(D)$. Then

$$g^{-p}dg^p = g^{-p}b^qg^p = (g^{-p}bg^p)^q = (bc)^q = b^qc^q = d.$$

Thus

$$[\Omega_1(D), g^p] = \langle 1 \rangle.$$

Since $p \neq q$, $[D, g^p] = \langle 1 \rangle$. The equality $G = \langle g \rangle D$ shows that $\langle g^p \rangle \leq \zeta(G)$. Thus every subgroup of $D\langle g^p \rangle$ is G -invariant.

Suppose that $g^p \neq 1$ and that D is not G -monolithic. Then a subgroup D has a family

$$\{V_\lambda \mid \lambda \in \Lambda\}$$

of non-trivial G -invariant subgroups such that

$$\bigcap_{\lambda \in \Lambda} V_\lambda = \langle 1 \rangle.$$

By Lemma 2.1 every subgroup $\langle g^p \rangle V_\lambda$ is G -invariant. Then

$$\langle g^p \rangle = \bigcap_{\lambda \in \Lambda} V_\lambda \langle g^p \rangle$$

is G -invariant. By Lemma 3.1 every subgroup of D is G -invariant. It follows that every subgroup of $D \langle g^p \rangle$ is G -invariant.

Suppose that $g^p = 1$ and that D is not G -monolithic. If $\text{sr}_p(D) > 2$ and D is not G -monolithic, then Lemma 3.8 implies that every subgroup of D is G -invariant. If $\text{sr}_p(D) = 1$, then D is a cyclic q -subgroup or a Prüfer q -subgroup. In each of these cases, every subgroup of D is G -invariant. Assume that $\text{sr}_p(D) = 2$. Then $\Omega_1(D)$ has order q^2 . Since D is not G -monolithic, $\Omega_1(D)$ contains two non-trivial G -invariant subgroups, having trivial intersection. It follows that

$$\Omega_1(D) = \langle d_1 \rangle \times \langle d_2 \rangle$$

and both subgroups $\langle d_1 \rangle$ and $\langle d_2 \rangle$ are G -invariant. Then

$$d_1^g = d_1^k, \quad d_2^g = d_2^m \quad (1 \leq k, m < q, \quad k \neq m).$$

Suppose that $D = \Omega_1(D)$. If $k = m$, then every subgroup of D is G -invariant. If $k \neq m$, G is a group of type (ii.a).

Consider the case when $D \neq \Omega_1(D)$. If

$$(\Omega_2(D))^q = \Omega_1(D),$$

Lemma 3.6 implies that every subgroup of D is G -invariant. Therefore suppose that $(\Omega_2(D))^q = E \neq \Omega_1(D)$. Without loss of generality we can assume that $E = \langle d_1 \rangle$. Using again Corollary 5.14 of the book [13] we obtain that E has a G -invariant complement C in D . Without loss of generality we can assume that $C = \langle d_2 \rangle$. Let b be an element of $\Omega_2(D)$ such that $b^q = d_1$. We have $\Omega_2(D)/E = \langle bE \rangle \times \langle d_2E \rangle$. The equality $d_2^g = d_2^m$ implies that $d_2^g E = d_2^m E$. By Lemma 2.1, every subgroup of $\Omega_2(D)/E$ is G -invariant. It follows that

$$(bE)^g = (bE)^m = b^m E.$$

It implies that $b^g = b^m u$ for some $u \in E$. Then

$$d_1^g = (b^q)^g = (b^g)^q = (b^m u)^q = b^{mq} u^q = b^{q^2} = d_1^m.$$

Thus $k = m$. It follows that every subgroup of $\Omega_1(D)$ is G -invariant. Using Lemma 3.5 we obtain that every subgroup of D is G -invariant.

Suppose that Sylow p -subgroup P of D is not trivial. Then

$$D = P \times B$$

where B is a Sylow p' -subgroup of D . Since G/D is a p -group, B is a Sylow p' -subgroup of G . Suppose that P is finite. Then factor-group G/B , being a finite p -subgroup, is nilpotent. We note that a nilpotent group, having a cyclic factor-group by the derived subgroup, is itself cyclic. On the other hand, the choice of B implies that it is a proper G -invariant subgroup of D , and we obtain a contradiction. Thus G is a group of type (ii.b).

Suppose now that P is infinite and divisible. By Corollary 3.7, every subgroup of P is G -invariant. Let x be an element of P , having prime order. Let d be an arbitrary element of D . By Lemma 2.1, $\langle d, x \rangle / \langle x \rangle$ is G -invariant, so that $\langle d, x \rangle$ is G -invariant. Suppose that $\gcd(|d|, p) = 1$. Then $\langle d \rangle$ is a Sylow p' -subgroup of $\langle d, x \rangle$, and therefore $\langle d \rangle$ is G -invariant. Thus, every subgroup of B is G -invariant too. It follows that every subgroup of D is G -invariant. By Lemma 2.1, every subgroup of G/B is normal or contranormal. Since G/B is a p -group, Lemma 2.4 implies that $p = 2$.

Suppose that P is infinite and not divisible. Then $S = P^p \neq P$ and P/S is an elementary abelian Sylow p -subgroup of D/S . Then

$$D/S = P/S \times B_1/S$$

where $B_1/S = BS/S$. Furthermore,

$$(D/S)^p = (B_1/S)^p = B_1/S$$

is a G -invariant subgroup having a trivial intersection with P/S . Then factor-group G/B_1 , being an extension of elementary abelian p -subgroup by a cyclic p -group, is nilpotent (see, for example, [2]). We note that a nilpotent group, having a cyclic factor-group by the derived subgroup, is itself cyclic. On the other hand, the choice of B_1 shows that it is a proper G -invariant subgroup of D , and we obtain a contradiction. This contradiction shows that a subgroup P must be divisible. We have already proved that in this case, every subgroup of D is G -invariant.

Suppose that $\langle g^2 \rangle$ is non-trivial. Suppose that $B \neq \langle 1 \rangle$. Then by Lem-

ma 2.1 every subgroup of $G^2[G, G]/P$ is G -invariant, so that $\langle g^2 \rangle P/P$ is G -invariant. Since $\langle g^2 \rangle P$ is a 2-subgroup, $\langle g^2 \rangle P \cap B = \langle 1 \rangle$. By Lemma 3.1 every subgroup of $\langle g^2 \rangle P$ is G -invariant. Then every subgroup of $G^2[G, G]$ is G -invariant.

Suppose now that $B = \langle 1 \rangle$, so that G is a 2-group. Since every subgroup of D is G -invariant, $G/C_G(D)$ is isomorphic to a subgroup of the multiplicative group of ring \mathbb{Z}_{2^∞} of 2-adic integers (see, for example, [21, Theorem 1.5.6]). We recall that $U(\mathbb{Z}_{2^\infty}) = C \times J$ where $|C| = 2$ and J is the additive group of 2-adic integer (see, for example, Chapter 4 and Theorem 6.5 of [10]). Since G is periodic, $G/C_G(D)$ is a group of order 2. It follows that $g^2 \in C_G(D)$, so that a subgroup $\langle g^2 \rangle D$ is abelian. The equality $G = \langle g \rangle D$ implies that $\langle g^2 \rangle \leq \zeta(G)$. The facts that every subgroup of $\Omega_1(D)$ is G -invariant and that G is a 2-group imply that $\Omega_1(D) \leq \zeta(G)$. It follows that $\zeta(G)$ contains $\Omega_1(D \langle g^2 \rangle)$. Using Lemma 3.5 we obtain that every subgroup of $\langle g^2 \rangle [G, G]$ is G -invariant. Thus G is a group of type (ii.c). \square

Lemma 3.10 *Let G be a soluble p -group whose non-normal subgroups are either contranormal or core-free, p is a prime. Suppose that G contains proper contranormal and non-trivial core-free subgroups. If G is monolithic, then $G = D \rtimes \langle v \rangle$ where D is a normal Prüfer 2-subgroup, $v^2 = 1$, $d^v = d^{-1}$ for all $d \in D$.*

PROOF — Let M be the monolith of G . Then M is a minimal normal subgroup of G . Since G is a soluble p -group, G is locally nilpotent. It follows that M has a prime order p and $M \leq \zeta(G)$ (see, for example, [4, Proposition 1.2.20]). If we suppose that $M = [G, G]$, then G is nilpotent. But a nilpotent group does not contain proper contranormal subgroups, and we obtain a contradiction. This contradiction shows that M is a proper subgroup of $[G, G]$.

Lemma 3.9 implies that G is a 2-group and $[G, G]$ is a divisible subgroup with $G/[G, G]$ is cyclic. Moreover, every subgroup of

$$D = G^2[G, G]$$

is G -invariant. Since $G^2[G, G]$ contains a divisible subgroup, it is abelian. Then the facts that G is monolithic and that every subgroup of $\Omega_1(D)$ is G -invariant imply that $M = \Omega_1(D)$. In particular, $\Omega_1(D)$ is cyclic. Being abelian and infinite, D must be a Prüfer 2-subgroup. Let g be an element of G such that $G = \langle g \rangle [G, G]$. As in Lemma 2.4 we can obtain that $d^g = d^{-1}$ for all $d \in D$.

By our assumption G contains a proper non-trivial core-free subgroup S . Then $S \cap D$ must be trivial. It follows that $|S| = 2$. Since $|G/D|=2$, $G=DS$. Let $S = \langle v \rangle$, then $d^v = d^g = d^{-1}$ for all $d \in D$. \square

Lemma 3.11 *Let G be a soluble p -group whose non-normal subgroups are either contranormal or core-free, p is a prime. Suppose that G contains proper contranormal and non-trivial core-free subgroups. If G is non-monolithic, then $p = 2$ and then $G = A \rtimes \langle v \rangle$ where A is a normal divisible subgroup, $v^2 = 1$, $a^v = a^{-1}$ for all $a \in A$.*

PROOF — Let \mathfrak{S} be the family of all proper non-trivial normal subgroups of G . Since G is not monolithic, $\bigcap \mathfrak{S} = \langle 1 \rangle$. If H is a proper non-trivial normal subgroup of G , then Lemma 2.1 implies that every subgroup of G/H is normal or contranormal. Using Lemma 2.4 we obtain that G/H is abelian or metabelian. In any case, H contains $[[G, G], [G, G]]$. Since it is true for every subgroup $H \in \mathfrak{S}$,

$$[[G, G], [G, G]] \leq \bigcap \mathfrak{S} = \langle 1 \rangle.$$

The fact that G contains a proper contranormal subgroup implies that G is not abelian. Thus G is metabelian. If we suppose that $[G, G]$ is a minimal normal subgroup of G , then $[G, G]$ has a prime order p and $[G, G] \leq \zeta(G)$ (see, for example, [4, Proposition 1.2.20]). Then G is nilpotent. But nilpotent group does not contain proper contranormal subgroup.

Thus $[G, G]$ contains a proper non-trivial G -invariant subgroup. Then Lemma 3.9 implies that $p = 2$, $G/[G, G]$ is cyclic, $[G, G]$ is divisible and every subgroup of $G^2/[G, G]$ is G -invariant.

Let g be an element of G such that $G = \langle g \rangle [G, G]$. Since

$$D = G^2/[G, G]$$

contains a divisible subgroup, it is abelian. Then $G/C_G(D)$ is isomorphic to a subgroup of the multiplicative group of ring \mathbb{Z}_{2^∞} of 2-adic integers (see, for example, [21, Theorem 1.5.6]). We recall that

$$U(\mathbb{Z}_{2^\infty}) = C \times J$$

where $|C| = 2$ and J is the additive group of 2-adic integer (see, for example, [10, Chapter 4, Theorem 6.5]). Since G is a 2-group, we obtain that $G/C_G(D)$ is a group of order 2. We note that a subgroup C of $U(\mathbb{Z}_{2^\infty})$ coincides with $\{1, -1\}$. It follows that $x^g = x^{-1}$

for each $x \in D$. The fact that G is not abelian implies that $D = C_G(D)$. If y is an element of G such that $y \notin D$, then $y = g^k a$ where k is an odd integer and $a \in D$. Then we obtain that $x^y = x^{-1}$ for each $x \in D$. We have

$$[x, y] = x^{-1} x^y = x^{-2}.$$

Since $[G, G]$ is divisible, $[[G, G], y] = [G, G]$. Then equality

$$G = [G, G]\langle y \rangle = [G, G]\langle y \rangle$$

implies $\langle y \rangle^G = G$. Hence $\langle y \rangle$ is contranormal in G . It means that every subgroup H which does not belong to D , is contranormal in G . We have already noted, that every subgroup of D is normal in G . Thus every subgroup of G is either normal or contranormal, and we can use Lemma 2.4. Taking into account the fact that G contains a proper non-trivial core-free subgroup, we obtain that

$$G = A \rtimes \langle v \rangle$$

where $A = [G, G]$ is a divisible subgroup, $v^2 = 1$ and $a^v = a^{-1}$ for all $a \in A$. \square

4 Non-primary periodic groups whose non-normal subgroups are either contranormal or core-free

Lemma 4.1 *Let G be a periodic soluble group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains proper contranormal and non-trivial core-free subgroups. If G is a monolithic group and $\text{Mon}(G) = M = [G, G]$, then $G = M \rtimes S$, M is an elementary abelian p -subgroup, p is a prime, S is a locally cyclic p' -subgroup, $C_G(M) = M$, and every complement to M in G is conjugate with S . In particular, if M is finite, then G is finite and $G = M \rtimes S$ where S is a cyclic Sylow p' -subgroup of G .*

PROOF — Let M be the monolith of G . Then M is a minimal normal subgroup of G . Since G is periodic and soluble, M is an elementary abelian p -subgroup for some prime p . If we suppose that the center of G contains M , then equality $M = [G, G]$ implies that G is nilpotent. But a nilpotent group does not contain proper contranormal subgroups. This contradiction shows that $M \cap \zeta(G) = \langle 1 \rangle$.

Suppose that $M \neq C_G(M)$. The facts that $C_G(M)$ is nilpotent and that G is monolithic imply that $C_G(M)$ is a p -subgroup. Since

$$C_G(M) \neq G,$$

Lemma 2.1 proves that every subgroup of $C_G(M)/M$ is G -invariant. Let a be an element of $C_G(M) \setminus M$, then $\langle M, a \rangle$ is normal in G . This subgroup is also abelian and the factor-group $G/C_G(\langle M, a \rangle)$ is abelian. Since G/M is abelian, factor $\langle M, a \rangle/M$ is G -central. On the other hand, M is not central and G -chief. Then $\langle M, a \rangle$ contains a G -invariant subgroup A such that

$$\langle M, a \rangle = M \times A$$

(see, for example, [4, Lemma 1.6.3]). This contradicts the fact that M is a monolith of G . This contradiction proves the equality

$$M = C_G(M).$$

In particular, it follows that if M is finite, then G is itself finite. Since G/M is periodic and abelian, equality $M = C_G(M)$ implies that G/M is a locally cyclic p' -subgroup (see, for example, [12, Theorem 2.3]). Moreover, $G = M \rtimes S$, and every another complement to M is conjugate with S (see, for example, [12, Theorem 14.18]). \square

Lemma 4.2 *Let G be a periodic soluble non-primary group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains proper contranormal and non-trivial core-free subgroups. If G is a monolithic group, $\text{Mon}(G) \neq [G, G]$ and $[G, G]$ is a p -subgroup for some prime p , then G is a group of one of the following types:*

- (i) $G = D \rtimes \langle g \rangle$ where D is a normal cyclic p -subgroup, $|g| = q$ is a prime, $q < p$, $C_G(D) = D$;
- (ii) $G = D \rtimes \langle g \rangle$ where D is a normal Prüfer p -subgroup, $|g| = q$ is a prime, $q < p$, $C_G(D) = D$;
- (iii) $G = D \rtimes \langle g \rangle$ where D is an extraspecial p -subgroup, p is a prime, $|g| = q$ is a prime, $q < p$, $q \neq 2$, moreover, $M = [D, D] = \zeta(D)$ is a monolith of G , and every subgroup of D/M is G -invariant.

PROOF — Let M be the monolith of G . Then M is a minimal normal subgroup of G . Since G is periodic and soluble, M is an elementary abelian p -subgroup for some prime p . Lemma 3.2 implies

that $G/[G, G]$ is a cyclic q -group for some prime q and every subgroup of $[G, G]/M$ is G -invariant. It follows that G/M is a hypercyclic group. Let s be the least prime from the set $\Pi(G/M)$. Then

$$G/M = P/M \rtimes S/M$$

where S/M is a Sylow s -subgroup of G/M , P/M is a Sylow s' -subgroup of G/M . From our conditions we obtain that $s = q$. Since G is not a q -group, $p \neq q$. The inclusion $P \leq [G, G]$ together with the fact that $[G, G]$ is a p -subgroup imply that $P = [G, G]$. It follows that S/M is a cyclic q -subgroup, so that there is an element g such that $S/M = \langle gM \rangle$.

Without loss of generality we may assume that g is a q -element. Moreover, $p < q$. Using again Lemma 3.2, we obtain that every subgroup of $\langle g^q, [G, G] \rangle/M$ is G -invariant. It follows that $[G, G]/M$ is abelian.

Since $M \neq [G, G]$, we can choose an element $d \in [G, G] \setminus M$. Then $\langle d, M \rangle$ is normal in G . This subgroup is nilpotent (see, for example, [2]). It follows that the intersection $M \cap \zeta(\langle d, M \rangle)$ is not trivial. Since this intersection is a normal subgroup of G , the fact that M is a minimal normal subgroup of G implies that

$$M = M \cap \zeta(\langle d, M \rangle).$$

It follows that $[d, M] = \langle 1 \rangle$. Since it is true for an arbitrary element $d \in [G, G]$, $M \leq \zeta([G, G])$. It follows that $[G, G] \leq C_G(M)$, in particular, $G/C_G(M)$ is a finite q -group. Then the fact that M is a minimal normal subgroup of G implies that M is finite.

Suppose that $h = g^q \neq 1$. As we have noted above, $K = \langle M, h \rangle$ is normal in G . The fact that M is the monolith of G implies that

$$C_K(M) = M.$$

We have $K = M \rtimes \langle h \rangle$, and every complement to M is conjugate with $\langle h \rangle$. Since K is normal in G , then using Proposition 8.2.12 of the book [4], we obtain that $G = M \rtimes V$ and every complement to M in G is conjugate with V . The isomorphism $V \simeq G/M$ shows that V contains a normal non-trivial Sylow p -subgroup

$$D_1 \simeq [G, G]/M.$$

In particular, D_1 is abelian. The inclusion $M \leq \zeta([G, G])$ implies that MD_1 is abelian, in particular, D_1 is M -invariant. Since D_1 is V -invariant, equality $G = MV$ implies that D_1 is G -invariant, and we obtain a contradiction with the fact that G is monolithic. This contradiction proves that g has prime order q .

As we have seen above, $\langle d, M \rangle$ is abelian and normal in G . Since G/M is hypercyclic, $G/C_G(\langle d, M \rangle)$ is hypercyclic too. Suppose that M is not cyclic. Then $\langle d, M \rangle$ contains a G -invariant subgroup E such that

$$\langle d, M \rangle = M \times E$$

(see [23]), and we obtain a contradiction with the fact that G is monolithic. This contradiction shows that M must be cyclic.

Let $M = \langle a \rangle$. If x, y are arbitrary elements of $D = [G, G]$, then

$$y^x = ya^m$$

where $1 \leq m \leq p$. It follows that

$$x^{-q}yx^q = y.$$

This means that $D^q \leq \zeta(D)$. Assume that

$$\Omega_1(\zeta(D)) \neq M.$$

Since $D = C_G(\Omega_1(\zeta(D)))$ and G/D is a cyclic p -group, $\Omega_1(\zeta(D))$ contains a non-trivial G -invariant subgroup E_1 such that

$$\Omega_1(\zeta(D)) = M \times E_1$$

(see, for example, [13, Corollary 5.14]), and we obtain a contradiction with the fact that G is monolithic. This contradiction shows that

$$\Omega_1(\zeta(D)) = M.$$

It follows that $\zeta(D)$ is either cyclic or quasicyclic. In particular, if D is abelian, then D is cyclic or quasicyclic. Then

$$G = D \rtimes \langle g \rangle$$

where $|g| = q$ is a prime, so that G is a group of type (i) or (ii).

Suppose that D is non-abelian. Then D/M is an extension of cyclic

or quasicyclic subgroup $\zeta(D)/M$ by an elementary abelian p -subgroup. Suppose that $\zeta(D) \neq M$. Then $D/M = \zeta(D)/M \times Y/M$ for some subgroup Y . If we suppose that Y is abelian, then the equality $D = \zeta(D)Y$ implies that D is abelian. Thus Y is not abelian.

The inequality $\zeta(D) \neq M$ shows that there exists an element $c \in \zeta(D)$ such that $c^p = a$. By such a choice, $cM \in \Omega_1(D/M)$. Lemma 2.1 implies that every subgroup of $\Omega_1(D/M)$ is G -invariant. It follows that there exists an integer r such that $1 \leq r < q$ and $(yM)^q = (yM)^r$ for all $yM \in \Omega_1(D/M)$ (see, for example, [21, Theorem 1.5.6]). If we suppose that $r = 1$, then it is not hard to see, that $gM \in C_{G/M}(D/M)$. Then G/M is abelian, and we obtain a contradiction. This contradiction shows that $r \neq 1$. Since Y is not abelian, there are elements $x, y \in Y$ such that $a = [x, y]$. We have $x^q = x^r u_1, y^q = y^r u_2$ where $u_1, u_2 \in M$, and hence

$$a^q = [x, y]^q = [x^q, y^q] = [x^r u_1, y^r u_2] = [x^r, y^r] = a^k$$

where $k = r^2$. On the other hand, $c^q = c^r u_3$ where $u_3 \in M$, and hence

$$a^q = (c^p)^q = (c^q)^p = (c^r u_3)^p = c^{pr} u_3^p = a^r.$$

Thus, $r \equiv r^2 \pmod{p}$. This contradiction proves that $\zeta(D) \neq M$, that is $D = Y$. Since D is not abelian, $[D, D] = M$, so that D is an extraspecial p -subgroup. If we suppose now that $q = 2$, then the fact that every subgroup of D/M is G -invariant implies that

$$D/M \leq \zeta(G/M).$$

In this case, G/M is abelian, and we obtain a contradiction with $M \neq [G, G]$. This final contradiction proves that G is a group of type (iii). \square

Lemma 4.3 *Let G be a periodic soluble group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains proper contranormal and non-trivial core-free subgroups. If G is a monolithic group with $\text{Mon}(G) \neq [G, G]$ and $[G, G]$ is not a primary group, then G is a group of one of the following types:*

- (i) $G = M \rtimes K$ where M is a finite elementary abelian p -subgroup, p is an odd prime, K is a quaternion group of order 8, M is a minimal normal subgroup of G , $C_G(M) = M$;
- (ii) $G = M \rtimes B$ where M is a minimal normal elementary abelian p -

subgroup, p is an odd prime, $B = K \rtimes \langle u \rangle$ where K is a normal Prüfer 2-subgroup, $u^2 = 1$, $au = a^{-1}$ for each $a \in K$;

- (iii) $G = M \rtimes B$ where M is a minimal normal elementary abelian p -subgroup, p is an odd prime, B is an infinite generalized quaternion group;
- (iv) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = D_1 \rtimes \langle g \rangle$ where D_1 is a locally cyclic p' -subgroup, $|g| = p$, every subgroup of D_1 is $\langle g \rangle$ -invariant, $C_V(D_1) = D_1$;
- (v) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = D_1 \rtimes \langle g \rangle$ where D_1 is a locally cyclic subgroup, g is a q -element, q is an odd prime, $p, q \notin \Pi(D_1)$, $C_{\langle g \rangle}(D_1) = \langle g^q \rangle$, and every subgroup of D_1 is $\langle g \rangle$ -invariant;
- (vi) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = D_1 \rtimes \langle g \rangle$ where D_1 is a locally cyclic subgroup, g is a 2-element, $2, p \notin \Pi(D_1)$, $C_{\langle g \rangle}(D_1) = \langle g^2 \rangle$, $x^g = x^{-1}$ for each $x \in D_1$;
- (vii) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = (S \times K) \rtimes \langle g \rangle$ where $S \times K$ is a locally cyclic subgroup, moreover, K is a Prüfer 2-subgroup, S is a $2'$ -subgroup, $|g| = 2$, $x^g = x^{-1}$ for each $x \in S \times K$;
- (viii) $G = M \rtimes V$ where M is a minimal normal elementary abelian p -subgroup, p is a prime, $V = S \rtimes (K \langle g \rangle)$ where S is a locally cyclic $2'$ -subgroup, K is a Prüfer 2-subgroup, $|g| = 4$, $g^2 \in \Omega_1(K)$, $\langle K, g \rangle$ is an infinite generalized quaternion group, $C_V(S) = S \times K$, $x^g = x^{-1}$ for each $x \in S \times K$.

PROOF — Let M be the monolith of G . Then M is a minimal normal subgroup of G . Since G is periodic and soluble, M is an elementary abelian p -subgroup for some prime p . Lemma 3.2 implies that $G/[G, G]$ is a cyclic q -group for some prime q and every subgroup of $[G, G]/M$ is G -invariant. In particular, $[G, G]/M$ is nilpotent.

Suppose that G/M is a q -group. Since $[G, G]$ is a not p -subgroup, $p \neq q$. The fact that G is monolithic implies that

$$C_G(M) = M.$$

Using Lemma 2.1 and Lemma 2.4 we obtain that either G/M is a Dedekind group or G/M is a group that was described in Lemma 2.4.

Since $M \neq [G, G]$, G/M is non-abelian. Suppose that G/M is a Dedekind group. Then

$$G/M = K_1/M \times K_2/M$$

where K_1/M is a quaternion group of order 8 and K_2/M is an elementary abelian 2-subgroup. We remark that

$$\zeta(G/M) = \zeta(K_1/M) \times K_2/M$$

is an elementary abelian subgroup. Since M is a monolith of G , $\zeta(G/M)$ must be locally cyclic [12, Theorem 3.1]. Since G/M is not abelian, G/M is a quaternion group. Since M is a minimal normal subgroup of G , M is finite, so that

$$G = M \rtimes K$$

where K is a quaternion group of order 8, so that G is a group of type (i).

Suppose now that

$$G/M = A/M \langle gM \rangle$$

where A/M is a divisible abelian 2-subgroup and

$$g^2M \in C_{G/M}(A/M).$$

Every subgroup of $\langle A, g^2 \rangle/M$ is G -invariant. In particular, every subgroup of $\Omega_1(\langle A, g^2 \rangle/M)$ is G -invariant. Since M is a minimal normal abelian p -subgroup of G , $\Omega_1(\langle A, g^2 \rangle/M)$ contains a subgroup W/M such that

$$\Omega_1(\langle A, g^2 \rangle/M)/(W/M)$$

is locally cyclic and $\text{Core}_{G/M}(W/M) = \langle 1 \rangle$ [12, Lemma 3.8]. The fact that every subgroup of $\Omega_1(\langle A, g^2 \rangle/M)$ is W -invariant shows that equality

$$\text{Core}_{G/M}(W/M) = \langle 1 \rangle$$

is possible only if $W/M = \langle 1 \rangle$. Thus $\Omega_1(\langle A, g^2 \rangle/M)$ is locally cyclic, and it follows that $\langle A, g^2 \rangle/M$ is locally cyclic. Then $g^2M \in A/M$. Finally, $G = M \rtimes B$ where $B \simeq A/M$ (see, for example, [3, Theorem 2.4.5]). Using Lemma 2.4 we obtain that either $B = K \rtimes \langle u \rangle$ where K is a normal Prüfer 2-subgroup, $u^2 = 1$, $a^u = a^{-1}$ for each $a \in K$, or B is an infinite generalized quaternion group, so that G

is a group of type (ii) or (iii).

Suppose now that $\Pi(G/M)$ contains at least two primes. Let s be the smallest prime from the set $\Pi(G/M)$. Since G/M is hypercyclic,

$$G/M = S/M \rtimes Q/M$$

where Q/M is a Sylow s -subgroup of G/M and S/M is a Sylow s' -subgroup of G/M . The fact that $G/[G, G]$ is a q -group implies that $s = q$. Without loss of generality we may suppose that $gM \in Q/M$.

Using Lemma 2.1 we obtain that every subgroup of

$$D/M = G^q[G, G]/M$$

is G -invariant. Thus, D/M is nilpotent. Then $D/M = P/M \times S/M$ where P/M is a Sylow p -subgroup of D/M , S/M is a Sylow p' -subgroup of D/M .

Let xM be an arbitrary element of P/M . By above noted a subgroup $\langle x, M \rangle$ is normal in G . This subgroup is nilpotent (see, for example, [2]). It follows that

$$M \cap \zeta(\langle x, M \rangle) \neq \langle 1 \rangle.$$

Since this intersection is a normal subgroup of G , the fact that M is a minimal normal subgroup of G implies that $M = M \cap \zeta(\langle x, M \rangle)$. It follows that $[x, M] = \langle 1 \rangle$. Since it is true for arbitrary $x \in P$, $M \leq \zeta(P)$.

By our assumption, S/M is not trivial. Then the fact that M is the monolith of G , implies that $C_S(M) = M$. Choose in S an element z such that

$$M \neq zM \in \zeta([G, G]/M).$$

Without loss of generality we may assume that z is a p' -element. Then $M = C_M(\langle z \rangle) \times [\langle z \rangle, M]$ (see, for example, [13, Proposition 5.19]). Since $\langle zM \rangle$ is normal in G/M , $C_M(\langle z \rangle)$ is G -invariant. If we suppose that $C_M(\langle z \rangle) \neq \langle 1 \rangle$, then $C_M(\langle z \rangle) = M$. But in this case, $z \in C_S(M)$ and we obtain a contradiction with $C_S(M) = M$. Thus

$$C_M(\langle z \rangle) = \langle 1 \rangle,$$

and hence $M = [\langle z \rangle, M]$. We note that $[\langle z \rangle, M] = [z, M]$. Then

$$S = M \rtimes U$$

and every complement to M in S is conjugate to U (see, for example, [4, Theorem 8.2.7]). Since S is normal in G , then using Proposition 8.2.12 of [4], we obtain that $G = M \rtimes V$, and every complement to M in G is conjugate to V . The isomorphism $V \simeq G/M$ shows that V contains a normal subgroup D_1 such that $|V/D_1| = q$ and every subgroup of D_1 is V -invariant.

Suppose now that P/M is not trivial. Then Sylow p -subgroup P_1 of D_1 is non-trivial. This subgroup is normal in V . By above proved, we have $M \leq \zeta(P)$. It follows that $[P_1, M] = \langle 1 \rangle$, in particular, P_1 is M -invariant. Then P_1 is G -invariant. The inclusion $P_1 \leq V$ implies that $P_1 \cap M = \langle 1 \rangle$, and we obtain a contradiction. This contradiction shows that D/M is a p' -group.

Thus we have $G = M \rtimes V$. Let D_2 be the socle of D_1 . Since M is a minimal normal abelian p -subgroup of G , D_2 contains a subgroup W such that D_2/W is locally cyclic and $\text{Core}_V(W) = \langle 1 \rangle$ [12, Lemma 3.8]. The fact that every subgroup of D_1 is V -invariant shows that equality

$$\text{Core}_V(W) = \langle 1 \rangle$$

is possible only if $W = \langle 1 \rangle$. Thus, D_2 is locally cyclic, and it follows that D_1 is locally cyclic.

If $p = q$, then $V = D_1 \rtimes \langle g \rangle$ where $|g| = p$, D_1 is abelian p' -subgroup and every subgroup of D_1 is $\langle g \rangle$ -invariant. Hence G is a group of type (iv).

Suppose that $p \neq q$. By Lemma 2.1, every subgroup of G/M is normal or contranormal, so that $G/M \simeq V$ satisfies the conditions of Lemma 2.5. If $q \neq 2$, then Lemma 2.5 implies that

$$V = D_1 \rtimes \langle g \rangle$$

where g is a q -element, D_1 is abelian subgroup, $p, q \notin \Pi(D_1)$,

$$C_{\langle g \rangle}(D_1) = \langle g^q \rangle,$$

and every subgroup of D_1 is $\langle g \rangle$ -invariant. Hence G is a group of type (v).

Finally suppose that $q = 2$. If a Sylow 2-subgroup of G/M is finite, then Lemma 2.5 shows that G is a group of type (v), moreover, $x^g = x^{-1}$ for each $x \in D_1$, so that G is a group of type (vi).

Suppose that a Sylow 2-subgroup of G/M is infinite. Taking into account Lemma 2.5 we conclude that $V = S \rtimes P$ where P is an infi-

nite Sylow 2-subgroup of G , $P = K\langle g \rangle$ is a group satisfying all conditions of Lemma 2.4, S is an abelian Sylow $2'$ -subgroup of G , $x^g = x^{-1}$ for each $x \in S$. From above proved we derive that $S \times K$ is a locally cyclic subgroup. By Lemma 2.4, we can have the following two types of V : either $V = (S \times K) \rtimes \langle g \rangle$ where $S \times K$ is a locally cyclic subgroup, K is a Prüfer 2-subgroup, S is a $2'$ -subgroup, $|g|=2$, $x^g = x^{-1}$ for each $x \in S \times K$, or $V = S \rtimes (K\langle g \rangle)$ where S is a locally cyclic $2'$ -subgroup, K is a Prüfer 2-subgroup, $|g| = 4$, $g^2 \in \Omega_1(K)$, $\langle K, g \rangle$ is an infinite generalized quaternion group, $C_V(S) = S \times K$, $x^g = x^{-1}$ for each $x \in S \times K$. Thus G is a group of types (vii) or (viii). \square

PROOF OF THEOREM D — This theorem is a direct consequence of Lemmas 3.10, 4.1, 4.2, and 4.3.

The next natural step is the study of the case when G is not a monolithic group.

Lemma 4.4 *Let G be a periodic soluble non-primary group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains proper contranormal and non-trivial core-free subgroups. If G is not a monolithic group but $[G, G]$ is G -monolithic, then G is a group of one of the following types:*

- (i) $G = M \rtimes (\langle c \rangle \times \langle g \rangle)$ where M is a normal subgroup of prime order $p \neq 2$, $|c| = s$ is a prime, $|g| = q$ is a prime, $q \neq s$, q divides $p - 1$, and we have $C_G(M) = M \times \langle c \rangle$;
- (ii) $G = M \rtimes (\langle c \rangle \times \langle g \rangle)$ where M is a normal subgroup of prime order $p \neq 2$, $|c| = |g| = q$ is a prime, q divides $p - 1$, and we have $C_G(M) = M \times \langle c \rangle$;
- (iii) $G = M \rtimes \langle g \rangle$ where M is a normal subgroup of prime order $p \neq 2$, g is an element of order q , q is a prime, q divides $p - 1$, and we have $C_G(M) = M$;
- (iv) $G = [G, G] \rtimes \langle g \rangle$ where $[G, G]$ is a normal cyclic p -subgroup, where p is an odd prime, $\langle g \rangle$ is a cyclic q -subgroup, q is a prime, and we have $C_G([G, G]) = [G, G]$;
- (v) $G = [G, G] \rtimes \langle g \rangle$ where $[G, G]$ is a normal Prüfer p -subgroup, where p is an odd prime, $\langle g \rangle$ is a cyclic q -subgroup, q is a prime, and we have $C_G([G, G]) = [G, G]$.

PROOF — Let M be the G -monolith of $[G, G]$. Then M is a minimal normal subgroup of G . Since G is periodic and soluble, M is an elementary abelian p -subgroup for some prime p . Since G is not monolithic, G contains a non-trivial normal subgroup H such that $H \cap M = \langle 1 \rangle$. It follows that

$$H \cap [G, G] = \langle 1 \rangle.$$

The last equality implies that $H \leq \zeta(G)$. Lemma 3.1 implies that every subgroup of $[G, G]$ is G -invariant. In particular, $[G, G]$ is a Dedekind group. Being Dedekind, $[G, G]$ is abelian or nilpotent of class nilpotency 2. The fact that $[G, G]$ is G -monolithic implies that $[G, G]$ must be a p -subgroup for some prime p . Since every subgroup of $[G, G]$ is G -invariant, $M = \langle a \rangle$ is a cyclic subgroup of order p .

Suppose that $M = [G, G]$. If we assume that $p = 2$, then $M \leq \zeta(G)$, so that G is nilpotent. However, a nilpotent group does not contain proper contranormal subgroups. This contradiction shows that $p \neq 2$. Let C be a proper contranormal subgroup of G . Since G/M is abelian, CM/M is a normal subgroup of G/M . On the other hand, CM/M is contranormal in G/M , which implies that

$$CM/M = G/M \quad \text{or} \quad CM = G.$$

Since M has a prime order and C is a proper subgroup of G , it follows that $C \cap M = \langle 1 \rangle$. Clearly, $C_1 = C_C(M)$ is normal in G , and equality

$$C_1 \cap [G, G] = \langle 1 \rangle$$

implies that $C_1 \leq \zeta(G)$. Assume that C_1 is non-trivial and chose an element $c \in C_1$ such that $|c| = s$ is a prime. By Lemma 2.1, every subgroup of $G/\langle c \rangle$ is normal or contranormal.

Using Lemma 2.2 we obtain that $(G/\langle c \rangle)/[G/\langle c \rangle, G/\langle c \rangle]$ is a cyclic q -group for some prime q . Since G is not abelian, $q \neq p$. Choose an element g such that $G = \langle g \rangle(M\langle c \rangle)$. Without loss of generality we can suppose that g is a q -element. Lemma 2.2 implies that $\langle g^q \rangle \langle c \rangle$ is normal in G . Equality $(\langle g^q \rangle \langle c \rangle) \cap M = \langle 1 \rangle$ together with Lemma 3.1 imply that every subgroup of $\langle g^q \rangle \langle c \rangle$ is G -invariant.

Let $s \neq q$ and suppose that $g^q \neq 1$. Then

$$(G/\langle g^q \rangle)/[G/\langle g^q \rangle, G/\langle g^q \rangle] = (G/\langle g^q \rangle)/(M\langle g^q \rangle/\langle g^q \rangle)$$

is not a primary group, so we obtain a contradiction with Lemmas 2.1 and 2.2. This contradiction shows that $g^q = 1$, and we obtain that G is a group of type (i).

Assume now that $s = q$ and $\langle g, c \rangle$ is not cyclic. Using the above arguments we obtain again that $g^q = 1$, thus G is a group of type (ii).

Suppose now that $\langle g, c \rangle$ is cyclic. Then $\langle g, c \rangle = \langle g \rangle$. As above we can prove that $\langle g^q \rangle$ is normal in G . Since G must contain proper non-trivial core-free subgroup, we obtain that $g^q = 1$, so that G is a group of type (iii).

Suppose now that $M \neq [G, G]$. Lemma 3.2 implies that $G/[G, G]$ is a cyclic q -group for some prime q . Since G is not a primary group, we have $p \neq q$. The fact that every subgroup of $[G, G]$ is G -invariant implies that G is hypercyclic. It follows that $q < p$. In particular, $p \neq 2$, so that $[G, G]$ is abelian. Using again the fact that every subgroup of $[G, G]$ is G -invariant, we obtain that $M = \Omega_1([G, G])$. In turn out, it follows that $[G, G]$ is a cyclic p -subgroup or $[G, G]$ is a Prüfer p -subgroup. Since $[G, G]$ is a normal Sylow p -subgroup of G , then

$$G = [G, G] \rtimes K$$

for some subgroup K and every complement to $[G, G]$ in G is conjugate to K (see, for example, [3, Theorem 2.4.5]). The isomorphism

$$K \simeq G/[G, G]$$

implies that K is a cyclic q -subgroup. Let $K = \langle g \rangle$. Moreover, equality

$$H \cap [G, G] = \langle 1 \rangle$$

implies that H is a q -subgroup. It follows that $\Omega_1(K)$ is a normal q -subgroup of G . Using Lemma 2.1 we obtain that every subgroup of $G^q[G, G]\Omega_1(K)/\Omega_1(K)$ is G -invariant. In particular, the subgroup $\langle g^q\Omega_1(K) \rangle$ is normal in $G/\Omega_1(K)$. Then $\langle g^q \rangle$ is normal in G . Since G must contain a proper non-trivial core-free subgroup, we conclude that $g^q = 1$. This means that $C_G([G, G]) = [G, G]$, so that G is a group of types (iv) or (v). \square

Lemma 4.5 *Let G be a soluble periodic non-primary group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains proper contranormal and non-trivial core-free subgroups, and $[G, G]$ is a q -subgroup for some prime q such that $\text{sr}_q(H) > 2$. If the derived subgroup $[G, G]$ is not G -monolithic, then $G = [G, G] \rtimes \langle g \rangle$ where g is an ele-*

ment of order p , p is a prime, $p < q$, $[G, G]$ is an abelian Sylow q -subgroup of G , $C_G([G, G]) = [G, G]$, and every subgroup of $[G, G]$ is G -invariant.

PROOF — Since $[G, G]$ is not G -monolithic, it contains a proper non-trivial abelian G -invariant subgroup. Lemma 3.2 shows that $G/[G, G]$ is a cyclic p -group for some prime p . Since G is not primary, $p \neq q$. Using Lemma 3.8, we obtain that every subgroup of $[G, G]$ is G -invariant. Then G is a hypercyclic group. Let r be the smallest prime from the set $\Pi(G)$. Then

$$G = S \rtimes P$$

where P is a Sylow r -subgroup of G and S is a Sylow r' -subgroup of G . Inclusion $[G, G] \leq S[P, P]$ together with the fact that $G/[G, G]$ is a p -group imply that $r = p$. It follows that $p < q$. In particular, $p \neq 2$. Then $[G, G]$ is abelian. Moreover, our conditions imply that $[G, G]$ is a Sylow q -subgroup of G . Then

$$G = [G, G] \rtimes K$$

for some subgroup K and every complement to $[G, G]$ in G is conjugate to K (see, for example, [3, Theorem 2.4.5]). The isomorphism

$$K \simeq G/[G, G]$$

implies that K is a cyclic p -subgroup. Let $K = \langle g \rangle$.

The fact that $[G, G]$ is not G -monolithic implies that $[G, G]$ contains a proper non-trivial G -invariant subgroup. Thus we can apply Lemma 3.9. By this lemma, every subgroup of $G^p/[G, G]$ is G -invariant. The equality

$$G = [G, G]\langle g \rangle$$

shows that $\langle g^p \rangle \leq \zeta(G)$. Since G must contain a proper non-trivial core-free subgroup, $g^p = 1$. Thus $C_G([G, G]) = [G, G]$. \square

Lemma 4.6 *Let G be a soluble periodic non-primary group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains proper contranormal and non-trivial core-free subgroups, and $[G, G]$ is a q -subgroup for some prime q such that $\text{sr}_q([G, G]) \leq 2$. If the derived subgroup $[G, G]$ is not G -monolithic, then G is a group of one of the following types:*

- (i) $G = (\langle a_1 \rangle \times \langle a_2 \rangle) \rtimes \langle g \rangle$ where $|a_1| = |a_2| = q$, $|g| = p$, p is a prime with $p < q$, $C_G([G, G]) = [G, G]$, $a_1^g = a_1^m$, $a_2^g = a_2^s$, $1 \leq m, s < q$ and $m \neq s$;

- (ii) $G = [G, G] \rtimes \langle g \rangle$ where g is an element of order p , p is a prime, $p < q$, the commutator subgroup $[G, G]$ is an abelian Sylow q -subgroup of G which is equal to $C_G([G, G])$ and every subgroup of $[G, G]$ is G -invariant.

PROOF — As in the proof of Lemma 4.5 we can show that $G/[G, G]$ is a cyclic p -group for some prime p . Since $[G, G]$ is not G -monolithic, it has a family $\{V_\lambda \mid \lambda \in \Lambda\}$ of proper non-trivial G -invariant subgroups such that $\bigcap_{\lambda \in \Lambda} V_\lambda = \langle 1 \rangle$. By Lemma 2.1, every subgroup of $[G, G]/V_\lambda$ is G -invariant. Then $[G, G]/V_\lambda$ is hypercyclic. It follows that $p < q$, in particular, $q \neq 2$. Being a Dedekind group, $[G, G]/V_\lambda$ is abelian. The equality $\bigcap_{\lambda \in \Lambda} V_\lambda = \langle 1 \rangle$ implies that $[G, G]$ is abelian. The fact that $[G, G]$ is not G -monolithic implies that $[G, G]$ contains a proper non-trivial G -invariant subgroup. Thus, we can apply Lemma 3.9. By this lemma, either G is a group of type (i), or every subgroup of $G^p[G, G]$ is G -invariant. Repeating the same arguments that we used in the proof of Lemma 4.5 we obtain that G is a group of type (ii). \square

Lemma 4.7 *Let G be a group whose non-normal subgroups are either contranormal or core-free, H be a proper non-trivial normal periodic subgroup of G . Suppose also that $\Pi(H)$ contains at least two different primes. If H is not G -monolithic, then every subgroup of H is G -invariant.*

PROOF — Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the family of all non-trivial G -invariant subgroups of H . Since H is not G -monolithic, $\bigcap_{\lambda \in \Lambda} V_\lambda = \langle 1 \rangle$. This equality together with Remak’s theorem imply that H is embedded in the cartesian product $\text{Cr}_{\lambda \in \Lambda} H/V_\lambda$. Lemma 2.1 shows that the factors H/V_λ are Dedekind groups for each index $\lambda \in \Lambda$. It follows that all these factors are nilpotent of nilpotency class at most 2. Therefore, H is also nilpotent of nilpotency class at most 2. Then

$$H = \text{Dr}_{p \in \Pi(H)} H_p$$

where H_p is a Sylow p -subgroup of H , $p \in \Pi(H)$. Since a set $\Pi(H)$ contains at least two different primes, Lemma 3.1 shows that every subgroup of H_p is G -invariant for each $p \in \Pi(H)$. Then every subgroup of H is G -invariant. \square

Lemma 4.8 *Let G be a soluble periodic group whose non-normal subgroups are either contranormal or core-free. Suppose that G contains proper contranormal and non-trivial core-free subgroups, and $\Pi([G, G])$ contains*

at least two different primes. If the derived subgroup $[G, G]$ is not G -monolithic, then G is a group of one of the following types:

- (i) $G = S \rtimes \langle g \rangle$ where g is an element of order 2, S is an abelian $2'$ -subgroup, $C_G(S) = S$, and $x^g = x^{-1}$ for every $x \in S$;
- (ii) $G = S \rtimes P$ where P is a Sylow 2-subgroup of G and S is an abelian $2'$ -subgroup, $P = P_1 \rtimes \langle g \rangle$ where D is a normal divisible abelian 2-subgroup, $[S, P_1] = \langle 1 \rangle$, $|g| = 2$, and $x^g = x^{-1}$ for every $x \in S \times P_1$;
- (iii) $G = S \rtimes \langle g \rangle$ where $|g| = p$, where p is the least prime of the set $\Pi(G)$, S is an abelian Sylow p' -subgroup of G , $C_G(S) = S$, and every subgroup of S is G -invariant.

PROOF — Lemma 3.2 shows that $G/[G, G]$ is a cyclic p -group for some prime p . Since $[G, G]$ is not G -monolithic, $G^p/[G, G]$ is not G -monolithic too. Lemma 4.7 shows that every subgroup of $G^p/[G, G]$ is G -invariant. In particular, $G^p/[G, G]$ is a Dedekind group. Being a Dedekind group, it is nilpotent of nilpotency class at most 2.

By our conditions, G contains a proper non-trivial core-free subgroup E . By above proved, every subgroup of $G^p/[G, G]$ is G -invariant. It follows that

$$E \cap G^p/[G, G] = \langle 1 \rangle.$$

Thus $|E| = p$ and $E = \langle g \rangle$ where $|g| = p$. Moreover, $G = [G, G] \rtimes \langle g \rangle$.

Since every subgroup of $[G, G]$ is G -invariant and $G/[G, G]$ is cyclic, the group G is hypercyclic. Let q be the smallest prime from the set $\Pi(G)$. Then $G = S \rtimes P$ where P is a Sylow q -subgroup of G and S is a Sylow q' -subgroup of G . The inclusion $[G, G] \leq S[P, P]$ together with the fact that $G/[G, G]$ is a p -group imply that $q = p$. Thus, p is a least prime number of a set $\Pi(G)$.

Consider the case when $p = 2$. Since $[G, G]$ is a Dedekind group, its Sylow $2'$ -subgroup S is abelian. If we suppose that $[S, g] \neq S$, then

$$G/([S, g] \times (P \cap [G, G]))$$

is abelian, and we obtain a contradiction. This contradiction implies the equality $S = [S, g]$. Since $S = [S, g] \times C_S(g)$ (see, for example, Proposition 5.19 of [13]), we obtain that $C_S(g) = \langle 1 \rangle$. Using the fact that every subgroup of S is G -invariant, we obtain that $x^g = x^{-1}$ for each $x \in S$. In particular, G is a group of type (i).

Suppose first that the derived subgroup $[G, G]$ is non-abelian. Then its Sylow 2-subgroup P_1 is a direct product of a quaternion group of order 8 and an elementary abelian 2-group. Now we can obtain that G/S is nilpotent. But a nilpotent group, having a cyclic factor-group by the derived subgroup, is itself cyclic. It follows that $P = \langle g \rangle$. Thus G is a group of type (i).

Suppose now that a derived subgroup $[G, G]$ is abelian and its Sylow 2-subgroup P_1 is non-trivial. By Lemma 2.1, every subgroup of 2-group G/S is either normal or contranormal. Lemma 2.4 implies that $[G, G]/S$ is divisible, so that P_1 is divisible. Repeating the arguments of the proof of Lemma 2.4 we obtain that $x^g = x^{-1}$ for each $x \in P_1$, so that G is a group of type (ii).

Consider now the case when p is odd. Using the above arguments we obtain that $P = \langle g \rangle$. Thus G is a group of type (iii). \square

PROOF OF THEOREM E — This theorem directly follows from Lemmas 3.11, 4.4, 4.6, 4.8.

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