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On Finite Groups Factorized by Weakly $\mathbb{X}$-Subnormal Subgroups

S.I. Lendzenkova

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Abstract

Let $\mathbb{X}$ be a nonempty set of natural numbers. A subgroup A of a finite group G is said to be $\mathbb{X}$-*subnormal* in G if there is a chain of subgroups

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_{n-1} \leqslant A_n = G,$$

where $|A_i : A_{i-1}| \in \mathbb{X}$, for all i. Besides, A is *weakly* $\mathbb{X}$-*subnormal* in G if $A = \langle U_1, U_2 \rangle$, where U_1 is subnormal and U_2 is $\mathbb{X}$-subnormal subgroups of G.

We establish the properties of weakly $\mathbb{X}$-subnormal subgroups and extend the known results on finite factorizable groups with $\mathbb{X}$-subnormal factors to finite factorizable groups with weakly $\mathbb{X}$-subnormal factors. In particular, we establish solvability and r-solvability of a finite group $G = AB$ with weakly $\mathbb{X}$-subnormal solvable or r-solvable subgroups A and B for some $\mathbb{X}$.

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1 Introduction

All groups in this paper are finite. We use the standard notation and terminology of [8]. In particular, $\mathbb{P}$ is the set of all primes.

Further in the text, $\mathbb{X}$ is a nonempty set of natural numbers closed with respect to divisors, i.e. $\mathbb{X}$ has the following property: If $x \in \mathbb{X}$

and a natural number y divides x, then $y \in \mathbb{X}$. Note that $1 \in \mathbb{X}$, and $\mathbb{P}^\infty := \{1\} \cup \mathbb{P}$ is one of such sets.

Recall that a subgroup A of a group G is said to be *semipermutable* [1] or *seminormal* in G if there exists a subgroup B in G such that $G = AB$ and A permutes with all subgroups U of B (that is, $AU = UA$). Besides, A is *weakly subnormal* in G (see [2]) if

$$A = \langle U_1, U_2 \rangle,$$

where U_1 is subnormal and U_2 is semipermutable subgroups of G. Moreover, A is $\mathbb{X}$-*subnormal* in G [12] (denoted by $\mathbb{X}$-sn G) if there is a chain of subgroups

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_{n-1} \leqslant A_n = G,$$

where $|A_i : A_{i-1}| \in \mathbb{X}$, for all i.

In this paper, we discuss some applications of the following generalization of $\mathbb{X}$-subnormality.

Definition 1.1 A subgroup A of a group G is *weakly* $\mathbb{X}$-*subnormal* in G if $A = \langle U_1, U_2 \rangle$, where U_1 is subnormal and U_2 is $\mathbb{X}$-subnormal subgroups of G.

It is clear that a $\mathbb{X}$-subnormal subgroup is weakly $\mathbb{X}$-subnormal in the group. In the alternating group of degree 6, the identity subgroup is subnormal, but not $\mathbb{X}$-subnormal for $\mathbb{X} = \mathbb{P}^\infty$. In addition, in a solvable group, each weakly subnormal subgroup is weakly $\mathbb{P}^\infty$-subnormal, see Lemma 3.5.

Groups factorized by weakly subnormal subgroups were studied in [2] and [11]. In particular, in [2, Theorem 1.4], supersolvability of a group $G = AB$ with the nilpotent derived subgroup and supersolvable weakly subnormal factors A and B is established. A metanilpotent group $G = AB$ with weakly subnormal multipliers of mutually simple indices is also supersolvable [11, Corollary 4.2]. Moreover, V.N. Tyutyanov and V.N. Knyagina studied groups factorized by r-solvable $\mathbb{X}$-subnormal subgroups (see [12]).

In this paper, we establish the properties of $\mathbb{X}$-subnormal and weakly $\mathbb{X}$-subnormal subgroups that allow us to extend many well-known results on groups with sets of $\mathbb{X}$-subnormal subgroups to groups with weakly $\mathbb{X}$-subnormal subgroups. In particular, we show solvability and r-solvability of a finite group $G = AB$ with weakly $\mathbb{X}$-subnormal solvable and, respectively, r-solvable subgroups A and B, for

some sets $\mathbb{X}$. So, for example, a group $G = AB$ is solvable if A and B are weakly $\mathbb{L}$-subnormal solvable subgroups, where $\mathbb{L}$ is the set that consists of all odd primes and numbers 2 and 4.

2 Auxiliary results

We use $\pi(G)$ to denote the set of all primes dividing the order of a group G, and $\mathbb{N}$ the set of all natural numbers. Let us denote by $O_\pi(G)$ the largest normal π-subgroup of G for $\pi \subseteq \pi(G)$. We write

$$X \leqslant Y, \qquad X < Y, \qquad X \lhd Y, \qquad X \lhd\lhd Y$$

if X is a subgroup, a proper subgroup, a normal subgroup, a subnormal subgroup of a group Y, respectively. Moreover, $|X|$ is the order of a group X and $|X : Y|$ is the index of $Y \leqslant X$. In addition, we use the following notation for a group X and its subgroup Y:

- Y_X is the largest normal subgroup of X contained in Y;
- Y_{sX} is the largest subnormal subgroup of X contained in Y.

Definition 2.1 Let G be a group and fix a prime $r \in \pi(G)$. If G contains a normal Sylow r-subgroup, then G is said to be r-*closed*. Moreover, G is called r-*solvable* if it has a normal series such that factors are either r-groups, or r'-groups.

Definition 2.2 A group G of order $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}$, with $p_i \in \mathbb{P}$ and $p_1 > p_2 > \dots > p_n$, $\alpha_i \in \mathbb{N}$, $i = 1, \dots, n$, has a *Sylow tower of supersolvable type* if G has a normal subgroup of the order $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_i^{\alpha_i}$, for each $i = 1, 2, \dots, n$.

Lemma 2.3 (see [10], Lemma 1.2) *Let M be a maximal subgroup of a group G, then $M_G = M_{sG}$.*

Lemma 2.4 (see [8],[7]) *Let N be a normal subgroup of a group G and H is subnormal in G. Then the following assertions hold*:

(1) $HN/N \lhd\lhd G/N$ *and* $H \leqslant O_{\pi(H)}(G)$;

(2) *If H is solvable, then H^G is solvable.*

3 Properties of $\mathbb{X}$-subnormal and weakly $\mathbb{X}$-subnormal subgroups

Lemma 3.1 *Let* $T \leqslant U \leqslant G$, *let* U **X**-*sn* G *and let* $N \lhd G$. *The following statements hold*:

(1) *If* T $\mathbb{X}$-*sn* U, *then* T $\mathbb{X}$-*sn* G;

(2) $(U \cap N)$ $\mathbb{X}$-*sn* N *and* UN/N $\mathbb{X}$-*sn* G/N;

(3) UN $\mathbb{X}$-*sn* G;

(4) U $\mathbb{X}$-*sn* UN.

Proof — Statements (1)–(3) are proved in [5, Lemma 1].

(4) Since U is $\mathbb{X}$-subnormal in G, then there exists a chain of subgroups

$$U = U_0 < U_1 < \ldots < U_{n-1} < G$$

such that $|U_i : U_{i-1}| \in \mathbb{X}$, for each i. Let $U_{n-1} \leqslant U^\star < G$, $U^\star$ is a maximal subgroup in G. Moreover,

$$|G : U_{n-1}| = |G : U^\star||U^\star : U_{n-1}|.$$

Since $|G : U_{n-1}| \in \mathbb{X}$ and $\mathbb{X}$ are closed with respect to the divisors, then

$$|G : U^\star| \in \mathbb{X} \quad \text{and} \quad |U^\star : U_{n-1}| \in \mathbb{X}.$$

Therefore, $U^\star$ is a maximal subgroup in G in which U is $\mathbb{X}$-subnormal and $|G : U^\star| \in \mathbb{X}$. Let us use induction on the group order. We can assume that UN is a proper subgroup of G. If $UN \leqslant U^\star$, then by induction, U is $\mathbb{X}$-subnormal in UN. Let UN not be contained in $U^\star$. Then N is not contained in $U^\star$, $G = U^\star N$ and

$$|G : U^\star| = |N : (N \cap U^\star)| \in \mathbb{X}.$$

By Dedekind's identity, $U \leqslant UN \cap U^\star = U(N \cap U^\star) \leqslant UN$. Since

$$|UN : U(N \cap U^\star)| = \frac{|U||N||U \cap N \cap U^\star|}{|U \cap N||U||N \cap U^\star|} = |N : (N \cap U^\star)| \in \mathbb{X},$$

then $|UN : U(N \cap U^\star)| \in \mathbb{X}$. Applying inductions to the group $U^\star$ in which $N \cap U^\star$ is normal and U is $\mathbb{X}$-subnormal, we conclude that U is $\mathbb{X}$-subnormal in $U(N \cap U^\star)$. The subgroup U is $\mathbb{X}$-subnormal in UN by (1). □

Definition 3.2 (see [3]) A subgroup A of a group G is $K\mathbb{X}$-*subnormal* if there is a chain of subgroups

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_{n-1} \leqslant A_n = G$$

such that, for each $i > 0$, either A_{i-1} is normal in A_i or

$$|A_i : A_{i-1}| \in \mathbb{X}.$$

Lemma 3.3 *Let G be a solvable group and $\pi(G) \subseteq \mathbb{X}$. Then every $K\mathbb{X}$-subnormal subgroup of G is $\mathbb{X}$-subnormal in G.*

Proof — Let $\pi(G) \subseteq \mathbb{X}$ and let U be $K\mathbb{X}$-subnormal subgroup of a solvable group G. Then there is a chain of subgroups such that either U_{i-1} is normal in U_i, or $|U_i : U_{i-1}| \in \mathbb{X}$, for each i. Note that, if $|U_i : U_{i-1}| \in \mathbb{X}$, for each i, then U is $\mathbb{X}$-subnormal in G. Suppose that for some i, U_{i-1} is normal in U_i. Since U_i is solvable, then its composition series

$$U_{i-1} = V_0 < V_1 < \ldots < V_{m-1} < V_m = U_i$$

has indices $|V_j : V_{j-1}| \in \pi(G) \subseteq \mathbb{X}$, for each j. Repeating this compaction for every pair $U_{i-1} \lhd U_i$, we get the chain of subgroups in which all indices belong to $\mathbb{X}$. Therefore, U is $\mathbb{X}$-subnormal in G. □

Lemma 3.4 (1) *Every weakly $\mathbb{X}$-subnormal subgroup of a group G is a $K\mathbb{X}$-subnormal subgroup of G.*

(2) *If G is a solvable group and $\pi(G) \subseteq \mathbb{X}$, then each weakly $\mathbb{X}$-subnormal subgroup of G is an $\mathbb{X}$-subnormal subgroup of G.*

Proof — (1) Use induction on the group order. Let $U = \langle U_1, U_2 \rangle$ be a weakly $\mathbb{X}$-subnormal subgroup of G and M is a maximal subgroup of G containing U. Here U_1 is subnormal in G, and U_2 is $\mathbb{X}$-subnormal in G. According to Lemma 2.3, $M_{sG} = M_G$, therefore

$$U_1 \leqslant M_G,\ U = \langle U_1, U_2 \rangle \leqslant M_G U_2 \leqslant M < G.$$

Since U_2 is $\mathbb{X}$-subnormal in G, then U_2M_G is $\mathbb{X}$-subnormal in G, according to Lemma 3.1 (3). By Lemma 3.1 (4), U_2 is $\mathbb{X}$-subnormal in U_2M_G. Therefore, U is weakly $\mathbb{X}$-subnormal in U_2M_G. By induction, U is $K\mathbb{X}$-subnormal in U_2M_G. So, we obtain U $K\mathbb{X}$-sn U_2M_G and U_2M_G $\mathbb{X}$-sn G, it follows that U is $K\mathbb{X}$-subnormal in G.

(2) Applying (1) and Lemma 3.3, we obtain the claim. □

Lemma 3.5 *If H is a seminormal subgroup of a solvable group G, then H is $\mathbb{P}^\infty$-subnormal in G.*

Proof — By the definition of a seminormal subgroup, there exists a subgroup K, such that $G = HK$ and HL is a proper subgroup of G, for each $L < K$. Since K is solvable, its composition series has prime indices

$$1 = K_0 \leqslant K_1 \leqslant \ldots \leqslant K_{n-1} \leqslant K_n = K,$$

$$K_i \lhd K_{i+1}, \quad |K_{i+1} : K_i| \in \mathbb{P}, \ \ \forall\, i.$$

Since H is seminormal in G, then $HK_i = K_iH$ is a subgroup, for any i, and there is a chain of subgroups

$$1 = HK_0 \leqslant HK_1 \leqslant HK_2 \leqslant \ldots \leqslant HK_{n-1} \leqslant HK_n = H = G,$$

such that

$$|HK_{i+1} : HK_i| = \frac{|H||K_{i+1}|}{|H \cap K_{i+1}|} : \frac{|H||K_i|}{|H \cap K_i|} = \frac{|K_{i+1} : K_i|}{|H \cap K_{i+1} : H \cap K_i|} \in \{1\} \cup \mathbb{P}.$$

We conclude that H is $\mathbb{P}^\infty$-subnormal in G. □

4 On the product of weakly $\mathbb{X}$-subnormal subgroups

Fix a natural number t and a prime number r. Let

$$\mathbb{P}_r^t = \left\{p^i \mid p \in \mathbb{P}\backslash\{r\},\ i \in \{0\} \cup \mathbb{N}\right\} \cup \left\{r^j \mid j \in \{0\} \cup \mathbb{N},\ j \leqslant t\right\}.$$

For $r = t = 2$, we get the set $\mathbb{P}_2^2$, which consists of numbers 1, 2, 4 and all natural powers of odd primes. Let

$$\mathbb{P}^t = \{p^k \mid p \in \mathbb{P},\ k \in \{0, 1, \ldots, t\}\}.$$

For $t = 2$, we get the set $\mathbb{P}^2 = \{p^k \mid p \in \mathbb{P},\ k \in \{0, 1, 2\}\}$. The set

$$\mathbb{L} = \{2, 4\} \cup \{2n - 1 \mid n \in \mathbb{N}\}$$

consists of all odd primes and numbers 2 and 4.
Since the sets $\mathbb{P}_r^t$, $\mathbb{P}_2^2$, $\mathbb{P}^t$, $\mathbb{P}^2$, $\mathbb{L}$ and $\mathbb{P}^\infty$ are closed with respect to divisors, then Lemmas 3.1, 3.3 and 3.4 are valid for each of these sets.

Lemma 4.1 *In a solvable group* G*, a subgroup* H *is* $\mathbb{L}$*-subnormal in* G *if and only if* H *is* $\mathbb{P}_2^2$*-subnormal in* G*.*

Proof — Let H be an $\mathbb{L}$-subnormal subgroup of a solvable group G. Then there is a chain of subgroups

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_{n-1} \leqslant H_n = G$$

such that $|H_i : H_{i-1}| \in \mathbb{L}$. Let $H_{i-1} \leqslant K \leqslant H_i$, K is a maximal subgroup of H_i. Moreover,

$$|H_i : H_{i-1}| = |H_i : K||K : H_{i-1}| \in \mathbb{L}.$$

Since the set $\mathbb{L}$ is closed with respect to divisors, then $|H_i : K| \in \mathbb{L}$. In view of G is solvable, we have

$$|H_i : K| \in \mathbb{P}^\infty.$$

Therefore, $|H_i : K| \in \mathbb{P}_2^2$. Conducting the same reasoning with each pair of subgroups $H_{i-1} < H_i$, we obtain a maximal subgroup chain with indices from $\mathbb{P}_2^2$. It follows that H is $\mathbb{P}_2^2$-subnormal in G.

The converse is true due to the fact that $\mathbb{P}_2^2 \subseteq \mathbb{L}$. □

Let us observe that factorized groups with $\mathbb{X}$-subnormal factors were studied in [3],[4],[6],[9] and [12]. The obtained properties of weakly $\mathbb{X}$-subnormal subgroups allow to extend a lot of the results of these papers to factorizable groups with weakly $\mathbb{X}$-subnormal factors.

Theorem 4.2 *If* A *and* B *are weakly* $\mathbb{L}$*-subnormal solvable subgroups of a group* $G = AB$*, then* G *is solvable.*

Proof — Let $A = \langle A_1, A_2 \rangle$, $B = \langle B_1, B_2 \rangle$, where A_1, B_1 are subnormal subgroups of G, and A_2, B_2 are $\mathbb{L}$-subnormal in G. Since A_1 is subnormal in G, then A_1^G is solvable by Lemma 2.4. Similarly, B_1^G is solvable. Let $N = A_1^G B_1^G$. Then N is a solvable normal subgroup of G. The quotient group

$$G/N = (AN/N)(BN/N) = (A_2N/N)(B_2N/N),$$

where A_2N/N and B_2N/N are $\mathbb{L}$-subnormal subgroups of G/N according to Lemma 3.1 (2). By [4, Theorem 2], we obtain that G/N is solvable, and G is solvable. □

Note that Theorem 4.2 can also be proved in view of Lemma 3.4 (1) and [3, Theorem 4.2].

Corollary 4.3 *If A and B are $\mathbb{L}$-subnormal solvable subgroups of a group*

$$G = AB,$$

then A and B are $\mathbb{P}_2^2$-subnormal in G.

Proof — According to Theorem 4.2, G is solvable, and according to Lemma 4.1, A and B are $\mathbb{P}_2^2$-subnormal in G. □

In view of $\mathbb{P}^2 \subseteq \mathbb{P}_2^2 \subseteq \mathbb{L}$, and Theorem 4.2, we obtain the following results.

Corollary 4.4 *If A and B are weakly $\mathbb{P}_2^2$-subnormal solvable subgroups of a group $G = AB$, then G is solvable.*

Corollary 4.5 *If A and B are weakly $\mathbb{P}^2$-subnormal solvable subgroups of a group $G = AB$, then G is solvable.*

Example 4.6 The simple group $PSL(2,7) = S_4(C_7C_3)$ can be represented as the product of the solvable subgroups S_4 and C_7C_3, where S_4 is the symmetric group of degree 4, and C_7C_3 is a noncyclic group of order 21. The subgroup S_4 is weakly $\mathbb{P}^\infty$-subnormal in $PSL(2,7)$, and C_7C_3 is weakly $\mathbb{P}^3$-subnormal in $PSL(2,7)$. It follows that, in Corollaries 4.4 and 4.5, the assumptions of weakly $\mathbb{P}_2^2$-subnormality and weakly $\mathbb{P}^2$-subnormality of factors cannot be weakened to weakly $\mathbb{P}_2^3$- and weakly $\mathbb{P}^3$-subnormality, respectively.

Corollary 4.7 *Let r be the largest prime divisor of the order of a group G, let A and B be solvable r-closed weakly $\mathbb{P}^2$-subnormal subgroups of G, and let $G = AB$. If $r > 3$, then G is r-closed.*

Proof — According to Corollary 4.5, the group G is solvable, and A and B are $\mathbb{P}^2$-subnormal by Lemma 3.4 (2). By [4, Theorem 1 (2)], we obtain that G is r-closed. □

Example 4.8 In Corollary 4.7, the requirement $r > 3$ cannot be removed. For example, the symmetric group S_4 of degree 4 can be represented as the product of a weakly $\mathbb{P}^\infty$-subnormal subgroup of order 8 and a weakly $\mathbb{P}^2$-subnormal subgroup of order 3, but S_4 is not 3-closed.

Corollary 4.9 *If A and B are weakly $\mathbb{P}^2$-subnormal supersolvable subgroups of a group $G = AB$ and H is a Hall $\{2,3\}'$-subgroup of the group G, then H has a Sylow tower of supersolvable type and H is normal in G.*

Proof — By Corollary 4.5, G is solvable and, by Lemma 3.4 (2), A and B are $\mathbb{P}^2$-subnormal in G. By [4, Theorem 1 (3)], we obtain that H is normal in G and H has a Sylow tower of supersolvable type. □

Theorem 4.10 *Let A and B be weakly $\mathbb{X}$-subnormal subgroups of a group*

$$G = AB.$$

If A and B are r-solvable, then G is r-solvable in each of the following cases:

(1) $\mathbb{X} = \mathbb{P}_2^2$;

(2) $\mathbb{X} = \mathbb{P}^\infty$ *and* $r \in \pi(G)\backslash\{2,3,7\}$.

Proof — According to Lemma 3.4 (1), A and B are $K\mathbb{X}$-subnormal in G. Hence, if we assume (1), then G is r-solvable by [3, Theorem 2.2]. If we assume (2), then G is r-solvable by [3, Theorem 3.1]. □

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Svetlana I. Lendzenkova
Department of Mathematics and Technologies of Programming
Francisk Skorina Gomel State University
Kirova Str. 119, 246019 Gomel (Belarus)
e-mail: slendenkova@mail.ru