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Finite Almost Simple Groups whose Holomorph Contains a Solvable Regular Subgroup

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Abstract

In our previous paper, we gave a complete list of the finite non-abelian simple groups whose holomorph contains a solvable regular subgroup. In this paper, we refine our previous work by considering all finite almost simple groups. In particular, our theorem yields a complete characterization of the finite almost simple groups which occur as the type of a Hopf–Galois structure on a solvable extension, or equivalently, the additive group of a skew brace having a solvable multiplicative group.

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1 Introduction

Let G and N be two finite groups of the same order. We shall say that the pair (G, N) is *realizable* if the holomorph of N contains a regular subgroup that is isomorphic to G. Recall that the *holomorph of* N is defined to be

$$\mathrm{Hol}(N) = \rho(N) \rtimes \mathrm{Aut}(N) = \lambda(N) \rtimes \mathrm{Aut}(N)$$

as a subgroup of the symmetric group of N, where λ and ρ are the left and right regular representations of N. By [2, 5] and [6], respectively, it is known that (G, N) is realizable if and only if there exists

a Hopf–Galois structure of type N on a Galois G-extension of fields, or equivalently, a skew brace with additive group N and multiplicative group isomorphic to G. This is exactly our motivation of studying realizability (see [4] and [7] for more details).

For the pair (G, N) to be realizable, it is conjectured that if G is insolvable then N is also insolvable (see [3, 9] for some progress on this). On the other hand, the converse is known to be false — there exist realizable pairs (G, N) for which G is solvable but N is insolvable. It seems to be a natural problem to try and determine the insolvable groups N that can occur in such a pair. In [8, Theorems 1.2 and 1.3], we proved the following.

Theorem 1 *Let N be any finite almost simple group. If (G, N) is realizable for some solvable group G, then the socle of N is isomorphic to*

(1) $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(8)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$, M_{11}; *or*

(2) $\mathrm{PSL}_2(q)$ *with $q \neq 2, 3$ a prime power.*

The converse is also true when N is a finite non-abelian simple group.

The purpose of this paper is to study the converse of Theorem 1 for an arbitrary finite almost simple group N that is not necessarily simple. It turns out that the proposition below (see [8, Corollary 2.2 and Proposition 2.7]) is enough to do the job apart from the case in which $N = \mathrm{PSU}_3(8)$. This case has already been dealt with in [8] and so may be ignored. In the following, $\mathrm{Inn}(N)$ denotes the inner automorphism group of N.

Proposition 2 *Let N be any finite group.*

(a) *If (G, N) is realizable for a solvable group G, then there exists a subgroup*

$$\mathrm{Inn}(N) \leqslant P \leqslant \mathrm{Aut}(N)$$

such that $P = AB$ is factorized by some solvable subgroups A, B that satisfy the equality $A \cdot \mathrm{Inn}(N) = B \cdot \mathrm{Inn}(N)$.

(b) *If N has trivial center and there exists a subgroup*

$$\mathrm{Inn}(N) \leqslant P \leqslant \mathrm{Aut}(N)$$

such that $P = AB$ is factorized by some solvable subgroups A, B that satisfy $A \cap B = 1$, $A \cdot \mathrm{Inn}(N) = B \cdot \mathrm{Inn}(N)$, and A splits over the intersection $A \cap \mathrm{Inn}(N)$, then (G, N) is realizable for some solvable group G.

The converse of Theorem 1 does not hold in general, but we have a very nice criterion and here is our main result.

Theorem 3 *Let* N *be a finite almost simple group whose socle* $\mathrm{Soc}(N)$ *is isomorphic to one of the non-abelian simple groups in Theorem 1. There is a solvable group* G *such that* (G, N) *is realizable if and only if one of the following conditions holds:*

(1) $\mathrm{Soc}(N) \simeq \mathrm{PSL}_2(q)$ *with* $q \neq 2, 3$ *a prime power;*

(2) $\mathrm{Soc}(N) \simeq \mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(8)$, $\mathrm{PSU}_4(2)$, M_{11}*;*

(3) $\mathrm{Soc}(N) \simeq \mathrm{PSL}_3(4)$ *with* $[N : \mathrm{Soc}(N)] \neq 2, 4$*;*

(4) $\mathrm{Soc}(N) \simeq \mathrm{PSL}_3(4)$ *with* $[N : \mathrm{Soc}(N)] = 2$ *and* $N \trianglelefteq \mathrm{Aut}(\mathrm{Soc}(N))$*;*

(5) $\mathrm{Soc}(N) \simeq \mathrm{PSU}_3(8)$ *with* $[N : \mathrm{Soc}(N)] \neq 2, 6$*;*

(6) $\mathrm{Soc}(N) \simeq \mathrm{PSU}_3(8)$ *with* $[N : \mathrm{Soc}(N)] = 6$ *and* $N \trianglelefteq \mathrm{Aut}(\mathrm{Soc}(N))$.

Remark 4 Our result yields a complete classification of the finite almost simple groups that can occur as the type of a Hopf–Galois structure on a solvable extension, or equivalently, the additive group of a skew brace having a solvable multiplicative group.

By definition, an almost simple group N corresponds to a subgroup of the outer automorphism group of its socle $\mathrm{Soc}(N)$. Let us note that

$$\mathrm{Out}(\mathrm{PSL}_3(3)) \simeq \mathrm{C}_2,\ \ \mathrm{Out}(\mathrm{PSL}_3(4)) \simeq \mathrm{D}_{12},\ \ \mathrm{Out}(\mathrm{PSL}_3(8)) \simeq \mathrm{C}_6,$$

$$\mathrm{Out}(\mathrm{PSU}_3(8)) \simeq \mathrm{C}_3 \times \mathrm{S}_3,\ \ \mathrm{Out}(\mathrm{PSU}_4(2)) \simeq \mathrm{C}_2,\ \ \mathrm{Out}(\mathrm{M}_{11}) \simeq \mathrm{C}_1.$$

Moreover, for a prime power $q = p^f$ with $q \neq 2, 3$, we have

$$\mathrm{Out}(\mathrm{PSL}_2(q)) \simeq \mathrm{C}_{\gcd(2, q-1)} \times \mathrm{C}_f.$$

Here, as usual C_n is the cyclic group of order n, D_{2n} is the dihedral group of order $2n$, and S_n is the symmetric group on n letters (see for example [10]).

Therefore, the converse of Theorem 1 holds with only five exceptions of N, as follows. Here, only the isomorphism class of N matters, so it suffices to consider the subgroups of $\mathrm{Out}(\mathrm{Soc}(N))$ up to conjugacy.

- The subgroups $\mathrm{PSL}_3(4) \leqslant N \leqslant \mathrm{Aut}(\mathrm{PSL}_3(4))$ corresponding to

$$\langle s \rangle, \ \langle rs \rangle, \ \langle r^3, s \rangle.$$

  Note that $\langle r^3 \rangle$ is also a subgroup of D_{12} of order 2, but this is normal and so falls under case (4) of Theorem 3. Here r and s are the usual generators of D_{12} with $r^6 = s^2 = 1$ and $srs = r^{-1}$.

- The subgroups $\mathrm{PSU}_3(8) \leqslant N \leqslant \mathrm{Aut}(\mathrm{PSU}_3(8))$ corresponding to

$$\{1\} \times \langle (1\,2) \rangle \text{ and } C_3 \times \langle (1\,2) \rangle.$$

  Note that $\{1\} \times S_3$ is also a subgroup of $C_3 \times S_3$ of order 6, but this is normal and so falls under case (6) of Theorem 3.

We now move on and use Proposition 2 to prove Theorem 3. We use the help of Magma [1] except for the infinite family $\mathrm{PSL}_2(q)$ with $q \neq 2, 3$.

2 Proof of cases (2) to (6)

We ran the following code in Magma [1].

```
T:=(the non-abelian simple group in question);
AutT:=PermutationGroup(AutomorphismGroup(T));
InnT:=Socle(AutT);
NN:=[N`subgroup:N in Subgroups(AutT)|InnT subset N`subgroup];
//A list of the finite almost simple groups N with socle T
//up to conjugacy class in Aut(T).
for N in NN do
  AutN:=Normalizer(AutT,N);
  //We identify Aut(N) with the normalizer of N in Aut(T)
  //and in particular Inn(N) with N.
  PP:=[P`subgroup:P in Subgroups(AutN)|N subset P`subgroup];
  //A list of the subgroups P between Inn(N) and Aut(N)
  //up to conjugacy class in Aut(N).
  PPa:=[]; PPb:=[];
  //A list to contain the P that satisfies the conditions
  //in Proposition 2(a) and 2(b), respectively.
  for P in PP do
    SolSub:=[S`subgroup:S in SolvableSubgroups(P)];
    if exists{<A,B>:A,B in SolSub|
```

```
        #P eq #A*#B/#(A meet B) and
        sub<P|A,N> eq sub<P|B,N>} then
      Append(~PPa,P);
    end if;
    if exists{<A,B>:A,B in SolSub|
        #(A meet B) eq 1 and
        #P eq #A*#B and
        sub<P|A,N> eq sub<P|B,N> and
        HasComplement(A,A meet N)} then
      Append(~PPb,P);
    end if;
  end for;
  if IsEmpty(PPa) then
    conclusion:="false";
    //Proposition 2(a): If PPa is empty then (G,N) is not
    //realizable for any solvable group G.
    //Thus the converse of Theorem 1 is "false" for N.
  elif not IsEmpty(PPb) then
    conclusion:="true";
    //Proposition 2(b): If PPb is non-empty then (G,N) is
    //realizable for some solvable group G.
    //Thus the converse of Theorem 1 is "true" for N.
  else
    conclusion:="inconclusive";
    //Proposition 2 is not sufficient to determine if the
    //converse of Theorem 1 holds for N.
  end if;
  if IsNormal(AutT,N) then
    normalORnot:="normal";
  else
    normalORnot:="not normal";
  end if;
  <#N/#InnT,normalORnot,conclusion>;
end for;
```

For $T = PSL_3(3)$, $PSL_3(8)$, $PSU_4(2)$, M_{11}, the output of `conclusion` is all `true` and this yields case (2) of Theorem 3.

For $T = PSL_3(4)$ and $[N : Soc(N)] \neq 2, 4$, the output of `conclusion` is all `true` and this yields case (3) of Theorem 3.

For $T = PSL_3(4)$ and $[N : Soc(N)] = 2, 4$, the output is as follows:

```
<2, "normal", "true">
<2, "not normal", "false">
<2, "not normal", "false">
```

```
<4, "not normal", "false">
```

We then deduce case (4) of Theorem 3.

For $T = PSU_3(8)$ and $[N : Soc(N)] \neq 1, 2, 6$, the output of `conclusion` is all `true` and this yields case (5) of Theorem 3 except when $N \simeq PSU_3(8)$, which is already covered by Theorem 1. Let us remark that the output of `conclusion` is `inconclusive` in this case and [8] dealt with it using a slightly different method.

For $T = PSU_3(8)$ and $[N : Soc(N)] = 2, 6$, the output is as follows:

```
<2, "not normal", "false">
<6, "normal", "true">
<6, "not normal", "false">
```

We then deduce case (6) of Theorem 3.

3 Proof of case (1)

Let $q = p^f$ be a prime power with $q \neq 2, 3$ and let

$$PSL_2(q) \leqslant N \leqslant Aut(PSL_2(q)).$$

Since $Out(PSL_2(q))$ is abelian, N is normal in $Aut(PSL_2(q))$, so we may identify $Aut(N)$ with $Aut(PSL_2(q))$, and in particular $Inn(N)$ with N. By Proposition 2 (b), it is then enough to show that there is a subgroup

$$N \leqslant P \leqslant Aut(PSL_2(q))$$

for which there exist solvable subgroups A, B of P satisfying

$$P = AB,\ A \cap B = 1,\ AN = BN,\ \text{and } A \text{ splits over } A \cap N. \tag{3.1}$$

Note that the last two conditions are trivial when P is taken to be N, since then $AN = N = BN$ and A obviously splits over $A \cap N = A$.

We may regard $PSL_2(q)$ as a normal subgroup of $PGL_2(q)$ via the natural embedding, and in particular $PGL_2(q)$ as a subgroup of $Aut(PSL_2(q))$ acting on $PSL_2(q)$ via conjugation. It is well-known that

$$Aut(PSL_2(q)) = PGL_2(q) \rtimes F, \tag{3.2}$$

where $F \simeq C_f$ is the subgroup consisting of the automorphisms induced by the field automorphisms of $\mathbb{F}_{p^f}$ over $\mathbb{F}_p$. By the proof of [8, Theorem 1.3], we know that there are solvable subgroups C, D of $\mathrm{PGL}_2(q)$ such that

$$\mathrm{PGL}_2(q) = CD, \;\; C \cap D = 1, \;\; C \cdot \mathrm{PSL}_2(q) = D \cdot \mathrm{PSL}_2(q).$$

Explicitly, we may take $C = \widetilde{C}/Z$ and $D = \widetilde{D}/Z$, where

$$\widetilde{C} = \left\{ \begin{bmatrix} x & -dy \\ y & x - cy \end{bmatrix} : x, y \in \mathbb{F}_q, \, (x, y) \neq (0, 0) \right\}$$

and

$$\widetilde{D} = \left\{ \begin{bmatrix} u & v \\ 0 & w \end{bmatrix} : u, w \in \mathbb{F}_q^\times, \, v \in \mathbb{F}_q \right\}.$$

Here Z denotes the center of $\mathrm{GL}_2(q)$ and $X^2 + cX + d$ is the minimal polynomial over $\mathbb{F}_q$ of some generator of $\mathbb{F}_{q^2}^\times$. Observe that

$$\theta \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} Z \cdot \theta^{-1} = \begin{bmatrix} \theta(x) & \theta(y) \\ \theta(z) & \theta(w) \end{bmatrix} Z$$

in the group (3.2) for all $\theta \in F$ and $\left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right] \in \mathrm{GL}_2(q)$. It in particular implies that D is normalized by F. Thus, we have that $DE = D \rtimes E$ is a subgroup of $\mathrm{Aut}(\mathrm{PSL}_2(q))$ for any subgroup E of F. In what follows, let us take E to be the projection of N onto F along $\mathrm{PGL}_2(q)$, and let

$$P = \mathrm{PGL}_2(q) \rtimes E.$$

Clearly P contains N. For the choices of A, B, we separate into cases.

First suppose that $q \equiv 0 \pmod 4$. Then $\mathrm{PGL}_2(q) = \mathrm{PSL}_2(q)$ and so

$$N = \mathrm{PGL}_2(q) \rtimes E = P.$$

Take $(A, B) = (C, D \rtimes E)$. That A, B are solvable, $P = AB$, and $A \cap B = 1$ are all clear. The last two conditions in (3.1) are trivial since $P = N$.

Next suppose that $q \equiv 1, 3 \pmod 4$. It was shown in [8, Section 4] that

$$\begin{cases} C \text{ splits over } C \cap \mathrm{PSL}_2(q) & \text{when } q \equiv 1 \pmod 4, \\ D \text{ splits over } D \cap \mathrm{PSL}_2(q) & \text{when } q \equiv 3 \pmod 4. \end{cases} \tag{3.3}$$

Here we shall take

$$(A, B) = \begin{cases} (C, D \rtimes E) & \text{when } q \equiv 1 \pmod 4, \\ (D \rtimes E, C) & \text{when } q \equiv 3 \pmod 4. \end{cases}$$

That A, B are solvable, $P = AB$, and $A \cap B = 1$ are all obvious. Since

$$C \cdot \mathrm{PSL}_2(q) = D \cdot \mathrm{PSL}_2(q) \text{ and } \mathrm{PSL}_2(q) \leqslant N,$$

we clearly have $CN = DN$. Note that $\mathrm{PGL}_2(q) = \mathrm{PSL}_2(q)D$ holds because D contains an element lying outside of $\mathrm{PSL}_2(q)$, for example

$$\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} Z, \text{ where } u \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2.$$

By definition, elements of E may be represented as $\pi\eta$ for $\pi \in \mathrm{PGL}_2(q)$ and $\eta \in N$. Since N contains $\mathrm{PSL}_2(q)$, we then deduce that E lies in

$$\mathrm{PGL}_2(q)N = \mathrm{PSL}_2(q)DN = DN.$$

Thus, we have $(D \rtimes E)N = DN = CN$ and this yields $AN = BN$. Finally, we check that A indeed splits over $A \cap N$.

- $q \equiv 1 \pmod 4$: Here $A = C$. Since $A \leqslant \mathrm{PGL}_2(q)$, we have

  $$A \cap N = \begin{cases} A \cap \mathrm{PSL}_2(q) & \text{if } N \cap \mathrm{PGL}_2(q) = \mathrm{PSL}_2(q), \\ A & \text{if } N \cap \mathrm{PGL}_2(q) = \mathrm{PGL}_2(q). \end{cases}$$

  In the former case A splits over $A \cap N$ by (3.3), and in the latter case A certainly splits over $A \cap N = A$.

- $q \equiv 3 \pmod 4$: Here $A = D \rtimes E$. Since $q = p^f$, the exponent f must be odd in this case, and so

  $$\mathrm{Out}(\mathrm{PSL}_2(q)) \simeq C_2 \times C_f \simeq C_{2f}$$

  is cyclic. We then see that there are only two possibilities for N, namely

  $$N = \mathrm{PSL}_2(q) \rtimes E \text{ or } N = \mathrm{PGL}_2(q) \rtimes E.$$

  In the latter case $N = P$ and A certainly splits over $A \cap N = A$.

In the former case, we observe that

$$A \cap N = (D \rtimes E) \cap (\mathrm{PSL}_2(q) \rtimes E) = (D \cap \mathrm{PSL}_2(q)) \rtimes E$$

and in particular

$$[A : A \cap N] = [D : D \cap \mathrm{PSL}_2(q)].$$

Here, instead of applying (3.3) directly, we shall need to use its proof given in [8, Section 4]. It was shown by noting that

$$[D : D \cap \mathrm{PSL}_2(q)] = 2, \ \ |D \cap \mathrm{PSL}_2(q)| = \frac{q(q-1)}{2}$$

and by using the Schur–Zassenhaus theorem. But then we have

$$[A : A \cap N] = 2, \ \ |A \cap N| = \frac{q(q-1)}{2} \cdot |E|.$$

Since $|E|$, being a divisor of f, is odd in this case, we can similarly use the Schur–Zassenhaus theorem to deduce that A splits over $A \cap N$.

In both cases, we have shown that A splits over $A \cap N$ for our choice of A.

We have thus verified the existence of P satisfying (3.1) in all cases. This proves case (1) of Theorem 3.

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