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Automorphisms of Zappa–Szép Products

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Abstract

In this paper, we have found the automorphism group of the Zappa–Szép product of two groups. Also, we have computed the automorphism group of the Zappa–Szép product of a cyclic group of order m and a cyclic group of order p^2, where p is a prime.

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1 Introduction

A group G is the *internal Zappa–Szép product* of its two subgroups H and K if $G = HK$ and $H \cap K = \{1\}$. The Zappa–Szép product is a natural generalization of the semidirect product of two groups in which neither of the factor is required to be normal. If G is the internal Zappa–Szép product of H and K, then K appears as a right transversal to H in G. Let $h \in H$ and $k \in K$. Then $kh = \sigma(k,h)\tau(k,h)$, where $\sigma(k,h) \in H$ and $\tau(k,h) \in K$. This determines the maps

$$\sigma : K \times H \to H \quad \text{and} \quad \tau : K \times H \to K.$$

These maps are the *matched pair* of groups. We denote $\sigma(k,h) = k \cdot h$ and $\tau(k,h) = k^h$. These maps satisfy the following conditions (see [3])

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(C1) $1 \cdot h = h$ and $k^1 = k$,

(C2) $k \cdot 1 = 1 = 1^h$,

(C3) $kk' \cdot h = k \cdot (k' \cdot h)$,

(C4) $(kk')^h = k^{k' \cdot h} k'^h$,

(C5) $k \cdot (hh') = (k \cdot h)(k^h \cdot h')$,

(C6) $k^{hh'} = (k^h)^{h'}$,

for all $h, h' \in H$ and $k, k' \in K$.

On the other hand, let H and K be two groups. Let

$$\sigma : K \times H \to H \quad \text{and} \quad \tau : K \times H \to K$$

be two maps defined by $\sigma(k, h) = k \cdot h$ and $\tau(k, h) = k^h$ satisfying the above conditions. Then, the external Zappa–Szép product $G = H \bowtie K$ of H and K is the group defined on the set $H \times K$ with the binary operation defined by

$$(h, k)(h', k') = \big(h(k \cdot h'), k^{h'} k'\big).$$

The internal Zappa–Szép product is isomorphic to the *external Zappa–Szép product* (see [3, Proposition 2.4, p. 4]). We will identify the external Zappa–Szép product with the internal Zappa–Szép product.

The Zappa–Szép product of two groups was introduced by G. Zappa in [13]. J. Szép studied such type of products in a series of papers (few of them are [7],[8],[9],[10]). From the QR decomposition of matrices, one concludes that the general linear group $GL(n, \mathbb{C})$ is a Zappa–Szép product of the unitary group and the group of upper triangular matrices. Z. Arad and E. Fisman in [1] studied the finite simple groups as a Zappa–Szép product of two groups H and K with the order of H and K coprime. In the same paper, they studied the finite simple groups as a Zappa–Szép product of two groups H and K with one of H or K is p-group, where p is a prime. From the main result of [4], one observes that a finite group G is solvable if and only if G is a Zappa–Szép product of a Sylow p-subgroup and a Sylow p-complement.

Note that, if either of the actions $k \cdot h$ or k^h is a group homomorphism, then the Zappa–Szép product reduces to the semidirect product of groups. M.J. Curran [2] and N.C. Hsu [5] studied the automorphisms of the semidirect product of two groups as the 2×2 matrices of maps satisfying some certain conditions. In this paper (with the same terminology as in [2] and [5]), we have found the automorphism group of the Zappa–Szép product of two groups as the 2×2 matrices of maps satisfying some certain conditions. As an application, we have found the automorphism group of the Zappa–Szép product of two cyclic groups in which one is of order p^2 and other is of order m, where p is a prime. Throughout the paper, $\mathbb{Z}_n$ denotes the cyclic group of order n and $U(n)$ denotes the group of units of n. Also, $\mathrm{Aut}(G)$ denotes the group of all automorphisms of a group G. Let U and V be groups. Then $\mathrm{CrossHom}(U, V)$ denotes the group of all crossed homomorphisms from U to V. Also, if U acts on V, then $\mathrm{Stab}_U(V)$ denotes the stabilizer of V in U.

2 Structure of the automorphism group

Let
$$G = H \bowtie K$$
be the Zappa–Szép product of two groups H and K. Let U, V, W be groups, and $\mathrm{Map}(U, V)$ denote the set of all maps between U and V. If $\phi, \psi \in \mathrm{Map}(U, V)$ and $\eta \in \mathrm{Map}(V, W)$, then $\phi + \psi \in \mathrm{Map}(U, V)$ is defined by
$$(\phi + \psi)(u) = \phi(u)\psi(u),$$
$\eta\phi \in \mathrm{Map}(U, W)$ is defined by
$$\eta\phi(u) = \eta\big(\phi(u)\big),$$
$\phi \cdot \psi \in \mathrm{Map}(U, V)$ is defined by
$$(\phi \cdot \psi)(u) = \phi(u) \cdot \psi(u)$$
and $\phi^{\psi} \in \mathrm{Map}(U, V)$ is defined by
$$\phi^{\psi}(u) = \phi(u)^{\psi(u)},$$
for all $u \in U$.

Let $\mathcal{A}$ be the set of all matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with $\alpha \in Map(H,H)$, $\beta \in Map(K,H)$, $\gamma \in Map(H,K)$, and $\delta \in Map(K,K)$ satisfying the following conditions for all $h, h' \in H$ and $k, k' \in K$:

(A1) $\alpha(hh') = \alpha(h)\big(\gamma(h) \cdot \alpha(h')\big)$;

(A2) $\gamma(hh') = \gamma(h)^{\alpha(h')}\gamma(h')$;

(A3) $\beta(kk') = \beta(k)\big(\delta(k) \cdot \beta(k')\big)$;

(A4) $\delta(kk') = \delta(k)^{\beta(k')}\delta(k')$;

(A5) $\beta(k)(\delta(k) \cdot \alpha(h)) = \alpha(k \cdot h)\big(\gamma(k \cdot h) \cdot \beta(k^h)\big)$;

(A6) $\delta(k)^{\alpha(h)}\gamma(h) = \gamma(k \cdot h)^{\beta(k^h)}\delta(k^h)$;

(A7) for any $h'k' \in G$, there exists a unique $h \in H$ and $k \in K$ such that $h' = \alpha(h)\big(\gamma(h) \cdot \beta(k)\big)$ and $k' = \gamma(h)^{\beta(k)}\delta(k)$.

Then, the set $\mathcal{A}$ forms a group with the binary operation defined by

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha'\alpha + \gamma'\alpha \cdot \beta'\gamma & \alpha'\beta + \gamma'\beta \cdot \beta'\delta \\ (\gamma'\alpha)^{\beta'\gamma} + \delta'\gamma & (\gamma'\beta)^{\beta'\delta} + \delta'\delta \end{pmatrix}.$$

The identity element is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the inverse of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}$, which is obtained using the factorization

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \widehat{\beta} \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix},$$

is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} (1-\widehat{\beta}\gamma)^{-1}\alpha^{-1} & -(1-\widehat{\beta}\gamma)^{-1}\widehat{\beta} \\ \delta^{-1}(-\gamma)(1-\widehat{\beta}\gamma)^{-1} & \delta^{-1}(-\gamma)(-(1-\widehat{\beta}\gamma)^{-1}\widehat{\beta}) + \delta^{-1} \end{pmatrix},$$

where $\widehat{\beta} = \alpha^{-1}\beta\delta^{-1}$.

Proposition 2.1 *Let* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}$. *Then* $\alpha(1) = 1 = \beta(1) = \gamma(1) = \delta(1)$.

PROOF — Let $h \in H$ be any element. Then, using (A1),

$$\alpha(h) = \alpha(h1) = \alpha(h)\big(\gamma(h) \cdot \alpha(1)\big)$$

which implies that $\gamma(h) \cdot \alpha(1) = 1 = \gamma(h) \cdot 1$ by (C2). Thus

$$\gamma(h)^{-1} \cdot \big(\gamma(h) \cdot \alpha(1)\big) = \gamma(h)^{-1} \cdot (\gamma(h) \cdot 1).$$

Hence, using (C1), $\alpha(1) = 1$.

Using (A2), $\gamma(h) = \gamma(h1) = \gamma(h)^{\alpha(1)}\gamma(1)$. Using (C1), $\gamma(1) = 1$. Using a similar argument, we get $\beta(1) = 1$ and $\delta(1) = 1$. $\square$

Let us define the kernel of the map $\alpha \in Map(H, H)$ as usual for groups, that is, $\ker(\alpha) = \{h \in H \mid \alpha(h) = 1\}$. Here, we should remember that the map α need not to be a homomorphism. $\ker(\beta), \ker(\gamma)$ and $\ker(\delta)$ are defined in the same sense.

Lemma 2.2 *The following holds:*

(i) $\ker(\alpha) \leqslant H$,	(ii) $\ker(\beta) \leqslant K$,
(iii) $\ker(\gamma) \leqslant H$,	(iv) $\ker(\delta) \leqslant K$,
(v) $\ker(\alpha) \cap \ker(\gamma) = \{1\}$,	(vi) $\ker(\beta) \cap \ker(\delta) = \{1\}$.

PROOF — (i) Let $h, h' \in \ker(\alpha)$. Then using (A1) and (C2),

$$\alpha(hh') = \alpha(h)(\gamma(h) \cdot \alpha(h')) = \gamma(h) \cdot 1 = 1.$$

Also, $1 = \alpha(1) = \alpha(h^{-1}h) = \alpha(h^{-1})(\gamma(h^{-1}) \cdot 1)$. Thus, $\alpha(h^{-1}) = 1$. Hence, hh', $h^{-1} \in \ker(\alpha)$ and so $\ker(\alpha) \leqslant H$.

(ii) One can easily prove it using a similar argument as in (i).

(iii) Let $h, h' \in \ker(\gamma)$. Then using (A2) and (C2),

$$\gamma(hh') = \gamma(h)^{\alpha(h')}\gamma(h') = 1^{\alpha(h')} = 1.$$

Also, $1 = \gamma(1) = \gamma(hh^{-1}) = \gamma(h)^{\alpha(h^{-1})}\gamma(h^{-1})$. Then, $\gamma(h^{-1}) = 1$ and so, hh', $h^{-1} \in \ker(H)$. Hence, $\ker(\gamma) \leqslant H$.

(iv) One can easily prove it using a similar argument as in (iii).

(v) Let $h \in \ker(\alpha) \cap \ker(\gamma)$. Then $\alpha(h) = 1 = \gamma(h)$. Therefore,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} h \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha(h) \\ \gamma(h) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}$ is a bijection, $h = 1$. Hence, (v) holds.

(vi) One can easily prove it using a similar argument as in (v). □

Theorem 2.3 *Let* $G = H \bowtie K$ *be the Zappa–Szép product of two groups* H *and* K, *and* $\mathcal{A}$ *be as above. Then there is an isomorphism of groups between* $\mathrm{Aut}(G)$ *and* $\mathcal{A}$ *given by*

$$\theta \longleftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\theta(h) = \alpha(h)\gamma(h)$ *and* $\theta(k) = \beta(k)\delta(k)$, *for all* $h \in H$ *and* $k \in K$.

Proof — Given $\theta \in \mathrm{Aut}(G)$, we define α, β, γ and δ by means of $\theta(h) = \alpha(h)\gamma(h)$ and $\theta(k) = \beta(k)\delta(k)$, for all $h \in H$ and $k \in K$. Now, for all $h, h' \in H$,

$$\begin{aligned} \theta(hh') = \theta(h)\theta(h') &= \alpha(h)\gamma(h)\alpha(h')\gamma(h') \\ &= \alpha(h)\big(\gamma(h) \cdot \alpha(h')\big)\gamma(h)^{\alpha(h')}\gamma(h'). \end{aligned}$$

Thus, $\alpha(hh')\gamma(hh') = \big(\alpha(h)\big(\gamma(h) \cdot \alpha(h')\big)\big)\big(\gamma(h)^{\alpha(h')}\gamma(h')\big)$. Therefore, by uniqueness of representation, we have (A1) and (A2). Using a similar argument, we get (A3) and (A4).

Now,

$$\begin{aligned} \theta(kh) = \theta\big((k \cdot h)(k^h)\big) = \theta(k \cdot h)\theta(k^h) &= \alpha(k \cdot h)\gamma(k \cdot h)\beta(k^h)\delta(k^h) \\ &= \alpha(k \cdot h)\big(\gamma(k \cdot h) \cdot \beta(k^h)\big)\gamma(k \cdot h)^{\beta(k^h)}\delta(k^h). \end{aligned}$$

Also,

$$\begin{aligned} \theta(kh) = \theta(k)\theta(h) &= \beta(k)\delta(k)\alpha(h)\gamma(h) \\ &= \beta(k)\big(\delta(k) \cdot \alpha(h)\big)\delta(k)^{\alpha(h)}\gamma(h) \end{aligned}$$

Therefore, by the uniqueness of representation,

$$\beta(k)\big(\delta(k) \cdot \alpha(h)\big) = \alpha(k \cdot h)\big(\gamma(k \cdot h) \cdot \beta(k^h)\big)$$

and
$$\delta(k)^{\alpha(h)}\gamma(h) = \gamma(k \cdot h)^{\beta(k^h)}\delta(k^h),$$
which proves (A5) and (A6). Finally, (A7) holds because θ is onto. Thus, to every $\theta \in \mathrm{Aut}(G)$ we can associate the matrix
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}.$$
This defines a map
$$T : \mathrm{Aut}(G) \longrightarrow \mathcal{A}$$
given by
$$\theta \longmapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Now, if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}$ satisfying the conditions (A1)–(A7), then we associate to it, the map
$$\theta : G \longrightarrow G$$
defined by
$$\theta(h) = \alpha(h)\gamma(h) \quad \text{and} \quad \theta(k) = \beta(k)\delta(k),$$
for all $h \in H$ and $k \in K$. Using (A1)–(A6), one can check that θ is an endomorphism of G. Also, by (A7), the map θ is onto. Now, let $hk \in \ker(\theta)$. Then $\theta(hk) = 1$. Therefore,
$$\alpha(h)(\gamma(h) \cdot \beta(k))\gamma(h)^{\beta(k)}\delta(k) = 1$$
and so, by the uniqueness of representation
$$\alpha(h)\big(\gamma(h) \cdot \beta(k)\big) = 1 \quad \text{and} \quad \gamma(h)^{\beta(k)}\delta(k) = 1.$$
Again, by the uniqueness of representation and using (C1), (C2), (C3) and (C6), we get
$$\alpha(h) = 1 = \gamma(h) \quad \text{and} \quad \beta(k) = 1 = \delta(k).$$
Therefore, by Lemma 2.2 (v) and (vi), $h = 1 = k$ and so, $\ker(\theta) = \{1\}$. Thus, θ is one-one and hence, $\theta \in \mathrm{Aut}(G)$. Thus, T is a bijection. Let α, β, γ and δ be the maps associated with θ and α', β', γ' and δ'

be the maps associated with θ'. Now, for all $h \in H$ and $k \in K$, we have

$$\begin{aligned}
\theta'\theta(h) &= \theta'\big(\alpha(h)\gamma(h)\big)\\
&= \alpha'\big(\alpha(h)\big)\gamma'\big(\alpha(h)\big)\beta'\big(\gamma(h)\big)\delta'\big(\gamma(h)\big)\\
&= \alpha'\big(\alpha(h)\big)\big(\gamma'\big(\alpha(h)\big)\cdot\beta'\big(\gamma(h)\big)\big)\gamma'\big(\alpha(h)\big)^{\beta'\left(\gamma(h)\right)}\delta'\big(\gamma(h)\big)\\
&= \big(\alpha'\alpha + (\gamma'\alpha\cdot\beta'\gamma)\big)(h)\big((\gamma'\alpha)^{\beta'\gamma} + \delta'\gamma\big)(h).
\end{aligned}$$

Therefore, if we write hk as $\begin{pmatrix} h \\ k \end{pmatrix}$ and the map θ as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then

$$\theta(hk) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \alpha(h)(\gamma(h)\cdot\beta(k)) \\ \gamma(h)^{\beta(k)}\delta(k) \end{pmatrix},$$

$$\theta'\theta(h) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} \alpha(h) \\ \gamma(h) \end{pmatrix} = \begin{pmatrix} \alpha'\alpha(h)(\gamma'\alpha(h)\cdot\beta'\gamma(h)) \\ (\gamma'\alpha(h))^{\beta'\gamma(h)}\delta'\gamma(h) \end{pmatrix}$$

and

$$\theta'\theta(k) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} \beta(k) \\ \delta(k) \end{pmatrix} = \begin{pmatrix} \alpha'\beta(k) + \gamma'\beta(k)\cdot\beta'\delta(k) \\ (\gamma'\beta(k))^{\beta'\delta(k)} + \delta'\delta(k) \end{pmatrix}.$$

Thus, using a similar argument,

$$\theta'\theta(hk) = \begin{pmatrix} \alpha'\alpha + (\gamma'\alpha\cdot\beta'\gamma) & \alpha'\beta + (\gamma'\beta\cdot\beta'\delta) \\ (\gamma'\alpha)^{\beta'\gamma} + \delta'\gamma & (\gamma'\beta)^{\beta'\delta} + \delta'\delta \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix},$$

for all $h \in H$ and $k \in K$. Therefore,

$$T(\theta'\theta) = \begin{pmatrix} \alpha'\alpha + (\gamma'\alpha\cdot\beta'\gamma) & \alpha'\beta + (\gamma'\beta\cdot\beta'\delta) \\ (\gamma'\alpha)^{\beta'\gamma} + \delta'\gamma & (\gamma'\beta)^{\beta'\delta} + \delta'\delta \end{pmatrix} = T(\theta)T(\theta').$$

Hence, T is an isomorphism of groups. □

From here on, we will identify the automorphisms of G with the matrices in $\mathcal{A}$. Let

$$P = \big\{\alpha \in \mathrm{Aut}(H) \mid k\cdot\alpha(h) = \alpha(k\cdot h) \text{ and } k^{\alpha(h)} = k^h,\\ \forall\, h \in H, k \in K\big\},$$

$$Q = \big\{\beta \in \mathrm{Map}(K,H) \mid \beta(kk') = \beta(k)\big(k\cdot\beta(k')\big), k = k^{\beta(k')},$$
$$\beta(k) = \beta(k^h),\ \forall\, h \in H, k \in K\big\},$$

$$R = \big\{\gamma \in \mathrm{Map}(H,K) \mid \gamma(hh') = \gamma(h)^{h'}\gamma(h'), h' = \gamma(h)\cdot h',$$
$$\gamma(k\cdot h) = \gamma(h),\ \forall\, h \in H, k \in K\big\},$$

$$S = \big\{\delta \in \mathrm{Aut}(K) \mid \delta(k)\cdot h = k\cdot h, \delta(k)^h = \delta(k^h), \forall\, h \in H, k \in K\big\},$$

$$X = \big\{(\alpha,\gamma,\delta) \in \mathrm{Map}(H,H)\times\mathrm{Map}(H,K)\times\mathrm{Aut}(K) \mid \alpha(hh') = \alpha(h)$$
$$(\gamma(h)\cdot\alpha(h')), \gamma(hh') = \gamma(h)^{\alpha(h')}\gamma(h'), \delta(k)\cdot\alpha(h) = \alpha(k\cdot h),$$
$$\delta(k)^{\alpha(h)}\gamma(h) = \gamma(k\cdot h)\delta(k^h), \forall\, h \in H, k \in K\big\},$$

$$Y = \big\{(\alpha,\beta,\delta) \in \mathrm{Aut}(H)\times\mathrm{Map}(K,H)\times\mathrm{Map}(K,K) \mid \beta(kk') = \beta(k)$$
$$(\delta(k)\cdot\beta(k')), \delta(kk') = \delta(k)^{\beta(k')}\delta(k'), \beta(k)(\delta(k)\cdot\alpha(h)) = \alpha(k\cdot h)$$
$$\beta(k^h), \delta(k)^{\alpha(h)} = \delta(k^h), \forall\, h \in H, k \in K\big\},$$

$$Z = \big\{(\alpha,\delta) \in \mathrm{Aut}(H)\times\mathrm{Aut}(K) \mid \delta(k)\cdot\alpha(h) = \alpha(k\cdot h),$$
$$\delta(k)^{\alpha(h)} = \delta(k^h),\ \forall\, h \in H, k \in K\big\}.$$

Then one can easily check that P, S, X, Y and Z are all subgroups of the group $\mathrm{Aut}(G)$. But Q and R need not be subgroups of the group $\mathrm{Aut}(G)$. However, if H and K are abelian groups, then Q and R are subgroups of $\mathrm{Aut}(G)$. Let

$$A = \left\{\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} \mid \alpha \in P\right\}, \qquad B = \left\{\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \mid \beta \in Q\right\},$$
$$C = \left\{\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix} \mid \gamma \in R\right\}, \qquad D = \left\{\begin{pmatrix}1 & 0\\ 0 & \delta\end{pmatrix} \mid \delta \in S\right\},$$
$$E = \left\{\begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix} \mid (\alpha,\gamma,\delta) \in X\right\}, \quad F = \left\{\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} \mid (\alpha,\beta,\delta) \in Y\right\},$$
$$M = \left\{\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix} \mid (\alpha,\delta) \in Z\right\}.$$

be the corresponding subsets of $\mathcal{A}$. Then one can easily check that A, D, E, F and M are subgroups of $\mathcal{A}$, and if H and K are abelian groups, then B and C are also subgroups of $\mathcal{A}$. Note that A and D normalizes B and C.

Theorem 2.4 *Let* $G = H \bowtie K$ *be the Zappa–Szép product of two abelian groups* H *and* K. *Let* A, B, C *and* D *be defined as above. Then, if* $1 - \beta\gamma \in P$, *for all maps* β *and* γ, *then* $ABCD = \mathcal{A}$ *and* $\mathrm{Aut}(G) \simeq ABCD$.

Proof — Let $\alpha \in P$, $\beta \in Q$, $\gamma \in R$ and $\delta \in S$. Then note that, $\alpha\beta\delta \in Q$

and $\begin{pmatrix} 1 & \beta \\ \gamma & 1 \end{pmatrix} \in \mathcal{A}$. Assume that $1-\beta\gamma \in P$. Now, if $\widehat{\beta} = \alpha^{-1}\beta\delta^{-1}$, then

$$\begin{pmatrix} 1 & \widehat{\beta} \\ \gamma & 1 \end{pmatrix} = \begin{pmatrix} 1-\widehat{\beta}\gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (1-\widehat{\beta}\gamma)^{-1}\widehat{\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \in ABC.$$

Thus, if $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}$, then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \widehat{\beta} \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \in A(ABC)D = ABCD.$$

Therefore, $\mathcal{A} \subseteq ABCD$. Clearly, $ABCD \subseteq \mathcal{A}$. Hence, $ABCD = \mathcal{A}$ and so, $\mathrm{Aut}(G) \simeq ABCD$. □

3 Automorphisms of Zappa–Szép products of groups $\mathbb{Z}_4$ and $\mathbb{Z}_m$

In [11], Yacoub classified the groups which are Zappa–Szép products of cyclic groups of order 4 and order m. He found that these are of the following type (see [11, Conclusion, p. 126]):

$$\begin{aligned} L_1 =& \langle a, b \mid a^m = 1 = b^4, ab = ba^r, r^4 \equiv 1 \pmod{m} \rangle, \\ L_2 =& \langle a, b \mid a^m = 1 = b^4, ab = b^3a^{2t+1}, a^2b = ba^{2s} \rangle, \end{aligned}$$

where in L_2, m is even. These are not non-isomorphic classes. The group L_1 may be isomorphic to the group L_2 depending on the values of m, r and t (see [11, Theorem 5, p. 126]). Clearly, L_1 is a semidirect product. Throughout this section G will denote the group L_2 and we will be only concerned about groups L_2 which are Zappa–Szép products but not a semidirect product. Note that $G = H \bowtie K$, where $H = \langle b \rangle$ and $K = \langle a \rangle$. For the group G, the mutual actions of H and K are defined by $a \cdot b = b^3, a^b = a^{2t+1}$ along with $a^2 \cdot b = b$ and $(a^2)^b = a^{2s}$, where t and s are the integers satisfying the conditions

(G1) $2s^2 \equiv 2 \pmod{m}$,

(G2) $4t(s+1) \equiv 0 \pmod{m}$,

(G3) $2(t+1)(s-1) \equiv 0 \pmod{m}$,

(G4) $\gcd(s, \frac{m}{2}) = 1$.

Lemma 3.1

$$(a^l)^b = \begin{cases} a^{2t+1+(l-1)s}, & \textit{if } l \textit{ is odd} \\ a^{ls}, & \textit{if } l \textit{ is even} \end{cases}.$$

Lemma 3.2 *Let* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}$. *Then*

(i) $\mathrm{Im}(\delta) \subseteq \langle a^r \rangle$, *where* r *is odd,*

(ii) $\beta(a^l) = \begin{cases} \beta(a), & \textit{if } l \textit{ is odd} \\ 1, & \textit{if } l \textit{ is even} \end{cases}$,

(iii) $\mathrm{Im}(\gamma) \subseteq \langle a^2 \rangle$,

(iv) $\alpha \in \mathrm{Aut}(H)$,

(v) $\beta\gamma = 0$, *where* 0 *is the trivial group homomorphism,*

(vi) $\gamma(h) \cdot \beta(k) = \beta(k)$, *for all* $h \in H$ *and* $k \in K$,

(vii) *If either* $s = 1$ *or* $\mathrm{Im}(\beta) \subseteq \langle b^2 \rangle$, *then* $\gamma(h)^{\beta(k)} = \gamma(h)$, *for all* $h \in H$ *and* $k \in K$.

Proof — (i) If possible, let $\delta(a) = a^r$, where r is even. Then, using (A3) and $a^2 \cdot b^j = b^j$, it follows that β is a homomorphism. Also, using $(a^2)^b = a^{2s}$, (C4) and (A4), if $\beta(a) = 1$ or b^2, then δ is defined by $\delta(a^l) = a^{rl}$, for all l. Similarly, if $\beta(a) = b$ or b^3, then δ is defined by

$$\delta(a^l) = \begin{cases} a^{\frac{l+1}{2}r + \frac{l-1}{2}rs}, & \text{if } l \text{ is odd} \\ a^{\frac{l}{2}r(s+1)}, & \text{if } l \text{ is even} \end{cases}.$$

One can easily observe that δ is neither one-one nor onto. But this is a contradiction by (A7). Hence, $\mathrm{Im}(\delta) \subseteq \langle a^r \rangle$, where r is odd.

(ii) If ν is odd, using (C3) and $a \cdot b = b^{-1}$, we have $a^\nu \cdot b^j = b^{-j}$, for all j. Thus using (A3), (C2) and part (i),

$$\beta(a^2) = \beta(a)\big(\delta(a) \cdot \beta(a)\big) = \beta(a)(\beta(a))^{-1} = 1$$

and
$$\beta(a^3) = \beta(a)\big(\delta(a)\cdot\beta(a^2)\big) = \beta(a)\big(\delta(a)\cdot 1\big) = \beta(a).$$
Inductively, we get the required result.

(iii) Suppose that $\gamma(b) = a^{\lambda}$, where λ is odd. Then using (A1),
$$\alpha(b) = b^i = \alpha(b^3) \quad \text{and} \quad \alpha(b^2) = 1 = \alpha(1),$$
where $0 \leqslant i \leqslant 3$. Thus the map α is neither one-one nor onto, but by (A7), the map α is a bijection. This is a contradiction. Therefore λ is even. Now, using (A2), for different choices of $\alpha(b)$ we find that $\gamma(b^2) \in \{a^{2\lambda}, a^{\lambda(s+1)}\}$. Since λ is even, $\gamma(b^2) \in \langle a^2\rangle$. Similarly, $\gamma(b^3) \in \{a^{3\lambda}, a^{\lambda(s+2)}\}$ and so, $\gamma(b^3) \in \langle a^2\rangle$. Hence, (iii) holds.

(iv) Using (iii) and (A1), one observes that α is an endomorphism of H. Also, by (A7), α is a bijection. Thus, α is an automorphism of H. Hence, (iv) holds.

(v) Using parts (ii) and (iii), $\beta\gamma(h) = 1$, for all $h \in H$. Thus, $\beta\gamma = 0$.

(vi) Using relation $a^2 \cdot b = b$ and part (iii), (vi) holds.

(vii) Using (C4) and (G1), we get
$$(a^{2l})^{b^j} = \begin{cases} a^{2ls}, & \text{if } j \text{ is odd} \\ a^{2l}, & \text{if } j \text{ is even} \end{cases} \tag{3.1}$$
Thus, if either $s = 1$ or $\mathrm{Im}(\beta) \subseteq \langle b^2\rangle$, then using part (iii) and Equation (3.1), (vii) holds. □

By Lemma 3.2 (ii), observe that, $\beta(k^h) = \beta(k)$, for all $k \in K$ and $h \in H$.

Lemma 3.3 *Let* $\beta \in Q$. *Then*
$$\beta \in \mathrm{Hom}(K, H) \quad \textit{and} \quad \mathrm{Im}(\beta) \leqslant \langle b^2\rangle.$$
Moreover, $\mathrm{Im}(\beta) = \langle b^2\rangle$ *if and only if* $2t(1+s) \equiv 0 \pmod{m}$, *where* $\gcd(s+1, \frac{m}{2}) \neq 1$.

Proof — Let $\beta(a) = b^i$. Using Lemma 3.2 (ii), we have $\beta\big(a^{2j}\big) = 1$ and $\beta\big(a^{2j+1}\big) = b^i$, for all j. So, it is sufficient to study only $\beta(a)$ in the following,
$$a = a^{\beta(a)} = a^{b^i}. \tag{3.2}$$

Clearly, Equation (3.2) holds trivially for $i = 0$. If $i = 1$, then by Equation (3.2), $a = a^{2t+1}$ which implies that $2t \equiv 0 \pmod{m}$. Therefore, in the defining relations of the group G, $ab = b^3a$ which shows that G is a semidirect product of the groups H and K. For $i = 3$,

$$a = a^{b^3} = a^{4t+2ts+1},$$

which gives that

$$4t + 2ts \equiv 0 \pmod{m}.$$

So, using (G2) and (G4), $2ts \equiv 0 \pmod{m}$ giving $t \equiv 0 \pmod{\frac{m}{2}}$. Thus, G is again a semidirect product of H and K. Now, for $i = 2$, using (C6) and Lemma 3.1,

$$a^{b^2} = \left(a^{2t+1}\right)^b = a^{2t+1+2ts}.$$

Then, $a^{b^2} = a$ if and only if $2t(1 + s) \equiv 0 \pmod{m}$.

Now, if $\gcd\left(s + 1, \frac{m}{2}\right) = 1$, then $t \equiv 0 \left(\mathrm{mod}\ \frac{m}{2}\right)$ and hence G is a semidirect product of the groups H and K. On the other hand, if $\gcd\left(s + 1, \frac{m}{2}\right) \neq 1$, then $t \not\equiv 0 \left(\mathrm{mod}\ \frac{m}{2}\right)$. Thus, G is a Zappa–Szép product of H and K. It follows that $\mathrm{Im}(\beta) = \langle b^2 \rangle$ if and only if $2t(1 + s) \equiv 0 \pmod{m}$ and $\gcd\left(s + 1, \frac{m}{2}\right) \neq 1$. Since $\mathrm{Im}(\beta) \subseteq \langle b^2 \rangle$, using Lemma 3.2 (ii), one can easily observe that $\beta \in \mathrm{Hom}(K, H)$. Hence, the result holds. □

Now, one can easily observe that for the given group G, we have that

$$k \cdot \alpha(h) = \alpha(k \cdot h),\ \beta(k) = \beta(k^h),\ h' = \gamma(h) \cdot h',$$
$$\delta(k) \cdot h = k \cdot h,\ \delta(k) \cdot \alpha(h) = \alpha(k \cdot h),$$
$$\beta(k)\big(\delta(k) \cdot \alpha(h)\big) = \alpha(k \cdot h)\beta(k^h)$$

always holds for all $h \in H$, $k \in K$, $\alpha \in P$, $\beta \in Q$, $\gamma \in R$, $\delta \in S$, $(\alpha, \gamma, \delta) \in X$, $(\alpha, \delta) \in Z$ and $(\alpha, \beta, \delta) \in Y$, respectively. Thus the subgroups P, Q, R, S, X, Y and Z reduce to the following,

$$P = \left\{\alpha \in \mathrm{Aut}(H) \mid k^{\alpha(h)} = k^h, \forall\, h \in H, k \in K\right\},$$
$$Q = \left\{\beta \in \mathrm{Hom}(K, H) \mid k = k^{\beta(k')}, \forall\, k \in K\right\} = \mathrm{Hom}\big(K, \mathrm{Stab}_H(K)\big),$$
$$R = \left\{\gamma \in \mathrm{CrossHom}\big(H, \mathrm{Stab}_K(H)\big) \mid \gamma(k \cdot h) = \gamma(h), \forall\, h \in H, k \in K\right\},$$
$$S = \left\{\delta \in \mathrm{Aut}(K) \mid \delta(k)^h = \delta(k^h), \forall\, h \in H, k \in K\right\},$$

$$X = \big\{(\alpha,\gamma,\delta) \in \mathrm{Aut}(H) \times \mathrm{Map}(H,K) \times \mathrm{Aut}(K) \mid \\ \gamma(hh') = \gamma(h)^{\alpha(h')}\gamma(h'), \delta(k)^{\alpha(h)}\gamma(h) = \gamma(k\cdot h)\delta(k^h), \forall h\in H, k\in K\big\},$$

$$Y = \big\{(\alpha,\beta,\delta) \in \mathrm{Aut}(H) \times \mathrm{Map}(K,H) \times \mathrm{Map}(K,K) \mid \beta(kk') = \beta(k) \\ \big(\delta(k)\cdot\beta(k')\big), \delta(kk') = \delta(k)^{\beta(k')}\delta(k'), \delta(k)^{\alpha(h)} = \delta(k^h), \forall h\in H, k\in K\big\},$$

$$Z = \big\{(\alpha,\delta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K) \mid \delta(k)^{\alpha(h)} = \delta(k^h), \forall h \in H, k \in K\big\}.$$

Theorem 3.4 *Let* A, B, C, D *be defined as above. Then* $\mathrm{Aut}(G) = ABCD$.

Proof — Using Lemma 3.2 (v), we have that $\beta\gamma = 0$, so $1 - \beta\gamma \in P$. Therefore, by Theorem 2.4, we have $\mathrm{Aut}(G) = ABCD$. □

Theorem 3.5 *Let*

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{A}.$$

Then, if $\beta \in Q$ *and* $(\alpha,\gamma,\delta) \in X$, *then* $\mathrm{Aut}(G) \simeq E \rtimes B \simeq (C \rtimes M) \rtimes B$.

Proof — Let $\beta \in Q$. Using Lemma 3.3, $\mathrm{Im}(\beta) \leqslant \langle b^2\rangle$. Let $k, k' \in K$ such that $\beta(k) = b^{2i}$ and $\beta(k') = b^{2j}$, for all i, j. Then

$$\begin{aligned}
\gamma\beta(kk') &= \gamma\big(\beta(k)\big(k\cdot\beta(k')\big)\big) = \gamma\big(\beta(k)\big)^{\alpha\left(k\cdot\beta(k')\right)}\gamma\big(k\cdot\beta(k')\big)\\
&= \gamma\big(b^{2i}\big)^{\alpha(k\cdot b^{2j})}\gamma\big(\beta(k')\big) = \big(a^{i\lambda(s+1)}\big)^{\alpha(b^{2j})}\gamma\big(\beta(k')\big)\\
&= \big(a^{i\lambda(s+1)}\big)^{b^{2j}}\gamma\big(\beta(k')\big) = a^{i\lambda(s+1)s^{2j}}\gamma\big(\beta(k')\big)\\
&= a^{i\lambda(s+1)}\gamma\big(\beta(k')\big) = \gamma(b^{2i})\gamma\big(\beta(k')\big) = \gamma\beta(k)\gamma\beta(k').
\end{aligned}$$

Thus $\gamma\beta \in \mathrm{Hom}(K,K)$, so $\gamma\beta + \delta \in \mathrm{Hom}(K,K)$. Now, let $\beta(a) = b^{2j}$ and $\delta(a) = a^r$, where $j \in \{0,1\}$ and $r \in U(m)$. Then, using Lemma 3.2, we have

$$(\gamma\beta+\delta)(a^l) = \begin{cases} a^{lr}, & \text{if } l \text{ is even}\\ a^{\lambda j(s+1)+lr}, & \text{if } l \text{ is odd}\end{cases}$$

One can easily observe that $\gamma\beta + \delta$ defined as above is a bijection. Thus $\gamma\beta + \delta \in \mathrm{Aut}(K)$.

Now, using (C3), (C4) and Lemma 3.2 (iii), we have

$$\begin{aligned}
(\gamma\beta+\delta)(a)\cdot\alpha(b) &= \gamma\beta(a)\delta(a)\cdot\alpha(b)\\
= \gamma\beta(a)\cdot\big(\delta(a)\cdot\alpha(b)\big) &= \gamma\beta(a)\cdot\alpha(a\cdot b) = \alpha(a\cdot b)
\end{aligned}$$

and

$$(\gamma\beta+\delta)(a)^{\alpha(b)}\gamma(b) = \big(\gamma\beta(a)\delta(a)\big)^{\alpha(b)}\gamma(b)$$
$$= \big(\delta(a)\gamma\beta(a)\big)^{\alpha(b)}\gamma(b) = \delta(a)^{\gamma(\beta(a))\cdot\alpha(b)}\gamma\big(\beta(a)\big)^{\alpha(b)}\gamma(b)$$
$$= \delta(a)^{\alpha(b)}\gamma(b^{2i})^{\alpha(b)}\gamma(b) = \delta(a)^{\alpha(b)}\gamma(b)\big(a^{i\lambda(s+1)}\big)^{\alpha(b)}$$
$$= \gamma(a\cdot b)\delta(a^b)a^{i\lambda(s+1)} = \gamma(a\cdot b)\delta(a^b)\gamma(b^{2i})$$
$$= \gamma(a\cdot b)\gamma\big(\beta(a)\big)\ \delta(a^b) = \gamma(a\cdot b)\gamma\big(\beta(a^{2t+1})\big)\delta(a^b)$$
$$= \gamma(a\cdot b)\gamma\big(\beta(a^b)\big)\delta(a^b) = \gamma(a\cdot b)(\gamma\beta+\delta)(a^b).$$

Thus, $(\alpha,\gamma,\gamma\beta+\delta)\in X$.

Using Lemma 3.2 (v), we have

$$\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}\alpha & (\alpha+\beta\gamma)(-\beta)+\beta\delta\\ \gamma & \gamma\beta+\delta\end{pmatrix} \tag{3.3}$$

Now, using Lemma 3.2 (ii), we have

$$\big((\alpha+\beta\gamma)(-\beta)+\beta\delta\big)(a) = (\alpha+\beta\gamma)\big(-\beta(a)\big)\beta\big(\delta(a)\big)$$
$$= (\alpha+\beta\gamma)(b^{-2j})\beta(a^r) = \alpha(b^{2j})\beta\big(\gamma(b^{2j})\big)b^{2j}$$
$$= b^{2ij}b^{2j} = b^{2j(i+1)} = 1.$$

Thus,

$$(\alpha+\beta\gamma)(-\beta)+\beta\delta = 0.$$

Therefore, by Equation (3.3),

$$\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}\alpha & 0\\ \gamma & \gamma\beta+\delta\end{pmatrix} \in E.$$

So, $E \lhd \mathcal{A}$. Now, if $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{A}$, then

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ \gamma & -\gamma\alpha^{-1}\beta+\delta\end{pmatrix}\begin{pmatrix}1 & \alpha^{-1}\beta\\ 0 & 1\end{pmatrix} \in EB.$$

Clearly, $E\cap B = \{1\}$. Thus, $\mathcal{A} = E \rtimes B$. Hence, $\mathrm{Aut}(G) \simeq E \rtimes B$.

Let $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in E$. Then

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta^{-1}\gamma & 1 \end{pmatrix} \in MC.$$

Clearly, $M \cap C = \{1\}$. Since $A \times D$ normalizes C, we have that C is normal in E. Therefore, $E = C \rtimes M$. Hence, $X \simeq C \rtimes M$ and so, $\mathrm{Aut}(G) \simeq (C \rtimes M) \rtimes B$. □

Now, we will find the structure and the order of the automorphism group $\mathrm{Aut}(G)$. For this, we will proceed by first taking t to be such that $\gcd(t, m) = 1$ and then by taking t to be such that $\gcd(t, m) = d$, where $d > 1$.

Theorem 3.6 *Let* 4 *divide* m *and* t *be odd such that* $\gcd(t, m) = 1$*. Then*

$$\mathrm{Aut}(G) \simeq \begin{cases} \left(\mathbb{Z}_{\frac{m}{2}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right)\right) \rtimes \mathbb{Z}_2, & \textit{if } s \in \left\{\frac{m}{2} - 1, m - 1\right\} \\ \mathbb{Z}_{\frac{m}{2}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right), & \textit{if } s \in \left\{\frac{m}{4} - 1, \frac{3m}{4} - 1\right\} \end{cases}$$

Proof — Let $\gcd(t, m) = 1$. Then, using (G2), we get $s \equiv -1 \left(\mathrm{mod}\ \frac{m}{4}\right)$ which implies that $s \in \left\{\frac{m}{4} - 1, \frac{m}{2} - 1, \frac{3m}{4} - 1, m - 1\right\}$. Now, using (G3), we get $t \equiv -1 \ (\mathrm{mod}\ \frac{m}{4})$. Then $t \in \left\{\frac{m}{4} - 1, \frac{m}{2} - 1, \frac{3m}{4} - 1, m - 1\right\}$.

Let $(\alpha, \gamma, \delta) \in X$ be such that $\alpha(b) = b^i$, $\gamma(b) = a^\lambda$ and $\delta(a) = a^r$, where $i \in \{1, 3\}$, λ is even, $0 \leqslant \lambda \leqslant m - 1$, and $r \in U(m)$. Using

$$\gamma(hh') = \gamma(h)^{\alpha(h')}\gamma(h'),$$

we get $\gamma(b^2) = a^{\lambda(s+1)}, \gamma(b^3) = a^{\lambda(s+2)}$ and $\gamma(b^4) = 1$. We consider two cases based on the image of the map α.

Case (i): $\alpha(b) = b$.

Using $\gamma(a \cdot b)\delta(a^b) = \delta(a)^b\gamma(b)$, we get

$$\begin{aligned} a^{\lambda(s+2)+(2t+1)r} &= \gamma(a \cdot b)\delta(a^b) = \delta(a)^b\gamma(b) \\ &= (a^r)^b a^\lambda = a^{2t+1+(r-1)s+\lambda} \end{aligned}$$

which implies that

$$\lambda(s + 1) \equiv (r - 1)(s - 2t - 1) \ (\mathrm{mod}\ m). \tag{3.4}$$

If $s \in \{\frac{m}{2}-1, m-1\}$, then Equation (3.4) holds for all values of t, λ and r. Now, if $(s,t) \in \{(\frac{m}{4}-1, \frac{m}{2}-1), (\frac{m}{4}-1, m-1)\}$, then by Equation (3.4), $r \equiv 1+\lambda \pmod 4$. Since λ is even, $r \equiv 1$ or $3 \pmod 4$. Again, if $(s,t) \in \{(\frac{m}{4}-1, \frac{m}{4}-1), (\frac{m}{4}-1, \frac{3m}{4}-1)\}$, then by Equation (3.4), $r \equiv 1-\lambda \pmod 4$. Since λ is even, $r \equiv 1$ or $3 \pmod 4$. Using a similar argument, we get the same results for $s = \frac{3m}{4}-1$. Thus, in this case, there are $\frac{m}{2}$ and $\phi(m)$ choices for the maps γ and δ respectively and these are $\gamma_\lambda(b) = a^\lambda$ and $\delta_r(a) = a^r$, for all $0 \leqslant \lambda \leqslant m-1$, λ is even, and $r \in U(m)$.

Case (ii): $\alpha(b) = b^3$.

Then

$$a^{\lambda(s+2)+(2t+1)r} = \gamma(a \cdot b)\delta(a^b) = \delta(a)^{\alpha(b)}\gamma(b)$$
$$= (a^r)^{b^3} a^\lambda = a^{4t+2ts+1+(r-1)s+\lambda}$$

which implies that

$$\lambda(s+1) \equiv 2t(s+1) + (r-1)(s-2t-1) \pmod m. \tag{3.5}$$

If $s \in \{\frac{m}{2}-1, m-1\}$, then Equation (3.5) holds for all values of t, λ and r. Now, if $(s,t) \in \{(\frac{m}{4}-1, \frac{m}{2}-1), (\frac{m}{4}-1, m-1)\}$, then by Equation (3.5), $r \equiv 3+\lambda \pmod 4$. Since λ is even, $r \equiv 1$ or $3 \pmod 4$. Again, if $(s,t) \in \{(\frac{m}{4}-1, \frac{m}{4}-1), (\frac{m}{4}-1, \frac{3m}{4}-1)\}$, then by Equation (3.5), we have $r \equiv 1+\lambda \pmod 4$. Since λ is even, $r \equiv 1$ or $3 \pmod 4$. Using a similar argument, we get the same results for $s = \frac{3m}{4}-1$. Thus, in this case also, there are $\frac{m}{2}$ and $\phi(m)$ choices for the maps γ and δ respectively and these are $\gamma_\lambda(b) = a^\lambda$ and $\delta_r(a) = a^r$, for all $0 \leqslant \lambda \leqslant m-1$, λ is even, and $r \in U(m)$.

Thus combining both the cases (i) and (ii), we get for all $\alpha \in \mathrm{Aut}(H)$, the choices for the maps γ and δ are $\gamma_\lambda(b) = a^\lambda$ and $\delta_r(a) = a^r$, where $0 \leqslant \lambda \leqslant m-1$, λ is even, and $r \in U(m)$. So, using Theorem 3.5,

$$X \simeq \mathbb{Z}_{\frac{m}{2}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right).$$

Now, if $s \in \{\frac{m}{2}-1, m-1\}$, then $2t(s+1) \equiv 0 \pmod m$. Therefore, using Lemma 3.3, $\mathrm{Im}(\beta) = \{b^2\}$ and so, $B \simeq \mathbb{Z}_2$. If $s \in \{\frac{m}{4}-1, \frac{3m}{4}-1\}$, then $2t(s+1) \not\equiv 0 \pmod m$. Therefore, using Lemma 3.3, $\mathrm{Im}(\beta) = \{1\}$

and so, B is a trivial group. Hence, by Theorem 3.5,

$$\mathrm{Aut}(G) \simeq E \rtimes B \simeq \begin{cases} \left(\mathbb{Z}_{\frac{m}{2}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right)\right) \rtimes \mathbb{Z}_2, & \text{if } s \in \left\{\frac{m}{2}-1, m-1\right\} \\ \mathbb{Z}_{\frac{m}{2}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right), & \text{if } s \in \left\{\frac{m}{4}-1, \frac{3m}{4}-1\right\} \end{cases}$$

The statement is proved. □

Theorem 3.7 *Let* $m = 2q$, *where* $q > 1$ *is odd and* $\gcd(t, m) = 1$. *Then we have* $\mathrm{Aut}(G) \simeq \left(\mathbb{Z}_{\frac{m}{2}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right)\right) \rtimes \mathbb{Z}_2$.

Proof — Using (G1), (G2), and (G3), we get $s, t \in \left\{\frac{m}{2}-1, m-1\right\}$. Then, the result follows on the lines of the proof of Theorem 3.6. □

Theorem 3.8 *Let* $m = 2^n$, $n \geqslant 3$. *Then*

(i) *if* t *is even, then* $\mathrm{Aut}(G) \simeq \left(\mathbb{Z}_4 \rtimes \left(\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})\right)\right) \rtimes \mathbb{Z}_2$,

(ii) *if* t *is odd, then*

$$\mathrm{Aut}(G) \simeq \begin{cases} \left(\mathbb{Z}_{2^{n-1}} \rtimes \left(\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})\right)\right) \rtimes \mathbb{Z}_2, & \text{if } s \in \left\{\frac{m}{2}-1, m-1\right\} \\ \mathbb{Z}_{2^{n-1}} \rtimes \left(\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})\right), & \text{if } s \in \left\{\frac{m}{4}-1, \frac{3m}{4}-1\right\} \end{cases}$$

Proof — We will find the automorphism group $\mathrm{Aut}(G)$ in two cases namely, when t is even and when t is odd.

Case (i): t is even.

Then

$$2(t+1)(s-1) \equiv 0 \pmod{2^n} \implies s \equiv 1 \pmod{2^{n-1}}.$$

Therefore, $s = 1, 2^{n-1}+1$. Moreover $4t(s+1) \equiv 0 \pmod{2^n}$ implies that $t \equiv 0 \pmod{2^{n-3}}$. Therefore,

$$t \in \left\{2^{n-3}, 2^{n-2}, 3\cdot 2^{n-3}, 2^{n-1}, 5\cdot 2^{n-3}, 3\cdot 2^{n-2}, 7\cdot 2^{n-3}, 2^n\right\}.$$

Note that, for $t = 2^{n-1}$ or $t = 2^n$, G is the semidirect product of H and K. So, we consider the other values of t.

Let $\gamma \in R$ be such that $\gamma(b) = a^\lambda$, where $0 \leqslant \lambda \leqslant m-1$ and λ is even. Then, since $s = 1$ and λ is even, by (A2), $\gamma \in \mathrm{Hom}(H, K)$. Now,

$$1 = \gamma(b^4) = a^{4\lambda}$$

implies that $\lambda \equiv 0 \pmod{2^{n-2}}$. Therefore

$$\lambda \in \{2^{n-2}, 2^{n-1}, 3 \cdot 2^{n-2}, 2^n\}.$$

Using $\gamma(a \cdot b) = \gamma(b)$, $a^{3\lambda} = \gamma(a \cdot b) = \gamma(b) = a^{\lambda}$ gives $\lambda \equiv 0 \pmod{2^{n-1}}$. Thus, $\lambda \in \{0, 2^{n-1}\}$ and so $C \simeq \mathbb{Z}_2$.

Now, let $(\alpha, \beta, \delta) \in Y$ be such that $\alpha(b) = b^i, \beta(a) = b^j, \delta(a) = a^r$, where $i \in \{1, 3\}$, $0 \leqslant j \leqslant 3$, $0 \leqslant r \leqslant 2^n - 1$ and r is odd. Using Lemma 3.2 (ii),

$$\beta(kk') = \beta(k)\big(\delta(k) \cdot \beta(k')\big)$$

holds for all $k, k' \in K$. Now, using $\delta(kk') = \delta(k)^{\beta(k')}\delta(k')$, we get

$$\delta(a^l) = \begin{cases} a^{(l-1)(jt+r)+r}, & \text{if } l \text{ is odd} \\ a^{l(jt+r)}, & \text{if } l \text{ is even} \end{cases}$$

Finally, using $\delta(k^h) = \delta(k)^{\alpha(h)}$, we have

$$a^{2it+r} = (a^r)^{b^i} = \delta(a)^{\alpha(b)} = \delta(a^b) = \delta(a^{2t+1}) = a^{2t(jt+r)+r}.$$

Thus, $2t(jt + r - i) \equiv 0 \pmod{2^n}$ which implies that

$$\begin{cases} r \equiv i \pmod 4, & \text{if } t \in \{2^{n-3}, 3 \cdot 2^{n-3}, 5 \cdot 2^{n-3}, 7 \cdot 2^{n-3}\} \text{ and } n \geqslant 5 \\ r \equiv i + 2j \pmod 4, & \text{if } t \in \{2^{n-3}, 3 \cdot 2^{n-3}, 5 \cdot 2^{n-3}, 7 \cdot 2^{n-3}\} \text{ and } n = 4 \\ r \equiv i \pmod 2, & \text{if } t \in \{2^{n-2}, 3 \cdot 2^{n-2}\} \end{cases}$$

Now, if $j \in \{0, 2\}$, then $r \equiv i \pmod 4$ and if $j \in \{1, 3\}$, then $r \equiv i$ or $i + 2 \pmod 4$. Thus, for all $\beta \in \mathrm{CrossHom}(K, H)$, the choices for the maps α and δ are $\alpha_i(b) = b^i$ and $\delta_r(a) = a^r$, where $i \in \{1, 3\}$ and $r \in U(m)$. Note that, if

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in F,$$

then

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\beta \\ 0 & 1 \end{pmatrix} \in MB.$$

Clearly, $M \cap B = \{1\}$ and M normalizes B. So, $B \triangleleft F$ and $F = B \rtimes M$. Therefore,

$$Y \simeq B \rtimes M \simeq \mathbb{Z}_4 \rtimes \big(\mathbb{Z}_2 \times U(m)\big).$$

Using Lemma 3.2 (v)–(vii),

$$\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}^{-1} = \begin{pmatrix}\alpha & \beta\\ \gamma\alpha + (\gamma\beta+\delta)(-\gamma) & \gamma\beta+\delta\end{pmatrix} \tag{3.6}$$

Now,

$$\begin{aligned}\big(\gamma\alpha + (\gamma\beta+\delta)(-\gamma)\big)(b) &= \gamma\alpha(b)(\gamma\beta+\delta)(-\gamma)(b)\\ &= \gamma(b^i)(\gamma\beta+\delta)\ (a^{-\lambda}) = a^{i\lambda}\gamma\big(\beta(a^{-\lambda})\big)\delta(a^{-\lambda})\\ &= a^{i\lambda}\gamma(1)a^{-\lambda(jt+r)} = a^{\lambda(i-jt-r)} = 1.\end{aligned}$$

Thus, $\gamma\alpha + (\gamma\beta+\delta)(-\gamma) = 0$. Also, one can easily observe that

$$(\alpha, \beta, \gamma\beta+\delta) \in Y.$$

Therefore, by Equation (3.6),

$$\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}^{-1} = \begin{pmatrix}\alpha & \beta\\ 0 & \gamma\beta+\delta\end{pmatrix} \in F.$$

So, $F \lhd \mathcal{A}$. Clearly, $F \cap C = \{1\}$. Also, if $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{A}$, then

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ \delta^{-1}\gamma & 1\end{pmatrix} \in FC.$$

Hence, $\mathcal{A} = F \rtimes C$ and so,

$$\mathrm{Aut}(G) \simeq F \rtimes C \simeq \big(\mathbb{Z}_4 \rtimes \big(\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})\big)\big) \rtimes \mathbb{Z}_2.$$

Case (ii): t is odd.

Then $\gcd(t, m) = 1$. Hence, the result follows from Theorem 3.6. □

Now, we discuss the structure of the automorphism group $\mathrm{Aut}(G)$ in the case when $\gcd(t, m) > 1$.

Theorem 3.9 *Let* $m = 4q$ *and* $\gcd(t, m) = 2^i d$, *where* $q > 1$ *is odd,* $i \in \{0, 1, 2\}$, *and* d *divides* q. *Then* $\mathrm{Aut}(G) \simeq \big(\mathbb{Z}_{\frac{m}{2d}} \rtimes (\mathbb{Z}_2 \times U(m))\big) \rtimes \mathbb{Z}_2$.

Proof — Let $q = du$, for some integer u. Then, using (G2), we get $s \equiv -1 \pmod u$, which implies that $s = lu - 1$, where $1 \leqslant l \leqslant 4d$.

Since $\gcd(s, \frac{m}{2}) = 1$, s is odd and so, l is even. Using (G1) and (G3), we get $l\big(u\frac{l}{2} - 1\big) \equiv 0 \pmod{d}$ and $t+1 \equiv u\frac{l}{2} \pmod{q}$. Now, one can easily observe that $\gcd(l, d) = 1$, which implies $u\frac{l}{2} - 1 \equiv 0 \pmod{d}$. Thus, $2t(s+1) \equiv 2ltu \equiv 0 \pmod{m}$ and $\gcd\big(s+1, \frac{m}{2}\big) \neq 1$. Therefore, using Lemma 3.3, $B \simeq \mathbb{Z}_2$.

Let $(\alpha, \gamma, \delta) \in X$ be such that $\alpha(b) = b^i$, $\gamma(b) = a^\lambda$ and $\delta(a) = a^r$, where $i \in \{1, 3\}$, $0 \leqslant \lambda \leqslant m-1$, λ is even, and $r \in U(m)$. Then, using

$$\gamma(hh') = \gamma(h)^{\alpha(h')}\gamma(h'),$$

we have

$$\gamma(b^2) = a^{\lambda(s+1)},\ \gamma(b^3) = a^{\lambda(s+2)} \quad \text{and} \quad \gamma(b^4) = 1.$$

Now, using

$$\delta(a)^{\alpha(b)}\gamma(b) = \gamma(a \cdot b)\delta(a^b) \quad \text{and} \quad 2t(s+1) \equiv 0 \pmod{m},$$

we have

$$\begin{aligned} a^{\lambda(s+2)+(2t+1)r} &= \gamma(b^3)\delta(a^{2t+1}) = \gamma(a \cdot b)\delta(a^b) = \delta(a)^{\alpha(b)}\gamma(b) \\ &= (a^r)^{b^i}a^\lambda = a^{2t+1+(r-1)s+\lambda+\frac{i-1}{2}2t(s+1)} = a^{2t+1+(r-1)s+\lambda}. \end{aligned}$$

Thus

$$\lambda(s+1) \equiv (r-1)(s-2t-1) \pmod{m}. \tag{3.7}$$

Since $2t(s+1) \equiv 0 \pmod{m}$, using (G3), we get

$$2(s-2t-1) \equiv 0 \pmod{m}.$$

Therefore, by Equation (3.7), $\lambda l u \equiv 0 \pmod{m}$. Using Lemma 3.2 (iii), we get $\lambda \equiv 0 \pmod{2d}$. Thus, using Theorem 3.5,

$$X \simeq \mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big).$$

Hence, $\mathrm{Aut}(G) \simeq E \rtimes B \simeq \big(\mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big)\big) \rtimes \mathbb{Z}_2$. □

Theorem 3.10 *Let* $m = 2q$ *and* $\gcd(t, m) = 2^i d$, *where* $q > 1$ *is odd,* $i \in \{0, 1\}$, *and* d *divides* q. *Then* $\mathrm{Aut}(G) \simeq \big(\mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big)\big) \rtimes \mathbb{Z}_2$.

Proof — Follows on the lines of the proof of Theorem 3.9. □

Theorem 3.11 *Let* $m = 2^n q$, t *be even and* $\gcd(m,t) = 2^i d$, *where* $1 \leqslant i \leqslant n$, $n \geqslant 3$, $q > 1$ *is odd and* d *divides* q. *Then*

$$\mathrm{Aut}(G) \simeq \begin{cases} \big(\mathbb{Z}_4 \rtimes (\mathbb{Z}_2 \times U(m))\big) \rtimes \mathbb{Z}_2, & \text{if } d = q \\ \mathbb{Z}_2 \times \big(\mathbb{Z}_{\frac{2q}{d}} \rtimes (\mathbb{Z}_2 \times U(m))\big), & \text{if } d \neq q \text{ and } n-2 \leqslant i \leqslant n \\ \mathbb{Z}_{\frac{4q}{d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big), & \text{if } d \neq q \text{ and } i = n-3 \end{cases}$$

Proof — We consider the following four cases to find the structure of $\mathrm{Aut}(G)$.

Case (i): $d = q$ and $\gcd(t+1, m) = u$.

Since $t+1$ is odd, u is odd and u divides q. Thus, u divides t and so, $u = 1$. Therefore, using (G2) and (G3),

$$s \equiv 1 \ \big(\mathrm{mod}\ \tfrac{m}{2}\big) \quad \text{and} \quad t \equiv 0 \ \big(\mathrm{mod}\ \tfrac{m}{8}\big).$$

Using a similar argument used in the proof of Theorem 3.8 (i), we get $\mathrm{Aut}(G) \simeq F \rtimes C \simeq \big(\mathbb{Z}_4 \rtimes (\mathbb{Z}_2 \times U(m))\big) \rtimes \mathbb{Z}_2$.

Case (ii): $n-2 \leqslant i \leqslant n$ and $q = du$, for some odd integer u.

Using (G2), $s \equiv -1 \pmod u$ and so, $s = lu - 1$, where $0 \leqslant l \leqslant 2^n d$. Since $\gcd\big(s, \frac{m}{2}\big) = 1$, s is odd and so, l is even. Now, using (G1),

$$\frac{l}{2}\Big(\frac{l}{2}u - 1\Big) \equiv 0 \pmod{2^{n-3} d}$$

and by (G3),

$$t \equiv \frac{l}{2}u - 1 \pmod{2^{n-2} q}.$$

Since t is even, $\frac{l}{2}$ is odd and $\gcd\big(\frac{l}{2}, d\big) = 1$. Thus,

$$\frac{l}{2}u \equiv 1 \pmod{2^{n-3} d} \quad \text{and} \quad t \equiv 2^i d \pmod{2^{n-2} q}.$$

One can easily observe that $2t(s+1) \equiv 0 \pmod m$. Therefore, using a similar argument as in the proof of Theorem 3.6, we get

$$\mathrm{Aut}(G) \simeq E \rtimes B \simeq \big(\mathbb{Z}_{\frac{2q}{d}} \rtimes (\mathbb{Z}_2 \times U(m))\big) \rtimes \mathbb{Z}_2.$$

Case (iii): $i = n-3$, $d \neq q$ and $q = du$, for some odd integer u.

Using (G2), $s \equiv -1 \pmod{2u}$, i.e. $s = 2lu - 1$, where $1 \leqslant l \leqslant 2^{n-1} d$.

Now, using (G1) and (G3),

$$l(lu-1) \equiv 0 \pmod{2^{n-3}d} \quad \text{and} \quad (t+1)(lu-1) \equiv 0 \pmod{2^{n-2}q}.$$

If l is even, then $t \equiv lu-1 \pmod{2^{n-2}q}$ gives that t is odd, which is a contradiction. Therefore, l is odd. Using

$$(t+1)(lu-1) \equiv 0 \pmod{2^{n-2}q},$$

one can easily observe that $\gcd(l, d) = 1$. Then

$$lu-1 = 2^{n-3}dl' \quad \text{and} \quad s = 2^{n-2}dl' + 1,$$

where $1 \leqslant l' \leqslant 8u$. Clearly, $\gcd(l', u) = 1$. Thus, $(t+1)l' \equiv 0 \pmod{2u}$. If l' is odd, then $(t+1) \equiv 0 \pmod{2u}$ which implies that t is odd. So, l' is even and so, $t = uq'-1$, $1 \leqslant q' < 2^{n-1}d$, q' is odd as t is even. Note that

$$\begin{aligned} s-2t-1 &= 2^{n-2}dl' - 2t = 2^{n-2}d\left(l' - \tfrac{t}{2^{n-3}d}\right) \\ &= 2^{n-2}d\left(\tfrac{lu-1}{2^{n-3}d} - \tfrac{uq'-1}{2^{n-3}d}\right) = 2^{n-2}du\left(\tfrac{l-q'}{2^{n-3}d}\right) \end{aligned}$$

Let $(\alpha, \gamma, \delta) \in X$ be such that $\alpha(b) = b^i$, $\gamma(b) = a^\lambda$ and $\delta(a) = a^r$, where $i \in \{1, 3\}$, $0 \leqslant \lambda \leqslant m-1$, λ is even and $r \in U(m)$. We consider two sub-cases based on the image of the map α.

Sub-Case (i): $\alpha(b) = b$.

Using $\delta(a)^{\alpha(b)}\gamma(b) = \gamma(a \cdot b)\delta(a^b)$, we have

$$\begin{aligned} a^{\lambda(s+2)+(2t+1)r} &= \gamma(a \cdot b)\delta(a^b) = \delta(a)^b\gamma(b) \\ &= (a^r)^b a^\lambda = a^{2t+1+(r-1)s+\lambda} \end{aligned},$$

which implies that

$$\lambda(s+1) \equiv (r-1)(s-2t-1) \pmod{m}.$$

Therefore

$$\lambda(2lu) \equiv 2^{n-2}du(r-1)\left(\frac{l-q'}{2^{n-3}d}\right) \pmod{2^n q},$$

which implies that

$$\lambda l \equiv 2^{n-3}d(r-1)\left(\frac{l-q'}{2^{n-3}d}\right) \pmod{2^{n-1}d}.$$

Now, if $\lambda \equiv 0 \pmod{2^{n-2}d}$, then $r \equiv 1$ or $3 \pmod 4$ and vice-versa. Thus, in this sub-case, the choices for the maps γ and δ are $\gamma_\lambda(b) = a^\lambda$ and $\delta_r(a) = a^r$, where λ is even and $\lambda \equiv 0 \pmod{2^{n-2}d}$, and $r \in U(m)$.

Sub-Case (ii): Let $\alpha(b) = b^3$.

Using $\delta(a)^{\alpha(b)}\gamma(b) = \gamma(a \cdot b)\delta(a^b)$, we get

$$\begin{aligned} a^{\lambda(s+2)+(2t+1)r} &= \gamma(a \cdot b)\delta(a^b) = \delta(a)^{\alpha(b)}\gamma(b) \\ &= (a^r)^{b^3}a^\lambda = a^{4t+2ts+1+(r-1)s+\lambda} \end{aligned}$$

which implies that

$$(\lambda - 2t)(s+1) \equiv (r-1)(s-2t-1) \pmod m.$$

Therefore

$$2lu(\lambda - 2t) \equiv 2^{n-2}du(r-1)\left(\frac{l-q'}{2^{n-3}d}\right) \pmod{2^n q},$$

which implies that

$$l(\lambda - 2t) \equiv 2^{n-3}d(r-1)\left(\frac{l-q'}{2^{n-3}d}\right) \pmod{2^{n-1}d}.$$

Now, if $\lambda \equiv 0 \pmod{2^{n-2}d}$, then $r \equiv 1$ or $3 \pmod 4$ and vice-versa. Thus, in this sub-case, the choices for the maps γ and δ are $\gamma_\lambda(b) = a^\lambda$ and $\delta_r(a) = a^r$, where λ is even and $\lambda \equiv 0 \pmod{2^{n-2}d}$, and $r \in U(m)$.

Combining both sub-cases (i) and (ii), we get for all $\alpha \in \mathrm{Aut}(H)$, the choices for the maps γ and δ are $\gamma_\lambda(b) = a^\lambda$ and $\delta_r(a) = a^r$, where λ is even and $\lambda \equiv 0 \pmod{2^{n-2}d}$, and $r \in U(m)$. Therefore, using Theorem 3.5,

$$X \simeq \mathbb{Z}_{\frac{4q}{d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big).$$

At last, since l is odd, $2t(s+1) \equiv 4tlu \not\equiv 0 \pmod m$. Therefore, using Lemma 3.3, $\mathrm{Im}(\beta) = \{1\}$. Thus, B is a trivial group. Hence, using Theorem 3.5, $\mathrm{Aut}(G) \simeq E \rtimes B \simeq \mathbb{Z}_{\frac{4q}{d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big)$.

Case (iv): Let $1 \leqslant i \leqslant n-4$. and $q = du$, for some odd integer u.

Using (G2), $s \equiv -1 \pmod{2^{n-i-2}u}$, that is, $s = 2^{n-i-2}lu - 1$, where $1 \leqslant l \leqslant 2^{i+2}d$. Now, using (G1) and (G3),

$$l(2^{n-i-3}lu - 1) \equiv 0 \pmod{2^i d}$$

and

$$(t+1)(lu2^{n-i-3} - 1) \equiv 0 \pmod{2^{n-2}q}.$$

Since $n-i-3 > 0$, $lu2^{n-i-3} - 1$ is odd. If l is even, then

$$t \equiv lu2^{n-i-3} - 1 \pmod{2^{n-2}q}$$

gives that t is odd, which is a contradiction. Now, if l is odd, then using $(t+1)(lu-1) \equiv 0 \pmod{2^{n-2}q}$, one can easily observe that $\gcd(l, d) = 1$. Thus,

$$2^{n-i-3}lu - 1 \equiv 0 \pmod{2^i d},$$

which is absurd. Hence, there is no such l exist and so, no such t and s exist and hence no group G exists as the Zappa–Szép product of H and K. □

Theorem 3.12 *Let* $m = 2^n q$, t *be odd and* $\gcd(t, m) = d$, *where* $n \geqslant 4$ *and* q *is odd. Then*

$$\mathrm{Aut}(G) \simeq \begin{cases} \big(\mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big)\big) \rtimes \mathbb{Z}_2, & \textit{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big), & \textit{if } 2t(s+1) \not\equiv 0 \pmod{m} \end{cases}$$

Proof — Let $q = du$, for some odd integer u. Then using (G2), we have

$$s \equiv -1 \pmod{2^{n-2}u}$$

which implies that $s = 2^{n-2}lu - 1$, where $1 \leqslant l \leqslant 4d$. Now, using (G1),

$$l(2^{n-3}ul - 1) \equiv 0 \pmod{d}.$$

Using (G3), we get

$$(t+1)(lu2^{n-3} - 1) \equiv 0 \pmod{2^{n-2}q}. \tag{3.8}$$

Case (i): l is even.

By Equation (3.8),

$$t \equiv lu2^{n-3} - 1 \pmod{2^{n-2}q}.$$

Note that

$$2t(s+1) \equiv 2t(2^{n-2}lu) \equiv 0 \pmod{m}$$

and

$$\lambda(s+1) = \lambda(lu2^{n-2}).$$

Thus $\lambda(s+1) \equiv 0 \pmod{m}$ if and only if $\lambda l \equiv 0 \pmod{4d}$, which is true for all $\lambda \equiv 0 \pmod{2d}$. Using a similar argument as in the proof of Theorem 3.6, we get

$$X \simeq \mathbb{Z}_{\frac{m}{2d}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right) \quad \text{and} \quad B \simeq \mathbb{Z}_2.$$

Hence, $\mathrm{Aut}(G) \simeq E \rtimes B \simeq \left(\mathbb{Z}_{\frac{m}{2d}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right)\right) \rtimes \mathbb{Z}_2$.

Case (ii): l is odd.

Using Equation (3.8), one can easily observe that $\gcd(l, d) = 1$ which means that $2^{n-3}lu - 1 = dl'$, where l' is odd, $\gcd(l', u) = 1$ and $1 \leqslant l' \leqslant 2^n u$. Thus, using Equation (3.8),

$$(t+1)dl' \equiv 0 \pmod{2^{n-2}q}.$$

Since $\gcd(l', u) = 1$, $t = 2^{n-2}uq' - 1$, where $1 \leqslant q' \leqslant 4d$. Now,

$$\begin{aligned} s - 2t - 1 &= 2dl' - 2t = 2d\left(l' - \tfrac{t}{d}\right) \\ &= 2d\left(\tfrac{2^{n-3}ul - 2^{n-2}uq'}{d}\right) = 2^{n-2}du\tfrac{l-2q'}{d} \end{aligned}$$

Let $(\alpha, \gamma, \delta) \in X$ be such that $\alpha(b) = b^i$, $\gamma(b) = a^\lambda$ and $\delta(a) = a^r$, where $i \in \{1, 3\}$, $0 \leqslant \lambda \leqslant m-1$, λ is even and $r \in U(m)$. We consider two sub-cases based on the image of the map α.

Sub-case (i): $\alpha(b) = b$.

Using $\delta(a)^{\alpha(b)}\gamma(b) = \gamma(a \cdot b)\delta(a^b)$, we get

$$\begin{aligned} a^{\lambda(s+2)+(2t+1)r} &= \gamma(a \cdot b)\delta(a^b) = \delta(a)^b\gamma(b) \\ &= (a^r)^b a^\lambda = a^{2t+1+(r-1)s+\lambda} \end{aligned}$$

which implies that

$$\lambda(s+1) \equiv (r-1)(s-2t-1) \pmod{m}.$$

Therefore

$$\lambda(lu2^{n-2}) \equiv 2^{n-2}q(r-1)\left(\frac{l-2q'}{d}\right) \pmod{2^n q},$$

which implies that

$$\lambda l \equiv d(r-1)\left(\frac{l-2q'}{d}\right) \pmod{4d}.$$

Now, if $\lambda \equiv 0 \pmod{2d}$, then $r \equiv 3 \pmod{4}$. Again, if $\lambda \equiv 0 \pmod{4d}$, then $r \equiv 1 \pmod{4}$. Thus, in this sub-case, the choices for the maps γ and δ are

$$\gamma_\lambda(b) = a^\lambda \quad \text{and} \quad \delta_r(a) = a^r,$$

where λ is even and $\lambda \equiv 0 \pmod{2d}$, and $r \in U(m)$.

Sub-case (ii): $\alpha(b) = b^3$.

Using $\delta(a)^{\alpha(b)}\gamma(b) = \gamma(a \cdot b)\delta(a^b)$, we get

$$\begin{aligned} a^{\lambda(s+2)+(2t+1)r} &= \gamma(a \cdot b)\delta(a^b) = \delta(a)^{\alpha(b)}\gamma(b) \\ &= (a^r)^{b^3}a^\lambda = a^{4t+2ts+1+(r-1)s+\lambda} \end{aligned}$$

which implies that

$$(\lambda - 2t)(s+1) \equiv (r-1)(s-2t-1) \pmod{m}.$$

Therefore

$$lu2^{n-2}(\lambda - 2t) \equiv 2^{n-2}q(r-1)\left(\frac{l-2q'}{d}\right) \pmod{2^n q},$$

which implies that

$$l(\lambda - 2t) \equiv d(r-1)\left(\frac{l-2q'}{d}\right) \pmod{4d}.$$

Now, if $\lambda \equiv 0 \pmod{2d}$, then $r \equiv 1 \pmod{4}$. Again, if $\lambda \equiv 0 \pmod{4d}$, then $r \equiv 3 \pmod{4}$. Thus, in this sub-case, the choices for the maps γ

and δ are
$$\gamma_\lambda(b) = a^\lambda \quad \text{and} \quad \delta_r(a) = a^r,$$
where λ is even and $\lambda \equiv 0 \pmod{2d}$, and $r \in U(m)$.

Combining both the sub-cases (i) and (ii), we get for all $\alpha \in \mathrm{Aut}(H)$, the choices for the maps γ and δ are

$$\gamma_\lambda(b) = a^\lambda \quad \text{and} \quad \delta_r(a) = a^r,$$

where λ is even and $\lambda \equiv 0 \pmod{2d}$, and $r \in U(m)$. Therefore, using Theorem 3.5,
$$E \simeq \mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big).$$
Also, since $2t(s+1) \not\equiv 0 \pmod{m}$, using Lemma 3.3, $\mathrm{Im}(\beta) = \{1\}$. Thus, B is a trivial group. Hence, using Theorem 3.5,

$$\mathrm{Aut}(G) \simeq E \rtimes B \simeq \mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big).$$

The statement is proved. $\square$

Theorem 3.13 *Let* $m = 8q$, t *be odd, and* $\gcd(t, m) = d$, *where* $q > 1$ *is odd. Then*

$$\mathrm{Aut}(G) \simeq \begin{cases} \big(\mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big)\big) \rtimes \mathbb{Z}_2, & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big), & \text{if } 2t(s+1) \not\equiv 0 \pmod{m} \end{cases}$$

Proof — Let $q = du$, for some odd integer u. Then using (G2), we have $s \equiv -1 \pmod{2u}$, which implies that $s = 2lu - 1$, where $1 \leqslant l \leqslant 4d$. Now, using (G1), $l(lu-1) \equiv 0 \pmod{d}$. Using (G3), we get

$$(t+1)(lu-1) \equiv 0 \pmod{2q}. \tag{3.9}$$

Case (i): l is even.

Then by Equation (3.9), $t \equiv lu - 1 \pmod{2q}$. Note that

$$2t(s+1) \equiv 2t(2lu) \equiv 0 \pmod{m} \quad \text{and} \quad \lambda(s+1) = \lambda(2lu).$$

Thus $\lambda(s+1) \equiv 0 \pmod{m}$ if and only if $\lambda l \equiv 0 \pmod{4d}$ which is true for all $\lambda \equiv 0 \pmod{2d}$. Thus, using a similar argument as in the proof of Theorem 3.6, we get $E \simeq \mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big)$ and $B \simeq \mathbb{Z}_2$. Hence, by Theorem 3.5, $\mathrm{Aut}(G) \simeq E \rtimes B \simeq \big(\mathbb{Z}_{\frac{m}{2d}} \rtimes \big(\mathbb{Z}_2 \times U(m)\big)\big) \rtimes \mathbb{Z}_2$.

Case (ii): l is odd.

Then using Equation (3.9), one can easily observe that $\gcd(l, d) = 1$ which means that $lu - 1 = dl'$, where $1 \leqslant l' \leqslant 8u$ and $\gcd(l', u) = 1$. Since $lu - 1$ is even, l' is even. Thus using Equation (3.9), we have that

$$(t+1)dl' \equiv 0 \pmod{2q}.$$

Since $\gcd(l', u) = 1$, $t = uq' - 1$, where $1 \leqslant q' \leqslant 8d$ and q' is even, as t is odd. Now,

$$\begin{aligned} s - 2t - 1 &= 2dl' - 2t = 2d\left(l' - \tfrac{t}{d}\right) \\ &= 2d\left(\tfrac{ul - uq'}{d}\right) = 2du\tfrac{l-q'}{d} \end{aligned}$$

Let $(\alpha, \gamma, \delta) \in X$ be such that $\alpha(b) = b^i$, $\gamma(b) = a^\lambda$ and $\delta(a) = a^r$, where $i \in \{1, 3\}$, $0 \leqslant \lambda \leqslant m - 1$, λ is even and $r \in U(m)$. We consider two sub-cases based on the image of the map α.

Sub-case (i): $\alpha(b) = b$.

Then

$$\begin{aligned} a^{\lambda(s+2)+(2t+1)r} &= \gamma(a \cdot b)\delta(a^b) = \delta(a)^b\gamma(b) \\ &= (a^r)^b a^\lambda = a^{2t+1+(r-1)s+\lambda} \end{aligned}$$

which implies that

$$\lambda(s+1) \equiv (r-1)(s-2t-1) \pmod{m}.$$

Therefore

$$\lambda(2lu) \equiv 2du(r-1)\left(\frac{l-q'}{d}\right) \pmod{8q},$$

which implies that

$$\lambda(l) \equiv d(r-1)\left(\frac{l-q'}{d}\right) \pmod{4d}.$$

Now, if $\lambda \equiv 0 \pmod{2d}$, then $r \equiv 3 \pmod{4}$. Again, if $\lambda \equiv 0 \pmod{4d}$, then $r \equiv 1 \pmod{4}$. Thus, in this sub-case, the choices for the maps γ and δ are

$$\gamma_\lambda(b) = a^\lambda \quad \text{and} \quad \delta_r(a) = a^r,$$

where λ is even and $\lambda \equiv 0 \pmod{2d}$, and $r \in U(m)$.

Sub-case (ii): $\alpha(b) = b^3$.
Then

$$\begin{aligned} a^{\lambda(s+2)+(2t+1)r} &= \gamma(a \cdot b)\delta(a^b) = \delta(a)^{b^3}\gamma(b) \\ &= (a^r)^{b^3} a^\lambda = a^{4t+2ts+1+(r-1)s+\lambda}, \end{aligned}$$

which implies that

$$(\lambda - 2t)(s+1) \equiv (r-1)(s-2t-1) \pmod{m}.$$

Therefore

$$2lu(\lambda - 2t) \equiv 2du(r-1)\left(\frac{l-q'}{d}\right) \pmod{8q},$$

which implies that

$$l(\lambda - 2t) \equiv d(r-1)\left(\frac{l-q'}{d}\right) \pmod{4d}.$$

Now, if $\lambda \equiv 0 \pmod{2d}$, then $r \equiv 1 \pmod{4}$. Again, if $\lambda \equiv 0 \pmod{4d}$, then $r \equiv 3 \pmod{4}$. Thus, in this sub-case, the choices for the maps γ and δ are

$$\gamma_\lambda(b) = a^\lambda \quad \text{and} \quad \delta_r(a) = a^r,$$

where λ is even and $\lambda \equiv 0 \pmod{2d}$, and $r \in U(m)$.

Combining both the sub-cases (i) and (ii), we get for all $\alpha \in \mathrm{Aut}(H)$, the choices for the maps γ and δ are $\gamma_\lambda(b) = a^\lambda$ and $\delta_r(a) = a^r$, where λ is even and $\lambda \equiv 0 \pmod{2d}$, and $r \in U(m)$. Therefore, using Theorem 3.5,

$$X \simeq \mathbb{Z}_{\frac{m}{2d}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right).$$

Also, since $2t(s+1) \not\equiv 0 \pmod{m}$, using Lemma 3.3, $\mathrm{Im}(\beta) = \{1\}$. Thus, B is a trivial group. Hence, by Theorem 3.5,

$$\mathrm{Aut}(G) \simeq E \rtimes B \simeq \mathbb{Z}_{\frac{m}{2d}} \rtimes \left(\mathbb{Z}_2 \times U(m)\right).$$

The statement is proved. $\square$

4 Automorphisms of Zappa–Szép products of groups $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_m$, p odd prime

In [12], Yacoub classified the groups which are Zappa–Szép products of cyclic groups of order m and order p^2, where p is an odd prime. He found that these are of the following type (see [12, Conclusion, p. 38])

$$\begin{aligned} M_1 =& \langle a, b \mid a^m = 1 = b^{p^2}, ab = ba^u, u^{p^2} \equiv 1 \pmod{m} \rangle, \\ M_2 =& \langle a, b \mid a^m = 1 = b^{p^2}, ab = b^t a, t^m \equiv 1 \pmod{p^2} \rangle, \\ M_3 =& \langle a, b \mid a^m = 1 = b^{p^2}, ab = b^t a^{pr+1}, a^p b = ba^{p(pr+1)} \rangle, \end{aligned}$$

and in M_3, p divides m. These are not non-isomorphic classes. The groups M_1 and M_2 may be isomorphic to the group M_3 depending on the values of m, r and t. Clearly, M_1 and M_2 are semidirect products. Throughout this section G will denote the group M_3 and we will be only concerned about groups M_3 which are Zappa–Szép products but not a semidirect product. Note that $G = H \bowtie K$, where $H = \langle b \rangle$ and $K = \langle a \rangle$. For the group G, the mutual actions of H and K are defined by

$$a \cdot b = b^t, \ a^b = a^{pr+1}, \ a^p \cdot b = b, \ (a^p)^b = a^{p(pr+1)},$$

where t and r are integers satisfying the conditions:

(G1) $\gcd(t-1, p^2) = p$, that is, $t = 1 + \lambda p$, where $\gcd(\lambda, p) = 1$,

(G2) $\gcd(r, p) = 1$,

(G3) $p(pr+1)^p \equiv p \pmod{m}$.

Lemma 4.1 $a^{(pr+1)^{ip\lambda}} = a^{i((pr+1)^{p\lambda}-1)+1}$, *for all* i.

Proof — One can easily prove the result using (G3). □

Lemma 4.2 (i) $a \cdot b^j = b^{jt}$, *for all* j,

(ii) $a^l \cdot b = b^{1+lp\lambda}$, *for all* l,

(iii) $a^{(b^j)} = a^{(pr+1)^j}$, *for all* j,

(iv) $(a^l)^b = a^{\frac{l(l-1)}{2}((pr+1)^{\lambda p}-1)+l(pr+1)}$, *for all* l,

(v) $a^l \cdot b^j = b^{jt^l}$, *for all* j *and* l,

(vi) $(a^l)^{b^j} = a^{\frac{jl(l-1)}{2}((pr+1)^{\lambda p}-1)+l(pr+1)^j}$, *for all* j *and* l.

Proof — (i) Using (C3) and (C5),

$$\begin{aligned} a \cdot b^2 = (a \cdot b)(a^b \cdot b) = b^t(a^{pr+1} \cdot b) &= b^t\big(a \cdot (a^{pr} \cdot b)\big) \\ &= b^t(a \cdot b) = b^{2t}. \end{aligned}$$

Similarly,

$$\begin{aligned} a \cdot b^3 &= (a \cdot b)(a^b \cdot b^2) = b^t(a^{pr+1} \cdot b^2) \\ &= b^t\big(a \cdot (a^{pr} \cdot b^2)\big) = b^t(a \cdot b^2) = b^{3t}. \end{aligned}$$

Inductively, we get $a \cdot b^j = b^{jt}$, for all j.

(ii) Using (C3) and part (i),

$$\begin{aligned} a^2 \cdot b &= a \cdot (a \cdot b) = a \cdot b^t \\ &= b^{t^2} = b^{1+2p\lambda}. \end{aligned}$$

Similarly,

$$\begin{aligned} a^3 \cdot b &= a \cdot (a^2 \cdot b) = a \cdot b^{t^2} \\ &= b^{t^3} = b^{1+3p\lambda}. \end{aligned}$$

Inductively, we get $a^l \cdot b = b^{1+lp\lambda}$, for all l.

(iii) First, note that, using (C4), we have

$$(a^{lp})^b = a^{lp(pr+1)}.$$

Now, using (C4) and (C6),

$$\begin{aligned} a^{(b^2)} = (a^b)^b = (a^{pr+1})^b &= a^{(a^{pr} \cdot b)}(a^{pr})^b \\ &= a^b a^{pr(pr+1)} = a^{(pr+1)^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} a^{(b^3)} = (a^b)^{b^2} = (a^{pr+1})^{b^2} = a^{(a^{pr} \cdot b^2)}(a^{pr})^{b^2} = a^{b^2}\big((a^{pr})^b\big)^b \\ = a^{(pr+1)^2}\big(a^{pr(pr+1)}\big)^b = a^{(pr+1)^2}a^{pr(pr+1)^2} = a^{(pr+1)^3}. \end{aligned}$$

Inductively, we get $a^{(b^j)} = a^{(pr+1)^j}$, for all j.

(iv) Using (C4), (G3) and part (iii), we get

$$\begin{aligned}(a^2)^b &= a^{(a\cdot b)}a^b = a^{(b^t)}a^{pr+1} = a^{(pr+1)^{(1+\lambda p)}}a^{pr+1}\\ &= a^{(pr+1)^{\lambda p}+pr(pr+1)^{\lambda p}+pr+1} = a^{((pr+1)^{\lambda p}-1)+2(pr+1)}.\end{aligned}$$

Using a similar argument, we get

$$\begin{aligned}(a^3)^b &= (a^2)^{(a\cdot b)}a^b = (a^2)^{b^t}a^{pr+1} = a^{(a\cdot b^t)}a^{(b^t)}a^{pr+1}\\ &= a^{(b^{1+2p\lambda})}a^{(b^{1+\lambda p})}a^{pr+1} = a^{(pr+1)^{1+2p\lambda}+(pr+1)^{1+p\lambda}+pr+1}\\ &= a^{(pr+1)^{2p\lambda}+pr(pr+1)^{2p\lambda}+(pr+1)^{p\lambda}+pr(pr+1)^{p\lambda}+pr+1}\\ &= a^{2((pr+1)^{p\lambda}-1)+1+pr+(pr+1)^{p\lambda}+pr+pr+1} \quad \text{(using Lemma 4.1)}\\ &= a^{3((pr+1)^{p\lambda}-1)+3(pr+1)}.\end{aligned}$$

Inductively, we get (iv).

(v) Follows inductively, using parts (i) and (ii).

(vi) Follows inductively, using parts (iii) and (iv). □

Lemma 4.3 *If for all* $l \neq 0$, $(pr+1)^{pl} \not\equiv 1 \pmod m$, *then*

(i) $\mathrm{Im}(\gamma) \subseteq \langle a^p \rangle$,

(ii) $\alpha \in \mathrm{Aut}(H)$.

Proof — (i) Let $\alpha(b)=b^i$ and $\gamma(b)=a^\mu$. Then using (A1) and Lemma 4.2 (v),

$$\alpha(b^2) = \alpha(b)\big(\gamma(b)\cdot\alpha(b)\big) = b^i(a^\mu\cdot b^i) = b^{i(1+t^\mu)}.$$

Inductively, we get

$$\begin{aligned}\alpha(b^u) &= b^{i(1+t^\mu+t^{2\mu}+\ldots+t^{(u-1)\mu})}\\ &= b^{i(1+(1+p\mu\lambda)+(1+2p\mu\lambda)+\ldots+(1+(u-1)p\mu\lambda))} = b^{i\left(u+\frac{u(u-1)}{2}p\mu\lambda\right)}\end{aligned}$$

for all $0 \leqslant u \leqslant p^2-1$. Now, using (A2) and Lemma 4.2 (vi),

$$\begin{aligned}\gamma(b^2) &= \gamma(b)^{\alpha(b)}\gamma(b) = (a^\mu)^{b^i}a^\mu\\ &= a^{\frac{i\mu(\mu-1)}{2}((pr+1)^{p\lambda}-1)+\mu(pr+1)^i+\mu}.\end{aligned}$$

Inductively, we get

$$\gamma(b^u) = a^{\left(i\frac{u(u-1)\mu(\mu-1)}{2}+i\mu^2\frac{u(u-1)(u-2)}{6}\right)((pr+1)^{p\lambda}-1)+\mu\sum_{\nu=0}^{u-1}(pr+1)^{i\nu}}$$

for all $0 \leqslant u \leqslant p^2-1$. Now, using (G3),

$$1 = \gamma\left(b^{p^2}\right) = a^{\mu\sum_{\nu=0}^{p^2-1}(pr+1)^{i\nu}} = a^{\mu\left(\frac{(pr+1)^{ip^2}-1}{(pr+1)^i-1}\right)},$$

which implies that

$$\mu\left(\frac{(pr+1)^{ip^2}-1}{(pr+1)^i-1}\right) \equiv 0 \pmod{m}. \tag{4.1}$$

If for all $l \neq 0$, $(pr+1)^{pl} \equiv 1 \pmod{m}$, then by Equation (4.1), μ can be anything. If for all $l \neq 0$, $(pr+1)^{pl} \not\equiv 1 \pmod{m}$, then by Equation (4.1) and (G3), $\mu \equiv 0 \pmod{p}$. Also, note that, in both the cases, namely

$$(pr+1)^{pl} \equiv 1 \pmod{m} \quad \text{and} \quad (pr+1)^{pl} \not\equiv 1 \pmod{m},$$

we have that

$$\gamma(b^u) = a^{\mu\sum_{\nu=0}^{u-1}(pr+1)^{i\nu}}.$$

Hence, if $(pr+1)^{pl} \not\equiv 1 \pmod{m}$, then $\gamma(b^u) = a^{\mu\sum_{\nu=0}^{u-1}(pr+1)^{i\nu}}$ belongs to $\langle a^p\rangle$.

(ii) Follows immediately using part (i). □

Lemma 4.4 *Let* $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{A}$. *Then, if* $\beta \in Q$, *then:*

(i) $\beta \in \mathrm{Hom}(K,H)$ *and* $\mathrm{Im}(\beta) \leqslant \langle b^p\rangle$;

(ii) $l(pr+1)^j \equiv l \pmod{m}$, *for all* l;

(iii) $\gamma(h)\cdot\beta(k) = \beta(k)$ *and* $\gamma(h)^{\beta(k)} = \gamma(h)$, *for all* $h \in H$ *and* $k \in K$;

(iv) $\gamma\beta = 0$, *where* 0 *is the trivial homomorphism in* $\mathrm{Hom}(K,K)$;

(v) $\gamma\beta + \delta \in \mathrm{Aut}(K)$ *and* $\gamma\beta+\delta \in S$;

(vi) $\beta\gamma \in \mathrm{Hom}(H,H)$;

(vii) $\alpha+\beta\gamma \in \mathrm{Aut}(H)$ *and* $\alpha+\beta\gamma \in P$.

PROOF — Let $\beta(a) = b^j$. Then using (A3),

$$\beta(a^2) = \beta(a)\big(a \cdot \beta(a)\big) = b^j(a \cdot b^j) = b^{j+jt}.$$

Inductively, we get

$$\begin{aligned}\beta(a^l) = b^{j(1+t+t^2+\ldots+t^{l-1})} &= b^{j(1+(1+\lambda p)+(1+2\lambda p)+\ldots+(1+(l-1)\lambda p))} \\ &= b^{j\left(l+\lambda p\frac{l(l-1)}{2}\right)}.\end{aligned}$$

(i) Since $\beta \in Q$, $\beta(k^h) = \beta(k)$. Therefore

$$b^j = \beta(a) = \beta(a^b) = \beta(a^{pr+1}) = b^{j(pr+1)},$$

which implies that

$$jpr + j \equiv j \pmod{p^2}.$$

Since $\gcd(r,p) = 1$, $j \equiv 0 \pmod p$. Thus, $\beta(a^l) = b^{jl} \in \langle b^p \rangle$, for all l. Hence, One can easily observe that β is a group homomorphism and $\mathrm{Im}(\beta) \leqslant \langle b^p \rangle$.

(ii) Since $\beta \in Q$, $k^{\beta(k')} = k$. Therefore, using Lemma 4.2 (vi),

$$a^l = (a^l)^{\beta(a)} = (a^l)^{b^j} = a^{\frac{jl(l-1)}{2}((pr+1)^{\lambda p}-1)+l(pr+1)^j}.$$

Now, using part (i) and (G3), we get $l(pr+1)^j \equiv l \pmod m$, for all l.

(iii) First, note that $a^l \cdot b^p = b^p$ and using part (ii), $(a^l)^{b^p} = a^l$, for all l. Hence, the result follows using part (i).

(iv) Using Lemma 4.3 (i), we have

$$\gamma(b^u) = a^{\mu\sum_{\nu=0}^{u-1}(pr+1)^{i\nu}},$$

for all u. Then, using part (ii), for all l, we get

$$\gamma\beta(a^l) = \gamma(b^{lj}) = a^{\mu\sum_{\nu=0}^{lj-1}(pr+1)^{i\nu}} = a^{\mu\left(\frac{(pr+1)^{ijl}-1}{(pr+1)^i-1}\right)} = 1.$$

Thus, $\gamma\beta = 0$.

(v) Follows directly using part (iv).

(vi) Using $\beta(k^h) = \beta(k)$ and part (i),

$$\begin{aligned}\beta\gamma(hh') &= \beta(\gamma(h)^{\alpha(h')}\gamma(h')) \\ &= \beta(\gamma(h)^{\alpha(h')})\beta(\gamma(h')) = \beta(\gamma(h))\beta(\gamma(h')).\end{aligned}$$

Hence, $\beta\gamma \in \mathrm{Hom}(K, K)$.

(vii) Using Lemma 4.3 (i), we have $\gamma(b^u) = a^{\mu \sum_{\nu=0}^{u-1} (pr+1)^{i\nu}}$, for all u. Also, using part (i), we have $\beta\gamma(b^u) = b^{uj\mu}$, for all u. Therefore,

$$(\alpha + \beta\gamma)(b^u) = b^{u(i+j\mu+p\mu\lambda\frac{u-1}{2})}.$$

Now, one can easily observe that $\alpha + \beta\gamma$ is a bijection. Hence, using part (vi), $\alpha + \beta\gamma \in \mathrm{Aut}(H)$.

Now, using part (i), (C5) and (C6),

$$\begin{aligned}k \cdot (\alpha + \beta\gamma)(h) &= k \cdot \alpha(h)\beta\gamma(h) = (k \cdot \alpha(h))\big(k^{\alpha(h)} \cdot \beta\big(\gamma(h)\big)\big) \\ &= \alpha(k \cdot h)\beta(\gamma(h)) = \alpha(k \cdot h)\beta\gamma(k \cdot h) = (\alpha + \beta\gamma)(k \cdot h)\end{aligned}$$

and

$$k^{(\alpha+\beta\gamma)(h)} = k^{\alpha(h)\beta\gamma(h)} = (k^{\alpha(h)})^{\beta\gamma(h)} = k^{\alpha(h)} = k^h.$$

Hence, $\alpha + \beta\gamma \in P$. □

Note that, using Lemma 4.4 (iii), multiplication in the group $\mathcal{A}$ reduces to the usual multiplication of matrices.

Theorem 4.5 *Let A, B, C, D be defined as above. Then $\mathrm{Aut}(G) = ABCD$.*

Proof — Using Lemma 4.4 (vii), $\alpha + \beta\gamma \in P$. In particular, $1 - \beta\gamma \in P$. Therefore, by Theorem 2.4, we have $\mathrm{Aut}(G) = ABCD$. □

Theorem 4.6 *Let G be as above. Then*

$$\mathrm{Aut}(G) \simeq \begin{cases} \big(\mathbb{Z}_m \rtimes (\mathbb{Z}_p \times \widetilde{D})\big) \rtimes \mathbb{Z}_p, & \textit{if } (pr+1)^p \equiv 1 \pmod m \\ \big(\mathbb{Z}_{\frac{m}{p}} \rtimes (\mathbb{Z}_p \times \widetilde{D})\big) \rtimes \mathbb{Z}_p, & \textit{if } (pr+1)^p \not\equiv 1 \pmod m \end{cases},$$

where $\widetilde{D}$ is a subgroup of $U(m)$ of order $\frac{\phi(m)}{p-1}$.

Proof — Let $\beta \in Q$. Using Lemma 4.4 (i), we have that $\beta(a^l) = b^{jl}$, where $j \equiv 0 \pmod p$. Thus, $B \simeq \mathbb{Z}_p$. Now, let $(\alpha, \gamma, \delta) \in X$ be such

that
$$\alpha(b) = b^i, \quad \gamma(b) = a^\mu \quad \text{and} \quad \delta(a) = a^s,$$
where $i \in \mathbb{Z}_{p^2}$, $\gcd(i, p^2) = 1$, $0 \leqslant \mu \leqslant m-1$, and $s \in U(m)$. Then using $\alpha(hh') = \alpha(h)\big(\gamma(h) \cdot \alpha(h')\big)$, $\gamma(hh') = \gamma(h)^{\alpha(h')}\gamma(h')$ and Lemma 4.3 (i), we have

$$\alpha(b^u) = b^{i\left(u + \frac{u(u-1)}{2} p\mu\lambda\right)} \text{ and } \gamma(b^u) = a^{\mu \sum_{\nu=0}^{u-1} (pr+1)^{i\nu}}. \tag{4.2}$$

Now, using $(\alpha, \gamma, \delta) \in X$, we obtain

$$\delta(k) \cdot \alpha(h) = \alpha(k \cdot h)$$

and
$$b^{it} = \alpha(b^t) = \alpha(a \cdot b) = \delta(a) \cdot \alpha(b) = a^s \cdot b^i = b^{it^s}.$$
Thus, $it^s \equiv it \pmod{p^2}$ which implies that $(1 + p\lambda)^{s-1} \equiv 1 \pmod{p^2}$. Therefore,
$$s \equiv 1 \pmod{p}.$$
Using $(\alpha, \gamma, \delta) \in X$, $\delta(k)^{\alpha(h)}\gamma(h) = \gamma(k \cdot h)\delta(k^h)$, (G3) and the fact that $s \equiv 1 \pmod{p}$, we get

$$\begin{aligned} a^{\mu \sum_{\nu=0}^{t-1} (pr+1)^{i\nu} + s(pr+1)} &= \gamma(b^t)\delta(a^{pr+1}) = \gamma(a \cdot b)\delta(a^b) \\ = \delta(a)^{\alpha(b)}\gamma(b) &= (a^s)^{b^i} a^\mu = a^{\frac{is(s-1)}{2}((pr+1)^{\lambda p} - 1) + s(pr+1)^i} a^\mu \\ &= a^{s(pr+1)^i + \mu}. \end{aligned}$$

Thus $\mu \sum_{\nu=0}^{t-1} (pr+1)^{i\nu} + s(pr+1) \equiv s(pr+1)^i + \mu \pmod{m}$. Therefore,

$$\begin{aligned} \mu + s(pr+1)^i &\equiv \mu\left(\tfrac{(pr+1)^{it} - 1}{(pr+1)^i - 1}\right) + s(pr+1) \pmod{m} \\ &\equiv \mu\left(\tfrac{(pr+1)^{i(1+p\lambda)} - 1}{(pr+1)^i - 1}\right) + s(pr+1) \pmod{m} \\ &\equiv \mu\left(\tfrac{(pr+1)^i (pr+1)^{ip\lambda} - 1}{(pr+1)^i - 1}\right) + s(pr+1) \pmod{m}. \end{aligned}$$

We consider two cases, namely

$$(pr+1)^p \equiv 1 \pmod{m} \quad \text{and} \quad (pr+1)^p \not\equiv 1 \pmod{m}.$$

Case (i): $(pr+1)^p \equiv 1 \pmod{m}$.

Then
$$\mu + s(pr+1)^i \equiv \mu + s(pr+1) \pmod{m},$$
which implies that $i \equiv 1 \pmod{p}$. Thus in this case, the choices for the maps α, γ and δ are

$$\alpha_i(b) = b^i, \quad \gamma_\mu(b) = a^\mu, \quad \text{and} \quad \delta_s(a) = a^s,$$

where $i \in U(p^2)$, $i \equiv 1 \pmod{p}$, $0 \leqslant \mu \leqslant m-1$, $s \in U(m)$, and $s \equiv 1 \pmod{p}$.

Case (ii): $(pr+1)^p \not\equiv 1 \pmod{m}$.

Then using Lemma 4.3, $\mu \equiv 0 \pmod{p}$. Therefore

$$\mu + s(pr+1)^i \equiv \mu + s(pr+1) \pmod{m},$$

which implies that $i \equiv 1 \pmod{p}$. Thus in this case, the choices for the maps α, γ and δ are

$$\alpha_i(b) = b^i, \quad \gamma_\mu(b) = a^\mu, \quad \text{and} \quad \delta_s(a) = a^s,$$

where $i \in U(p^2)$, $i \equiv 1 \pmod{p}$, $0 \leqslant \mu \leqslant m-1$, $\mu \equiv 0 \pmod{p}$, $s \in U(m)$ and $s \equiv 1 \pmod{p}$.

From both the cases (i) and (ii), we observe that for all μ,

$$i \equiv 1 \pmod{p} \quad \text{and} \quad s \equiv 1 \pmod{p}.$$

Using these conditions, first, we find the structure of $\mathrm{Aut}(G)$.

Since $A \times D$ normalizes C, we have that M normalizes C. So, clearly, $C \triangleleft E$ and $M \cap C = \{1\}$. Now, if $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in E$, then

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta^{-1}\gamma & 1 \end{pmatrix} \in MC$$

Thus $E = C \rtimes M$. Now, using Lemma 4.4 (iii) and (iv), we get

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha + \beta\gamma & (\alpha+\beta\gamma)(-\beta) + \beta\delta \\ \gamma & \delta \end{pmatrix}. \quad (4.3)$$

Using Lemma 4.4 (i) and (ii), we have

$$\big((\alpha+\beta\gamma)(-\beta)+\beta\delta\big)(a) = (\alpha+\beta\gamma)(-\beta)(a)(\beta\delta)(a)$$

$$= (\alpha+\beta\gamma)(b^{-j})\beta(a^{s}) = \alpha(b^{-j})\beta\big(\gamma(b^{-j})\big)b^{sj}$$

$$= b^{-ij}\beta\left(a^{\mu\sum_{\nu=0}^{-j-1}(pr+1)^{i\nu}}\right)b^{sj}$$

$$= b^{j(s-i)}\beta\left(a^{\mu\left(\frac{(pr+1)^{-ij}-1}{(pr+1)^{i}-1}\right)}\right) = \beta(1) = 1.$$

Thus, $(\alpha+\beta\gamma)(-\beta)+\beta\delta = 0$. Also, using Lemma 4.4 (vii), one can easily observe that $(\alpha+\beta\gamma,\gamma,\delta) \in X$. Therefore, by Equation (4.3),

$$\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}\alpha+\beta\gamma & 0\\ \gamma & \delta\end{pmatrix} \in E.$$

Thus $E \lhd \mathcal{A}$. Clearly, $E \cap B = \{1\}$. Now, if $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{A}$, using $\gamma\alpha\beta = 0$, we get

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & \alpha^{-1}\beta\\ 0 & 1\end{pmatrix} \in EB$$

Hence, $\mathcal{A} = E \rtimes B$ and so, $\mathrm{Aut}(G) \simeq E \rtimes B \simeq \big(C \rtimes (A \times D)\big) \rtimes B$.

Thus,

$$X \simeq \mathbb{Z}_m \rtimes (\mathbb{Z}_p \times \widetilde{D}) \quad \text{and} \quad \mathrm{Aut}(G) \simeq \big(\mathbb{Z}_m \rtimes (\mathbb{Z}_p \times \widetilde{D})\big) \rtimes \mathbb{Z}_p$$

in the Case (i), and

$$X \simeq \mathbb{Z}_{\frac{m}{p}} \rtimes (\mathbb{Z}_p \times \widetilde{D}) \quad \text{and} \quad \mathrm{Aut}(G) \simeq \big(\mathbb{Z}_{\frac{m}{p}} \rtimes (\mathbb{Z}_p \times \widetilde{D})\big) \rtimes \mathbb{Z}_p$$

in the Case (ii), where $\widetilde{D}$ is a subgroup of $U(m)$ of order $\frac{\phi(m)}{p-1}$. $\square$

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