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About Nilpotent Profinite Groups Largely Satisfying a Word Equation

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Abstract

We show that in a nilpotent profinite group, the set of answers of a word equation has nonempty interior, provided that this set has a positive Haar measure. This fact shows that the conjecture proposed by Lévai and Pyber is true for nilpotent profinite groups.

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1 Introduction

The following conjecture was proposed by Lévai and Pyber in [7].

Conjecture 1.1 *The solutions of equation* $x^n = 1$ *in a profinite group* G *has positive Haar measure if and only if it's interior is nonempty.*

This conjecture is Problem 14.53 in [8]. In that problem, it is pointed out that it would be interesting to see whether similar

results hold for profinite groups in which the set of solutions of some word equation has positive measure.

In [1], the authors have shown that the correctness of this conjecture results in the existence of the universal upper bound on the index of generalized Hughes–Thompson subgroup of finite groups whenever it is non-trivial. It is known that the latter is widely open even for all primes $n = p > 5$. Recall that, for any finite group G and any positive integer n, the generalized Hughes–Thompson subgroup is defined as the subgroup

$$\langle x \in G : x^n \neq 1 \rangle$$

of G (see Chapter 7 of [6]).

So far, we know that Conjecture 1.1 holds for $n = 2$ and $n = 3$ (see [7] and [1]); in these two cases, first it is shown that the profinite group G is virtually nilpotent and then the correctness of the conjecture is proven for nilpotent groups. In this article, we show that if a nilpotent profinite group that satisfies a word equation on a set of positive Haar measure, then this set should have nonempty interior (see Theorem 3.1). We also show that if G is a compact group satisfies the equation $x^n = 1$ on a set of positive Haar measure, then $Z(G)$ and G/G' are of finite exponent, where $Z(G)$ and G' stands for the center and commutator subgroup of G, respectively.

2 Preliminaries

Let us begin by introducing the following notion of largeness, which was introduced in [4].

Definition 2.1 A subset X of a group G is called k-*large*, for a positive integer k, if the intersection of any k left translates of X is non-empty.

We also require the following definition.

Definition 2.2 Let m and n be positive integers. Then for all $\mathbf{x} = (x_1, \ldots, x_m)$ in G^m and $\Sigma = (\sigma_1, \ldots, \sigma_n)$ in $\mathrm{Aut}(G)^n$, we define $y := x^{\Sigma} \in G^{mn}$ by

$$y_{(i-1)n+j} = x_i^{\sigma_j}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Let $w : G^{mn} \to G$ be a word map on G. We call the following map, a *twisted word map*:

$$G^m \to G,\ x \mapsto w\big(x^{\Sigma}\big)$$

The following theorem differs from [5, Theorem 6.5] only by a minor modification.

Theorem 2.3 *Let G be a nilpotent group of class at most k and*

$$w : G^r \times G^s \to G,$$

with $r > 0$, a twisted word map on G. If the set

$$\big\{x \in G^r : w(x, g) = c\big\}$$

is 2^k-large, for constants $g \in G^s$ and $c \in G$, then it should be whole of G^r.

With the notations in the above theorem, we write "G satisfies 2^k-largely an equation $w(x, g) = c$" instead of the expression "the set $\{x \in G^r : w(x, g) = c\}$ is 2^k-large".

Remark 2.4 If it is added in [5, Definition 6.1.] that for a supercommutator v and an automorphism ϕ of the group G, v^{ϕ} is also a supercommuter with the same variable set as v, and also the hypothesis of subgroup H being normal in [5, Lemma 6.3.] is replaced by "being characteristic", the proofs in section 6 of [5] will still be correct, and therefore, in the key Theorem 6.5., we can replace "word map" with "twisted word map", as we did in Theorem 2.3.

3 Nilpotent compact groups largely satisfying a word equation

In the following, we denote the normalized Haar measure of a compact group G by m_G, and we drop the index G, if there is no ambiguity. Also, we assume all automorphisms used to define twisted word maps to be continuous.

In this section, we prove the main result of the article which states that Conjecture 1.1 holds for nilpotent profinite groups.

Theorem 3.1 *Let G be a nilpotent profinite group. Assume that G satisfies a word equation $w = 1$ on a set of positive Haar measure, i.e.*

$$m_{G^d}\big(\{g \in G^d : w(g) = 1\}\big) > 0.$$

Then $\{g \in G^d : w(g) = 1\}$ has nonempty interior.

We start with the following simple result.

Lemma 3.2 *Let G be a compact group and $A \subseteq G$ be a measurable subset. If $m(A) > 1 - \frac{1}{n}$, then A is n-large.*

Proof — By [1, Lemma 2.1] we have $m\big(\bigcap_{k=1}^{n} g_k A\big) > 0$, so, in particular, $\bigcap_{k=1}^{n} g_k A$ is nonempty. □

By the above lemma and Theorem 2.3 we obtain the following result.

Theorem 3.3 *Let G be a nilpotent compact group and $w : G^d \to G$ a twisted word map. Then*

$$m_{G^d}\big(\{g \in G^d : w(g) = c\}\big) \in \left[0, \frac{2^k - 1}{2^k}\right] \cup \{1\},$$

where k is the nilpotency class of G.

Remark 3.4 For large values of k, the gap obtained in the above corollary becomes very small. The author does not know if it can be improved or not, and has no idea about it; but this

gap, even if very small, plays an essential role in proving the main theorem of the paper.

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1 — Let G be a k-step nilpotent group. Put

$$\mathfrak{A} := \{g \in G^d : w(g) = 1\}.$$

By [1, Lemma 2.4], there exists $t := (t_1, \dots, t_d) \in \mathfrak{A}$ and open normal subgroup N of G^d, such that

$$m_N\big(N \cap t^{-1}\mathfrak{A}\big) > 1 - 2^{-k}.$$

But,

$$N \cap t^{-1}\mathfrak{A} = \big\{x \in N : w(tx) = 1\big\},$$

and one can easily see, using the elementary equations

$$(tx)^n = t^n \prod_{j=1}^{n} x^{t^{n-j}}, \quad ts = st[t, s]$$

that the map

$$N^d \to N,\ x \mapsto w(tx)$$

is a twisted word map on N. Now, Theorem 3.3 implies that

$$w(tx) = 1,$$

for all $x \in N$, whence

$$N \cap t^{-1}\mathfrak{A} = N$$

and the proof is complete. □

In the two following propositions, we give some results about the nilpotent quotients and closed nilpotent normal subgroups of a compact group satisfying largely in a word.

Proposition 3.5 *Let* G *be a compact group and* w *be a word on* G.

Assume that

$$m_{G^d}\big(\{g \in G^d : w(g) = c\}\big) > 1 - 2^{-d}.$$

Then w is an equation on all compact quotients of G which are nilpotent of class at most k.

Proof — Let H be an k-step nilpotent group and $\psi : G \to H$ be a quotient map. By [2, Lemma 2.3.], we obtain

$$m_{H^d}\big(\{h \in H^d : w(h) = \psi(c)\}\big) \geqslant m_{G^d}\big(\{g \in G^k : w(g) = c\}\big).$$

Therefore, the result is obtained by Theorem 3.3. □

Proposition 3.6 *Let G be a compact group and w be a word on G. Assume that*

$$m_{G^d}\big(\{g \in G^d : w(g) = 1\}\big) > 1 - 2^{-k},$$

and N is a closed k-step nilpotent normal subgroup of G. Then a coset of N satisfies the word equation $w = 1$.

Proof — Put $\mathfrak{A} := \{g \in G^d : w(g) = 1\}$. By [3, Theorem 2.49], we obtain

$$m_{G^d}(\mathfrak{A}) = \int_{\frac{G^d}{N^d}} m_{N^d}\big(N^d \cap g^{-1}\mathfrak{A}\big)\, d\overline{g},$$

where $d\overline{g}$ is the integration with respect to m_{G^d/N^d}. This equation reveals the existence of $t := (t_1, \dots, t_k) \in \mathfrak{A}$ such that

$$m_{N^d}\Big(N^d \cap t^{-1}\mathfrak{A}\Big) > 1 - 2^{-k}.$$

But,

$$N^d \cap t^{-1}\mathfrak{A} = \big\{x \in N^d : w(tx) = 1\big\},$$

and, as we have seen in the proof of Theorem 3.1, the map

$$N^d \to N,\ x \mapsto w(tx)$$

is a twisted word map on N, therefore, the result is obtained from Theorem 3.3. □

As a result of the above propositions, we have: As a consequence of the above propositions, we have the following result.

Corollary 3.7 *Let* G *be a compact group,* w *be a word on* G *and*

$$m_{G^d}\big(\{g \in G^d : w(g) = 1\}\big) > \frac{1}{2}.$$

Then w *is an equation on* $G/\overline{G'}$ *and* $Z(G)$*, where* $\overline{G'}$ *is the closure of* G' *in* G.

Proof — Only note that $w(tx) = 1$ and $w(t) = 1$ implies $w(x) = 1$ provided $x \in Z(G)^d$. □

Let us conclude the paper with the following result.

Theorem 3.8 *Let* G *be a compact group such that the solutions set of* $x^n = 1$ *has positive Haar Measure* α*. Then* $G/\overline{G'}$ *and* $Z(G)$ *are of finite exponents at most* $n\lfloor\frac{1}{\alpha}\rfloor!$.

Proof — Put

$$\mathfrak{A} := \{x \in G : w(x) = x^n = 1\}.$$

For $t \in Z(G)$, $w(t\mathfrak{A}) = t^n$, so the subgroup $\{t^n : t \in Z(G)\}$ should be finite of size at most $\lfloor\frac{1}{\alpha}\rfloor$, because $w^{-1}(\{t^n\})$, $t \in Z(G)$ is of Haar measure at least α.

In a similar manner, $\{t^n\overline{G'} : t \in G\}$ should be a finite subgroup of $G/\overline{G'}$ of size at most $\lfloor\frac{1}{\alpha}\rfloor$. □

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