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Rota–Baxter Operators on Dihedral and Alternating Groups

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Abstract

Rota–Baxter operators on algebras, which appeared in 1960, have connections with different versions of the Yang–Baxter equation, pre- and postalgebras, double Poisson algebras, etc. In 2020, the notion of Rota–Baxter operator on a group was defined by L. Guo, H. Lang, Yu. Sheng.

In 2023, V. Bardakov and the second author showed that all Rota–Baxter operators on simple sporadic groups are splitting, i.e. they are defined via exact factorizations. In this work, we clarify for which n there exist non-splitting Rota–Baxter operators on the alternating group A_n. For the corresponding n, we describe all non-splitting Rota–Baxter operators on A_n. Besides, we describe Rota–Baxter operators on dihedral groups D_{2n} providing the general construction which lies behind all non-splitting Rota–Baxter operators on A_n and D_{2n}.

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1 Introduction

In 1960 [3], G. Baxter introduced the notion of Rota–Baxter operator on an algebra A as a linear operator R satisfying the relation

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy), \tag{1.1}$$

where $x, y \in A$ and the scalar λ from the ground field F is fixed, and it is called a weight of R.

The identity (1.1) for $\lambda = 0$ may be considered as a generalization of integration by parts formula. When $\lambda \neq 0$, the canonical example is a projection of the whole algebra A on its subalgebra A_1 provided a decomposition $A = A_1 \oplus A_2$ of A into a direct vector-space sum of its subalgebras A_1, A_2.

After more than 60 years of the study, a lot of connections and applications of Rota–Baxter operators were found: with different versions of the Yang–Baxter equation [1], [6], [31], pre- and postalgebras [1], [2], [21], double Poisson algebras [17], [18], [28], etc.

Recently, in 2020 (published in 2021), the analogous notion of Rota–Baxter operator (of weight ± 1) on groups was suggested by L. Guo, H. Lang, and Yu. Sheng [22]. Since this pioneer article appeared, a dozen papers have been published, and the direction of Rota–Baxter groups is rapidly developing. On the one hand, the correspondence between Rota–Baxter operators on Lie groups and Rota–Baxter operators on Lie algebras is studied, see [22] and [24]. On the other hand, the definition of Rota–Baxter operator on groups was extended on Clifford semigroups [11], Hopf algebras [16], and others. There is a deep connection [5] between Rota–Baxter operators on groups and so called skew left braces [20], which, in turn, serve as a source of set-theoretical solutions to the quantum Yang–Baxter equation. The cohomology theory for Rota–Baxter (Lie) groups was presented in [14] and [24].

Given a group G, one has always trivial RB-operators

$$B_e \colon g \to e \quad \text{and} \quad B_{-1} \colon g \to g^{-1}.$$

The analog of the projection operator mentioned above for algebras is the following: Given an exact factorization $G = HL$, the map

$$B \colon hl \to l^{-1}$$

is a Rota–Baxter operator on G. We will call it a *splitting RB-operator*. The criterion, when a given RB-operator is a splitting one, was stated in [4]. It actually says that R is a splitting RB-operator on a group G if and only if $\mathrm{Im}(B\widetilde{B}) = 1$, where

$$\widetilde{B}\colon G \to G,\ \widetilde{B}(g) = g^{-1}B(g^{-1})$$

is another RB-operator on G defined with the help of B. Different constructions of non-splitting Rota–Baxter operators on groups were suggested therein. However, it occurs that there are no non-splitting RB-operators on all simple sporadic groups, see [4].

In the current work, we continue the mentioned in [4] direction of description of RB-operators on all finite simple groups; here we deal with alternating groups A_n. The main result is the complete classification of (non-splitting) Rota–Baxter operators on A_n. For any RB-operator B on a group G, one has a factorization

$$G = \mathrm{Im}(B) \cdot \mathrm{Im}(\widetilde{B}).$$

Thus, we may apply the known description of factorizations of finite simple groups [27]. Furthermore, we use sharply 2-transitive groups and involve the Zassenhaus theorems on them (see [23], Theorems XII.9.7 and XII.9.8). As a consequence, we show in Theorem 7.1 that there are non-splitting RB-operators R on A_n if and only if $n \in \{q^m,\ q^m + 1\}$, where:

(i) q is a prime-power.

(ii) Each prime divisor of m divides $q - 1$.

(iii) $q \equiv 3 \pmod 4$ and $m \equiv 2 \pmod 4$.

When n is so, we clarify the general construction, which lies behind R (see Lemma 5.1). Note that there is an infinite number of pairs q, m satisfying (i)–(iii).

We prove that if G is a finite group, B is an RB-operator on

it, $R = \mathrm{Im}(B\widetilde{B})$ is an abelian subgroup of G, and

$$G = H \cdot L = \mathrm{Ker}(B) \cdot \mathrm{Im}(B),$$

then $B(hl) = C(l)$ for an RB-operator C on L. Moreover, $\mathrm{Im}(\widetilde{C})$ normalizes H and C is a homomorphism from L onto its abelian subgroup isomorphic to R. We apply this result not only for alternating groups but also for dihedral ones. We show that for any non-splitting RB-operator B on A_n, we have $\mathrm{Im}(B\widetilde{B}) \simeq \mathbb{Z}_2$. Regarding to dihedral groups D_{2n}, we prove that there are no non-splitting RB-operators on D_{2n}, when n is odd. When n is even, we show that for any non-splitting RB-operator B on D_{2n}, we have either $\mathrm{Im}(B\widetilde{B}) \simeq \mathbb{Z}_2$ or $\mathrm{Im}(B\widetilde{B}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let us highlight that we suggest a new kind of equivalence on a set of Rota–Baxter operators. In [4], it was noted that given a Rota–Baxter group (G, B) and $\varphi \in \mathrm{Aut}(G)$, one has again an RB-operator $B^{(\varphi)}$ on G, which acts by the formula

$$B^{(\varphi)}(g) = \varphi^{-1} B \varphi(g).$$

Therefore, it is natural to classify Rota–Baxter operators on a group G up to such action of $\mathrm{Aut}(G)$. We present an equivalence obtained under the action of a subgroup of $\mathrm{Aut}(G \times G)$ isomorphic to

$$(\mathrm{Aut}(G) \times \mathrm{Inn}(G)) \rtimes \mathbb{Z}_2.$$

As far as we know, there is no analogous action for Rota–Baxter operators on algebras. We prove that such equivalence preserves main properties of Rota–Baxter operators, e.g. the property of splitting one. Moreover, the mentioned above construction of an RB-operator G such that $\mathrm{Im}(B\widetilde{B})$ is abelian, is preserved too. So, we find all non-splitting Rota–Baxter operators on A_n up to the equivalence.

The work is organized as follows. Firstly, we provide the required preliminaries about Rota–Baxter groups in Section 2 and about sharply doubly transitive groups in Section 3, re-

spectively. In Section 4, we define the new equivalence of RB-operators (see Proposition 4.2). In Section 5, we derive the construction of a Rota–Baxter operator B such that $R = \mathrm{Im}(B\widetilde{B})$ is abelian and $G = \mathrm{Ker}(B) \cdot \mathrm{Im}(B)$ (see Lemma 5.1).

In Section 6, we describe Rota–Baxter operators on dihedral groups (see Theorem 6.1). In Section 7, we classify, up to equivalence, Rota–Baxter operators on alternating groups A_n (see Theorem 7.1). In particular, for small n, we use a computer algebra system GAP [15] to compute RB-operators on A_n. We use the reformulation of a Rota–Baxter operator B in terms of its graph in $G \times G$ [24] (see Proposition 4.1) and write a program with a starting point based on the program from [30].

2 Preliminaries

Let G be a group. A map $B\colon G \to G$ is called a *Rota–Baxter operator of weight* 1 if

$$B(g)B(h) = B(gB(g)hB(g)^{-1}) \tag{2.1}$$

holds, for all $g, h \in G$ (see [22]).

Note that a notion of Rota–Baxter operator on an abelian G coincides with homomorphism.

Proposition 2.1 (see [22]) *Let* (G, B) *be a Rota–Baxter group. Then the following hold*:

(a) $\mathrm{Im}(B)$ *and* $\mathrm{Ker}(B)$ *are subgroups of* G.

(b) $B(e) = e$.

(c) $B(g)^{-1} = B(B(g)^{-1}g^{-1}B(g))$ *for every* $g \in G$.

(d) $B(h) = B(gh)$ *for all* $h \in G$ *and* $g \in \mathrm{Ker}(B)$.

Proposition 2.2 (see [22]) *Let* (G, B) *be a Rota–Baxter group. Then the following hold*:

(a) *The map* $\widetilde{B}$ *defined as* $\widetilde{B}(g) = g^{-1}B(g^{-1})$ *is a Rota–Baxter operator on* G.

(b) *If* $\varphi \in \mathrm{Aut}(G)$, *then* $B^{(\varphi)} = \varphi^{-1}B\varphi$ *is a Rota–Baxter operator on* G.

Proposition 2.3 (see [22]) *Let* $(G, \cdot, B)$ *be a Rota–Baxter group. Then the following hold*:

(a) *The pair* $(G, \circ)$ *with the product*

$$g \circ h = gB(g)hB(g)^{-1}, \quad g, h \in G, \tag{2.2}$$

is a group.

(b) *The operator* B *is a Rota–Baxter operator on the group* $(G, \circ)$.

(c) *The map* $B\colon (G, \circ) \to (G, \cdot)$ *is a homomorphism of Rota–Baxter groups.*

Hereinafter, we will denote the group $(G, \circ)$ as $G^{(\circ)}$.

Proposition 2.4 (see [22]) *If* (G, B) *is an RB-group, then the following hold*:

(a) $\mathrm{Ker}(B)$ *and* $\mathrm{Ker}(\widetilde{B})$ *are normal in* $G^{(\circ)}$.

(b) $\mathrm{Ker}(B) \trianglelefteq \mathrm{Im}(\widetilde{B})$ *and* $\mathrm{Ker}(\widetilde{B}) \trianglelefteq \mathrm{Im}(B)$ *in* G.

(c) *We have the isomorphism of the quotient groups*

$$\mathrm{Im}(\widetilde{B})/\mathrm{Ker}(B) \simeq \mathrm{Im}(B)/\mathrm{Ker}(\widetilde{B}). \tag{2.3}$$

(d) *We have the factorization*

$$G = \mathrm{Im}(\widetilde{B})\mathrm{Im}(B). \tag{2.4}$$

From now on, denote by B_+ the map on the group G defined as follows,

$$B_+(g) = gB(g).$$

Proposition 2.5 (see [4, Lemma 5 (c)]) *If* (G, B) *is a Rota–Baxter group, then*

$$BB_+ = B_+B.$$

Remark 2.6 Because of Proposition 2.5, one may note

$$\mathrm{Im}(B\widetilde{B}) = \mathrm{Im}(\widetilde{B}B).$$

Indeed,

$$(B\widetilde{B})(g) = (BB_+)\big(g^{-1}\big) = (B_+B)\big(g^{-1}\big) = \widetilde{B}\big(\big(B(g^{-1})^{-1}\big).$$

However, the equality $B\widetilde{B} = \widetilde{B}B$ is not true, see Example 6.5 below.

Given an integer n, denote by $a^{\circ(n)}$ the nth power of a under the product $\circ$.

Proposition 2.7 (see [4, Proposition 14]) *If* (G, B) *is a Rota–Baxter group, we have the equality*

$$a^{\circ(k)} = (B_+(a))^k(B(a))^{-k},$$

for all $a \in G$ *and* $k \in \mathbb{Z}$.

Let us deduce an easy but important equality.

Lemma 2.8 *Let* (G, B) *be an RB-group. Then*

$$\mathrm{Im}(B\widetilde{B}) = \mathrm{Im}(\widetilde{B}B) = \mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B}).$$

Proof — The first equality follows from Remark 2.6 and it implies the inclusion $\mathrm{Im}(B\widetilde{B}) \subseteq \mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B})$. Now, suppose that $x \in \mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B})$, i.e., $x = B(g) = hB(h)$ for some $g, h \in G$. By Proposition 2.1 (c), we have

$$h = B(g)B(h)^{-1} = B(g)B(B(h)^{-1}h^{-1}B(h)) = B(z),$$

for $z = g \circ B(h)^{-1}h^{-1}B(h)$. Since $\mathrm{Im}(B)$ is a subgroup, we obtain that $h^{-1} \in \mathrm{Im}(B)$. Hence, $x = \widetilde{B}(h^{-1}) \in \mathrm{Im}(\widetilde{B}B)$. □

Example 2.9 (see [22, Lemma 2.6]) Let G be a group. Given an exact factorization $G = HL$, a map $B\colon G \to G$ defined as follows,

$$B(hl) = l^{-1}$$

is a Rota–Baxter operator on G.

We call such RB-operator on G a *splitting Rota–Baxter operator*.

Proposition 2.10 (see [4, Proposition 16]) *Let G be a group. Then, an RB-operator on G, $B\colon G \to G$, is a splitting Rota–Baxter operator on G if and only if* $\mathrm{Im}(\widetilde{B}B)=1$.

Proposition 2.11 (see [4, Proposition 20]) *Given a semidirect product $G = H \rtimes L$, let C be a Rota–Baxter operator on L. Then the map $B\colon G \to G$ defined by the formula*

$$B(hl) = C(l),$$

where $h \in H$ and $l \in L$, is a Rota–Baxter operator.

Corollary 2.12 *Given an exact factorization $G = HL$, let C be a Rota–Baxter operator on L. If $\mathrm{Im}(\widetilde{C}) \leqslant L$ normalizes H, then the map $B\colon G \to G$ defined by the formula*

$$B(hl) = C(l),$$

where $h \in H$ and $l \in L$, is a Rota–Baxter operator.

Proof — Let $h, h' \in H$ and $l, l' \in L$. Then

$$\begin{aligned} B\big(hlB(hl)h'l'(Bhl)^{-1}\big) &= B\big(hlC(l)h'l'C(l)^{-1}\big) \\ = B\big(hh''lC(l)l'C(l)^{-1}\big) &= C\big(lC(l)l'C(l)^{-1}\big) \\ = C(l)C(l') &= B(hl)B(h'l'), \end{aligned}$$

where $h'' = h'^{(lC(l))^{-1}} = h^{\widetilde{C}(l^{-1})^{-1}} \in H$. □

Proposition 2.13 (see [4, Proposition 21]) *Let G be a group and H an abelian subgroup of G. Then any homomorphism $B\colon G \to H$ is a Rota–Baxter operator.*

Lemma 2.14 *Let* B *be an RB-operator on a finite group* G *and let* $c \in Z(G)$. *Then the following hold*:

(a) *If* B *is invertible and* $B(x) = c$, *then* x *commutes with* $\mathrm{Im}(\widetilde{B})$.

(b) *We have that* $|B(c)|$ *divides* $|c|$.

Proof — (a) Since B is an isomorphism from $G^{(\circ)}$ onto G, we get $x \in Z(G^{(\circ)})$, i.e. $x \circ g = g \circ x$, for every $g \in G$. Hence,

$$xB(x)gB(x)^{-1} = xg = gB(g)xB(g)^{-1}.$$

Thus,

$$x = B_+(g)x(B_+(g))^{-1} = x^{(B_+(g))^{-1}}.$$

We apply that $\mathrm{Im}(\widetilde{B}) = \mathrm{Im}(B_+)$.

(b) Let $|c| = k$. By (2.1), we have

$$B(z)(B(c))^k = B(z)B(c) \cdot (B(c))^{k-1} = B(zc)(B(c))^{k-1}$$

$$= \ldots = B(zc^k) = B(z).$$

Hence, $(B(c))^k = e$. □

3 Sharply 2-transitive groups

In this section, we provide necessary information about sharply 2-transitive groups. The structure of a sharply 2-transitive group is described in the following theorems.

Theorem 3.1 (see [23, Theorem XII.9.7]) *Suppose that* q *is a prime-power and* m *is a natural number with the following properties*:

(1) *Each prime divisor of* m *divides* $q - 1$.

(2) *If* 4 *divides* m, *then* 4 *divides* $q - 1$.

Let $GF(q^m)^* = \langle w \rangle$ *and suppose that* $(m, t) = 1$. *Let* a, b *be the following elements of the group of semilinear mappings on* $GF(q^m)$*:*

$$xa = w^m x, \quad xb = w^t x^q. \tag{3.1}$$

Then $N = \langle a, b \rangle$ *is a regular permutation group on* $GF(q^m)^*$ *and* $\langle a \rangle$ *is an irreducible normal subgroup of* N. *If* F *is the group of all translations on* $GF(q^m)$, N *normalizes* F *and* $L(m, q, t) = FN$ *is a sharply doubly transitive permutation group of degree* q^m.

Note that the groups $L(m, q, t)$ are isomorphic for fixed m, q and different admissible t.

Theorem 3.2 (see [23, Theorem XII.9.8]) *Let* L *be a sharply doubly transitive permutation group. Then* L *is a Frobenius group with elementary abelian Frobenius kernel* F. *To within similarity,* L *permutes* $GF(p^r)$, *and* F *is the group of translations on* $GF(p^r)$. *If* p^r *is distinct from* $5^2, 7^2, 11^2, 23^2, 29^2$, *and* 59^2, L *is permutation isomorphic to one of the groups* $L(m, q, t)$ *defined in* Theorem 3.1.

We want to find the conditions when L is contained in the alternating group and L has a subgroup of index 2.

In the remarks after [34, Theorem A], the authors say that the sharply 2-transitive group consists of even permutations only if $p = 2$, or r is even and p is odd. Note that this condition is necessary but not sufficient. For example, if $n = 13^2$, then the symmetric group S_{169} has two sharply 2-transitive groups L_1 and L_2, where

$$L_1 \simeq (\mathbb{Z}_{13} \times \mathbb{Z}_{13}) : \mathbb{Z}_{168} \quad \text{and} \quad L_2 \simeq (\mathbb{Z}_{13} \times \mathbb{Z}_{13}) : (\mathbb{Z}_{21} : \mathbb{Z}_8)$$

It is clear that the generator of $\mathbb{Z}_{168}$ in L_1 and the generator of $\mathbb{Z}_8$ in L_2 are odd permutations, and so they do not belong to A_{169}.

Corollary 3.3 *Let* $L = FN$ *be a sharply doubly transitive permutation group of degree* $n = q^m$. *If* $L \leqslant A_n$ *and* L *has a subgroup of index* 2, *then one and only one of the following holds*:

(1) $q \equiv 3 \pmod 4$ *and* $L = L(m, q, t)$ *is defined in* Theorem 3.1. *Moreover,* L *has two non-isomorphic subgroups of index* 2*, except* $L = L(2, 3, 1)$ *which has only one non-isomorphic subgroup of index* 2*.*

(2) $n = 7^2$ *and* $N \simeq 2^-S_4 \simeq SL_2(3).2$*. Moreover,* L *has one non-isomorphic subgroup of index* 2*.*

(3) $n = 23^2$ *and* $N \simeq 2^-S_4 \times \mathbb{Z}_{11} \simeq SL_2(3).2 \times \mathbb{Z}_{11}$*. Moreover,* L *has one non-isomorphic subgroup of index* 2*.*

Proof — At first, consider the general case $L = L(m, q, t) = FN$. The Frobenius group of even degree does not have subgroups of index 2, and therefore q must be odd. Since q is odd, we have $F \leqslant A_n$, where $n = q^m$. It follows from the proof of Theorem 3.1 that

$$q^m - 1 = m(q-1)l$$

for some integer l, and the order of a is equal to $(q-1)l$. Moreover,

$$b^{-1}ab = a^q \quad \text{and} \quad b^m = a^{tl}.$$

Therefore, a is the product of m cycles of length $(q-1)l$. Thus, we have that $a \in A_n$ if and only if m is even.

It follows from the equality (3.1) that $m \,||b|$. Therefore,

$$|b| = \frac{m|a|}{(|a|, tl)} = \frac{m|a|}{l(q-1,t)} = \frac{q^m - 1}{l(q-1,t)} = \frac{n-1}{l(q-1,t)}.$$

Put $s = l(q-1,t)$. The element b is the product of s cycles of length $(n-1)/s$. Thus, $b \in A_n$ if and only if s is even. Since $(m,t) = 1$ and m must be even, we have t is odd. Therefore, s is even if and only if l is even.

Therefore, $N \leqslant A_n$ if and only if m and $l = (q^m - 1)/m(q-1)$ are even. It follows from [33, Lemma 1.7 (iii)] that l is even if and only if $q \equiv 3 \pmod 4$. Since $q \equiv 3 \pmod 4$, the property (2) of Theorem 3.1 is equivalent to $m \equiv 2 \pmod 4$.

Now $L = FN \leqslant A_n$ and N has the following presentation:

$$N = \left\langle a, b \mid a^{(q-1)l} = b^{(q-1)m} = 1, a^{tl} = b^m, b^{-1}ab = a^q \right\rangle.$$

The group N has three normal subgroups of index two:

$$S_1 = \left\langle a, b^2 \right\rangle, \quad S_2 = \left\langle a^2, b \right\rangle, \quad \text{and} \quad S_3 = \left\langle a^2, ab \right\rangle.$$

If $q^m = 3^2$, then $N = Q_8$ has three normal subgroups of index 2, isomorphic to $\mathbb{Z}_4$. If $q^m > 9$, then the group S_1 is not isomorphic to S_2 or S_3, however, $S_2 \simeq S_3$. Hence, L has two non-isomorphic subgroups FS_1 and FS_2 of index 2.

Now consider the exceptional cases when p^r is one of integers $5^2, 7^2, 11^2, 23^2, 29^2$, or 59^2. According to [19, Theorem 2.4] the group $L = FN$ is a Frobenius group, where F is the elementary abelian group of order p^2 and N is one of the following groups:

(i) $p = 5$ and $N \simeq SL_2(3)$.

(ii) $p = 7$ and $N \simeq 2^-S_4 \simeq SL_2(3).2$.

(iii) $p = 11$ and $N \simeq SL_2(3) \times \mathbb{Z}_5$.

(iv) $p = 11$ and $N \simeq SL_2(5)$.

(v) $p = 23$ and $N \simeq 2^-S_4 \times \mathbb{Z}_{11} \simeq SL_2(3).2 \times \mathbb{Z}_{11}$.

(vi) $p = 29$ and $N \simeq SL_2(5) \times \mathbb{Z}_7$.

(vii) $p = 59$ and $N \simeq SL_2(5) \times \mathbb{Z}_{29}$.

Note that the groups $SL_2(3)$ and $SL_2(5)$ have no subgroups of index 2. Thus, the only possibilities are (ii) and (v), wher N has one subgroup of index 2. □

4 Equivalence of Rota–Baxter operators

Proposition 4.1 (see [24]) *Let* B *be a map defined on a finite group* G *and construct* $H_B = \{(B(g), gB(g)) \mid g \in G\} \subset G \times G$.

Then the following hold:

(a) *If* B *is an RB-operator on* G*, then* H_B *is a subgroup of* $G \times G$.

(b) *If* H *is a subgroup of* $G \times G$*, then there exists an RB-operator* B *on* G *such that* $H = H_B$ *if and only if* $|H| = |G|$ *and*

$$\{ba^{-1} \mid (a, b) \in H\} = G.$$

Note that $\{ba^{-1} \mid (a, b) \in H\} = G$ is equivalent to the condition

$$H \cap \{(g, g) \mid g \in G\} = \{e\}.$$

Given an element x of a group G, the automorphism $g \mapsto g^x$ is denoted by α_x. Let us consider the following subgroup of $\mathrm{Aut}(G \times G)$:

$$\begin{aligned} Q(G) = &\{(\varphi, \varphi\,\alpha_x) \mid \varphi \in \mathrm{Aut}(G),\, x \in G\} \\ &\cup \{\tau((\varphi, \varphi\,\alpha_x)) \mid \varphi \in \mathrm{Aut}(G),\, x \in G\}, \end{aligned}$$

where $\tau((\varphi, \varphi\,\alpha_x))\colon (g, h) \to (\varphi\,\alpha_x(h), \varphi(g))$.

Proposition 4.2 *Let* B *be an RB-operator on a finite group* G*. For every* $\Phi \in Q(G)$*, there exists an RB-operator* B' *on* G *such that* $\Phi(H_B) = H_{B'}$.

Proof — It is clear that $|\Phi(H_B)| = |G|$, it remains to check that $\{ba^{-1} \mid (a, b) \in \Phi(H_B)\} = G$. Let us prove the statement for $\Phi = (\varphi, \varphi\,\alpha_x)$, the proof for $\Phi = \tau((\varphi, \varphi\,\alpha_x))$ is analogous. Thus, we have to find for every $g \in G$ such $(a, b) \in H_B$ that

$$\varphi(\alpha_x(b))\varphi(a)^{-1} = g.$$

Equivalently, we search for $(a, b) \in H_B$ such that $bxa^{-1} = h$, where $h = x\varphi^{-1}(g)$ may be any element from G. We may find $(a_0, b_0) \in H_B$ such that $b_0 a_0^{-1} = x$. Hence, we need to get a pair $(a, b) \in H_B$ satisfying the equality

$$bxa^{-1} = bb_0 a_0^{-1} a^{-1} = bb_0 (aa_0)^{-1} = h,$$

and it is possible, since $h = \widetilde{b}\widetilde{a}^{-1}$ for some $(\widetilde{a}, \widetilde{b}) \in H_B$. Thus, we obtain that $(a, b) = (\widetilde{a}, \widetilde{b})(a_0, b_0)^{-1} \in H_B$. □

Let us call a pair of RB-operators B, B' on a group G *equivalent*, if $\Phi(H_B) = H_{B'}$ for some $\Phi \in Q(G)$. Let us interpret Proposition 2.2 in terms of equivalent RB-operators. If $\Phi = \tau((id, id))$, we have $B' = \widetilde{B}$. If $\Phi = (\varphi, \varphi)$, then $B' = B^{(\varphi)}$.

Lemma 4.3 *Let G be a finite group and B, B' be a pair of equivalent RB-operators on G. Then the following hold*:

(a) $G^{(\circ)}(B) \simeq G^{(\circ)}(B')$.

(b) B *is trivial if and only if* B' *is trivial.*

(c) *Up to action of* $B \to \widetilde{B}$, *we have*

$$\mathrm{Im}(B) \simeq \mathrm{Im}(B') \quad \textit{and} \quad \mathrm{Im}(\widetilde{B}) \simeq \mathrm{Im}(\widetilde{B'}).$$

(d) $\mathrm{Im}(B\widetilde{B}) \simeq \mathrm{Im}(B'\widetilde{B'})$.

(e) B *is splitting if and only if* B' *is splitting.*

(f) *If* $G = \mathrm{Ker}(B) \cdot \mathrm{Im}(B)$, *then up to the action of* $B \to \widetilde{B}$, *one has*

$$G = \mathrm{Ker}(B') \cdot \mathrm{Im}(B').$$

Proof — (a) Define $\kappa\colon G \to H_B, g \mapsto (B(g), gB(g))$. Note that κ is bijective and $\kappa(g \circ h) \to \kappa(g)\kappa(h)$, thus, $G^{(\circ)}(B) \simeq H_B$, hence $G^{(\circ)}(B) \simeq H_B \simeq H_{B'} \simeq G^{(\circ)}(B')$.

(b), (c) They are trivial.

(d) Let us consider $\Phi = (\varphi, \alpha_x\varphi) \in Q(G)$, $\varphi \in \mathrm{Aut}(G)$, and $x \in G$. Note that $\mathrm{Im}(\widetilde{B^{(\varphi)}B^{(\varphi)}}) \simeq \mathrm{Im}(B\widetilde{B})$. Thus, it remains to study the case when B' comes from B with the help of $\Phi = (id, \alpha_x)$. Then $\mathrm{Im}(\widetilde{B}'B') = \alpha_x(\mathrm{Im}(\widetilde{B}B))$, i.e. the required subgroups are isomorphic.

(e) It follows from (d) and Proposition 2.10.

(f) It follows from the fact that $G = H \cdot L$ is an exact factorization if and only if $G = H \cdot \alpha_x(L)$ is an exact factorization, for any $x \in G$. □

Remark 4.4 Let us show that one may not generalize Proposition 4.2 on the whole $\mathrm{Aut}(G \times G)$. For this, take $G = A_4$ and identify $A_4 \times A_4$ with the corresponding subgroup in S_8. Then the map B defined on $A_4 = V \rtimes L$ by Proposition 2.11 for

$$V = \langle (12)(34), (13)(24) \rangle, \quad L = \langle (234) \rangle$$

and $B_{|_L} = \mathrm{id}$ is a Rota–Baxter operator. We have

$$H_B = \langle (234)(687), (13)(24), (12)(34) \rangle.$$

Let us consider $\psi = \mathrm{conj}_{(23)} \in \mathrm{Aut}(S_8)$, which induces the automorphism of $G \times G$. Then

$$\psi(H_B) = \langle (243)(687), (13)(24), (12)(34) \rangle$$

and now, $\psi(H_B)$ has a non-trivial intersection with the diagonal subgroup of $G \times G$. Thus, the condition (b) from Proposition 4.1 does not hold. Note that ψ has the form (φ, id) for the outer automorphism φ of A_4.

5 Properties of some explicit construction

Lemma 5.1 *Let* B *be an RB-operator on a finite group* G *such that*

(i) $R = \mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B})$ *is abelian.*

(ii) $G = \mathrm{Ker}(B) \cdot \mathrm{Im}(B)$.

Then $\widetilde{B}_{|_{\mathrm{Im}(B)}}$ *is a homomorphism onto the abelian subgroup* R *of* $\mathrm{Im}(B)$, *and* B *is defined on* G *by the formula*

$$B(hl) = B_{|_{\mathrm{Im}(B)}}(l),$$

where $h \in \mathrm{Ker}(B)$ *and* $l \in \mathrm{Im}(B)$.

Proof — Due to Condition (ii), for any $g \in G$, we have a decomposition $g = kB(s)$, where $k \in \mathrm{Ker}(B)$ and $s \in G$. By Proposition 2.1 (d), $B(g) = B^2(s)$, which implies $\mathrm{Ker}(B) = \mathrm{Ker}(B^2)$.

By Proposition 2.4, $\mathrm{Ker}(\widetilde{B}) \trianglelefteq \mathrm{Im}(B)$, so we may consider the groups $\mathrm{Im}(B)/\mathrm{Ker}(\widetilde{B})$, $\mathrm{Im}(\widetilde{B})/\mathrm{Ker}(B)$, and $B, \widetilde{B}$ are RB-operators on it. Moreover, $\widetilde{B}$ is a homomorphism from $(\mathrm{Im}(B))^{(\circ_{\widetilde{B}})}$ into

$$R = \mathrm{Im}(B\widetilde{B}) = \mathrm{Im}(\widetilde{B}B),$$

as well as B is a homomorphism from $(\mathrm{Im}(\widetilde{B}))^{(\circ_B)}$ into R. Here we define $\circ_B$ and $\circ_{\widetilde{B}}$ by (3). Since $\mathrm{Ker}(\widetilde{B}) = \mathrm{Ker}(\widetilde{B}) \cap \mathrm{Im}(B)$, the isomorphism theorem implies $\mathrm{Im}(B)^{(\circ_{\widetilde{B}})}/\mathrm{Ker}(\widetilde{B})^{\circ_{\widetilde{B}}} \simeq R$.

Let φ be a canonical homomorphism from $\mathrm{Im}(\widetilde{B})$ onto $\mathrm{Im}(\widetilde{B})/\mathrm{Ker}(B)$. We have the following sequence of homomorphisms:

$$\mathrm{Im}(B)^{(\circ_{\widetilde{B}})} \xrightarrow{\widetilde{B}} \mathrm{Im}(\widetilde{B}) \xrightarrow{\varphi} \mathrm{Im}(\widetilde{B})/\mathrm{Ker}(B),$$

and define $B' = \widetilde{B} \circ \varphi$ as the composition of them. Let $B(h)$ be an element of $\mathrm{Ker}(B')$. Then

$$B'(B(h)) = \mathrm{Ker}\, B \cdot \widetilde{B}B(h) = \mathrm{Ker}\, B.$$

Since $\mathrm{Ker}(B) = \mathrm{Ker}(B^2)$ and $\mathrm{Im}(B\widetilde{B}) = \mathrm{Im}(\widetilde{B}B)$, we have that $B(h)$ belongs to $\mathrm{Ker}(\widetilde{B})$. Thus, $\mathrm{Ker}(\widetilde{B}) = \mathrm{Ker}(B')$ and by the isomorphism theorem

$$\mathrm{Im}(\widetilde{B})/\mathrm{Ker}(B) \simeq \mathrm{Im}(B)^{(\circ_{\widetilde{B}})}/\mathrm{Ker}(\widetilde{B})^{(\circ_{\widetilde{B}})} \simeq R.$$

Hence, $\mathrm{Im}(\widetilde{B})/\mathrm{Ker}(B)$ is abelian and $\mathrm{Im}(B)/\mathrm{Ker}(\widetilde{B})$ is abelian too due to (2.3). Since $\widetilde{B}$ is an RB-operator on the abelian group $\mathrm{Im}(B)/\mathrm{Ker}(\widetilde{B})$, we derive that $\widetilde{B}$ is an endomorphism of $\mathrm{Im}(B)/\mathrm{Ker}(\widetilde{B})$.

Choose an independent set $r_1, \ldots, r_n$ of generators of an abelian group R and find $t_i \in \mathrm{Im}(B)$ such that $\widetilde{B}(t_i) = r_i$, $i = 1, \ldots, n$. Since $\widetilde{B}$ is a endomorphism of $\mathrm{Im}(B)/\mathrm{Ker}(\widetilde{B})$, $\widetilde{B}(t_it_j) = k_{ij}r_ir_j$

for some $k_{ij} \in \mathrm{Ker}(\widetilde{B})$. By (2.1), we write down

$$r_i r_j = \widetilde{B}(t_i)\widetilde{B}(t_j) = \widetilde{B}(t_i r_i t_j r_i^{-1}).$$

Since $\widetilde{B}$ is well-defined on the abelian group $\mathrm{Im}(B)/\mathrm{Ker}(\widetilde{B})$ and $\mathrm{Ker}(\widetilde{B})t_i t_j = \mathrm{Ker}(\widetilde{B})t_i r_i t_j r_i^{-1}$, we obtain $k_{ij} = e$ and $\widetilde{B}(t_i t_j) = r_i r_j$. Similarly, we show that $\widetilde{B}(t_i t_j t_k) = r_i r_j r_k$ and so on. Thus, $\widetilde{B}$ is a homomorphism from $\mathrm{Im}(B)$ to R. □

Now, we provide the conditions under which we may apply Lemma 5.1. Let B be an RB-operator on a finite group G. If $\mathrm{Ker}(B) = \mathrm{Im}(\widetilde{B})$, then B is a splitting RB-operator by Proposition 2.10. Consider the case

$$\mathrm{Ker}(\widetilde{B}) < \mathrm{Im}(B) \quad \text{and} \quad \mathrm{Ker}(B) < \mathrm{Im}(\widetilde{B}).$$

Denote $H = \mathrm{Im}(B)$, $L = \mathrm{Im}(\widetilde{B})$, and $R = H \cap L$. Then

$$\begin{aligned} |G| = |\mathrm{Im}(\widetilde{B})\mathrm{Im}(B)| &= \frac{|\mathrm{Im}(B)| \cdot |\mathrm{Im}(\widetilde{B})|}{|R|} \\ &= |\,\mathrm{Ker}(B)| \cdot |\mathrm{Im}(B)| = |\,\mathrm{Ker}(\widetilde{B})| \cdot |\mathrm{Im}(\widetilde{B})|, \end{aligned} \tag{5.1}$$

and

$$|R| = |\mathrm{Im}(B) : \mathrm{Ker}(\widetilde{B})| = |\mathrm{Im}(\widetilde{B}) : \mathrm{Ker}(B)| > 1.$$

Lemma 5.2 *Let B be a non-splitting RB-operator on a finite group G and*

$$R = \mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B}).$$

Then the following hold:

(a) $R \nleqslant \mathrm{Ker}(B)$ *or* $R \nleqslant \mathrm{Ker}(\widetilde{B})$.

(b) *if* $R \cdot \mathrm{Ker}(B) = \mathrm{Im}(\widetilde{B})$, *then* $G = \mathrm{Im}(B) \cdot \mathrm{Ker}(B)$ *is an exact factorization.*

Proof — (a) Observe that if $R \leqslant \mathrm{Ker}(B)$ and $R \leqslant \mathrm{Ker}(\widetilde{B})$, then $R \leqslant \mathrm{Ker}(B) \cap \mathrm{Ker}(\widetilde{B}) = 1$, a contradiction.

(b) Since $R \leqslant \mathrm{Im}(B)$, we have

$$G = \mathrm{Im}(B) \cdot \mathrm{Im}(\widetilde{B}) = \mathrm{Im}(B) \cdot R \cdot \mathrm{Ker}(B) = \mathrm{Im}(B) \cdot \mathrm{Ker}(B),$$

where the last factorization is exact. □

Corollary 5.3 *Let* B *be a non-splitting RB-operator on a finite group* G, $R = \mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B})$ *and* $|R|$ *is prime. Then, up to the action* $B \to \widetilde{B}$, B *is defined on* G *by* Lemma 5.1.

Proof — By Lemma 5.2 (a), we may suppose that $R \nleqslant \mathrm{Ker}(B)$. Then $|R \cap \mathrm{Ker}(B)| = 1$ and thus, $R \cdot \mathrm{Ker}(B) = \mathrm{Im}(\widetilde{B})$. By (b) in Lemma 5.2, we get $G = \mathrm{Im}(B) \cdot \mathrm{Ker}(B)$. Therefore, all conditions of Lemma 5.1 are satisfied. □

Remark 5.4 Let us observe that Corollary 5.3 has a simple form, when $|R| = 2$ and $R \nleqslant \mathrm{Ker}(B)$. Indeed, we have the exact factorization $G = \mathrm{Ker}(B) \cdot \mathrm{Im}(B)$. Choose $t \in \mathrm{Im}(B) \setminus \mathrm{Ker}(\widetilde{B})$ and denote $r = \widetilde{B}(t)$. Hence, each $x \in G$ has the form $x = kl$, where $k \in \mathrm{Ker}(B)$ and $l \in \mathrm{Im}(B)$. Since $|\mathrm{Im}(B) : \mathrm{Ker}(\widetilde{B})| = 2$, we get $l = t^{\delta}y$, for some $y \in \mathrm{Ker}(\widetilde{B})$ and $\delta \in \{0, 1\}$. Therefore, B acts on G as follows:

$$B(x) = B(kl) = B(l) = l^{-1}\widetilde{B}(l^{-1}) = l^{-1}\widetilde{B}(y^{-1}t^{\delta}) = l^{-1}r^{\delta}. \quad (5.2)$$

6 Rota–Baxter operators on dihedral groups

Below, we define the dihedral group D_{2n} as follows,

$$D_{2n} = \langle r, s \mid r^n = s^2 = e,\ r^s = r^{-1} \rangle.$$

Theorem 6.1 *The following statements hold*:

(a) *There are no non-splitting RB-operators on* D_{2n}, *where* n *is odd.*

(b) *Let* n *be even. Then all non-splitting RB-operators* B *on* D_{2n} *come from an exact factorization* $D_{2n} = HL$, *where* $H = \mathrm{Ker}(B)$

and $L = \mathrm{Im}(B)$, *up to the action* $B \to \widetilde{B}$. *Moreover,* $\widetilde{B}_{|_L}$ *is a homomorphism onto* $\mathbb{Z}_2$ *or* $\mathbb{Z}_2^2$. *Hence,* B *on* D_{2n} *is defined by* Corollary 2.12.

Proof — (a) We want to show that $r^m s \in \mathrm{Ker}(B\widetilde{B})$, for all m such that $0 \leqslant m < n$. It would imply that $\mathrm{Ker}(B\widetilde{B})$ contains at least $n+1$ elements; since $\mathrm{Ker}(B\widetilde{B})$ is a subgroup, we prove that $B\widetilde{B}(g) = e$, for all $g \in G$. By Proposition 2.10, B is splitting. One of $B(r^m s), \widetilde{B}(r^m s)$ lies in $\langle r \rangle$, say $B(r^m s) = r^k$, for some k. Therefore

$$r^{2k} = B(r^m s)B(r^m s) = B(r^m s r^k r^m s r^{-k}) = B(r^{-2k}),$$

and $r^{2k} \in \mathrm{Ker}(\widetilde{B})$. Since n is odd, we get that $r^k \in \mathrm{Ker}(\widetilde{B})$, and we are done.

(b) Let n be even. For $n = 2$, one may check the statement directly. Let $n > 2$. Define $R = \mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B})$. When $|R| = 2$, the statement follows from Remark 5.4. Hence, we may suppose that $|R| > 2$.

Assume that $\mathrm{Im}(B) = \langle r^{d_1} \rangle$ and $\mathrm{Im}(\widetilde{B}) = \langle r^{d_2}, s \rangle$, where $d_1 \mid n$ and $d_2 \mid n$. Since $\mathrm{Ker}(B) \trianglelefteq \mathrm{Im}(\widetilde{B}) \simeq D_{2n/d_2}$, we have

$$\mathrm{Ker}(B) = \langle r^{d_2 d_3} \rangle,$$

for some d_3. Thus, $\mathrm{Im}(\widetilde{B})/\mathrm{Ker}(B)$ is dihedral and noncyclic, whereas $\mathrm{Im}(B)/\mathrm{Ker}(\widetilde{B})$ is cyclic, a contradiction to (2.3).

It remains to study the situation when both $\mathrm{Im}(B)$ and $\mathrm{Im}(\widetilde{B})$ are dihedral groups.

Case 1. $|\mathrm{Im}(B)|, |\mathrm{Im}(\widetilde{B})| < |G|$.

Denote $H = \mathrm{Im}(B)$ and $L = \mathrm{Im}(\widetilde{B})$. Applying Lemma 2.8 and induction, we have

$$|\mathrm{Im}(B^2\widetilde{B})| = |\mathrm{Im}(B|_H) \cap \mathrm{Im}(\widetilde{B}|_H)| = 2^a, \ |\mathrm{Im}(B\widetilde{B}^2)|$$

$$= |\mathrm{Im}(B|_L) \cap \mathrm{Im}(\widetilde{B}|_L)| = 2^b,$$

for $a, b \in \{0, 1, 2\}$. By Lemma 2.8 and by (2.4), we have

$$R = \mathrm{Im}(B\widetilde{B}) = B(R) \cdot \widetilde{B}(R) = \mathrm{Im}(B^2\widetilde{B}) \cdot \mathrm{Im}(B\widetilde{B}^2).$$

Hence, similarly to (5.1), we deduce that $|R| = 2^c$, for $0 \leqslant c \leqslant 4$.

Case 1A. Suppose that R contains some reflection, so, up to the action of $\mathrm{Aut}(D_{2n})$, we may assume that $\mathrm{Im}(B) = \langle r^{d_1}, s\rangle$ and $\mathrm{Im}(\widetilde{B}) = \langle r^{d_2}, s\rangle$, where $d_1 \mid n$ and $d_2 \mid n$.

According to (2.4), we have $(d_1, d_2) = 1$ and so, $n = d_1 d_2 d_3$, for suitable d_3. Since

$$|R| \cdot |G| = |\mathrm{Im}(B)| \cdot |\mathrm{Im}(\widetilde{B})| = \frac{2n}{d_1} \cdot \frac{2n}{d_2} = 2d_3 \cdot 2n,$$

we derive that $|R| = 2d_3$. Hence, $\mathrm{Ker}(\widetilde{B})$ is a normal subgroup of index $2d_3$ in $\mathrm{Im}(B)$, i.e. $\mathrm{Ker}(\widetilde{B}) = \langle r^{d_1 d_3}\rangle$. Analogously, we obtain that $\mathrm{Ker}(B) = \langle r^{d_2 d_3}\rangle$.

Since we suppose that $|R| > 2$, it remains to study only the case $|R| = 2^c$, for $2 \leqslant c \leqslant 4$. Hence, $d_3 = 2^{c-1} > 1$. Thus, we obtain that $(d_1, d_3) = 1$ or $(d_2, d_3) = 1$. It means that we have an exact factorization, say, $G = \mathrm{Ker}(B) \cdot \mathrm{Im}(B)$. Such factorization implies that $\mathrm{Im}(B^2) = \mathrm{Im}(B)$. By induction,

$$|R| = |(\widetilde{B}B)(G)| = |(\widetilde{B}B^2)(G)| = |(\widetilde{B}B)(H)| = |(B\widetilde{B})(H)| \leqslant 4,$$

and we have exactly the case $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, the statement follows from Lemma 5.1.

Case 1B. Now, consider the case, where R does not contain any reflection. Denote $\mathrm{Im}(B) = \langle r^{d_1}, s\rangle$ and $\mathrm{Im}(\widetilde{B}) = \langle r^{d_2}, r^k s\rangle$, where $d_1 \mid n$, $d_2 \mid n$, and $d_2 \nmid k$. Again, by (2.4), we have either $(d_1, d_2) = 1$ or $(d_1, d_2) = 2$. In the first case,

$$R = \mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B}) = 1,$$

a contradiction. In the second one, we get $R = \{e, r^{n/2}\}$, a contradiction with the assumption $|R| > 2$.

Case 2. Only one of $|\mathrm{Im}(B)|$ and $|\mathrm{Im}(\widetilde{B})|$ is equal to $|G|$.

Suppose that $\mathrm{Im}(B) = G$. Choose $a \in G$ such that $B(a) = r$. Denote $x = a^{\circ(n/2)}$, then $B(x) = r^{n/2}$. By Proposition 2.7, we may rewrite x as

$$x = \big(B_+(a)\big)^{n/2}\big(B(a)\big)^{-n/2} = (ar)^{n/2}r^{n/2}.$$

Hence, we have either $a = r^m s$, with $0 \leqslant m < n$, or $a = r^m$, with $1 \leqslant m < n$. Assume that $a = r^m s$ and $n/2$ is odd. Then we have $x = r^{n/2+m-1}s$. By Lemma 2.14, x is centralized by $\mathrm{Im}(\widetilde{B})$. Hence,

$$\mathrm{Im}(\widetilde{B}) \leqslant C_{D_{2n}}(x) = \big\{e, r^{m-1}s, r^{n/2}, r^{n/2+m-1}s\big\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Moreover, we have the exact factorization $G = \mathrm{Ker}(B) \cdot \mathrm{Im}(B)$, thus, we may again apply Lemma 5.1.

If $a = r^m s$ and $n/2$ is even or $a = r^m$, we get that $x = r^{n/2}$. Define $K = \langle r^{n/2}\rangle$, the normal subgroup of order 2 in D_{2n}. Since $B(r^{n/2}) = r^{n/2}$, we may consider induced RB-operators B and $\widetilde{B}$ on G/K. Note that $\widetilde{B}(r^{n/2}) = e$. By induction, $\mathrm{Im}(\widetilde{B}|_{G/K})$ is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $r^{n/2} \notin \mathrm{Im}(\widetilde{B})$, then

$$\mathrm{Im}(\widetilde{B}) \leqslant \mathbb{Z}_2 \times \mathbb{Z}_2,$$

and the statement holds by Lemma 5.1 due to the exact factorization $G = \mathrm{Ker}(B) \cdot \mathrm{Im}(B)$. If $r^{n/2} \in \mathrm{Im}(\widetilde{B})$, then

$$\mathrm{Im}(\widetilde{B}) \leqslant \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

(since $r^{n/2}$ is central, we get an additional factor $\mathbb{Z}_2$). The case $\mathrm{Im}(\widetilde{B}) \simeq \mathbb{Z}_2^3$ is impossible, because $\mathbb{Z}_2^3$ may not be a subgroup of any dihedral group. Hence, we again apply Lemma 5.1.

Case 3. $|\mathrm{Im}(B)| = |\mathrm{Im}(\widetilde{B})| = |G|$.

Recall that the map B_+, defined as $B_+(g) = gB(g)$, is a homomorphism from $G^{(\circ)}$ in G. Since $|\mathrm{Im}(\widetilde{B})| = |\mathrm{Im}(B_+)| = |G|$,

we have $\mathrm{Ker}(B_+) = e$ and B_+ is an isomorphism. Therefore, the map $\varphi = B^{-1}B_+$ is an automorphism of $G^{(\circ)} \simeq D_{2n}$. It is trivial to check that φ is fixed-point free. Since $n \geqslant 3$, the center of D_{2n} is preserved by every automorphism. Hence, D_{2n} does not admit fixed-point free automorphisms, and we arrive at a contradiction. □

In [14], the correspondence between extensions of RB-groups and the second cohomology group was studied. As examples, the authors considered RB-operators on S_3 and D_8.

Corollary 6.2 *All RB-operators on* $S_3 \simeq D_6$ *are splitting.*

The next example appeared in [32, Example 1.13] in the context of skew left braces, we reformulate it in terms of RB-operators.

Example 6.3 Define a map $B\colon S_3 \to S_3$ as follows,

$$B\colon (ij) \to (23), \quad B\colon (123)^k \to e, \quad i \neq j,\ 1 \leqslant i,j,k \leqslant 3.$$

Then B is a Rota–Baxter operator on S_3. By Corollary 6.2, we know that B is splitting, indeed,

$$S_3 = \langle (123) \rangle \cdot \langle (23) \rangle = \mathrm{Ker}(B) \cdot \mathrm{Ker}(\widetilde{B}).$$

Thus, $S_3^{(\circ)} \simeq \mathbb{Z}_6$.

Example 6.4 In [32, Example 1.18], we have a Rota–Baxter operator B on D_8 defined as follows,

$$B(r^k) = e, \quad B(r^k s) = r, \quad 0 \leqslant k < 4.$$

Note that $\widetilde{B}$ is invertible and $D_8^{(\circ)} \simeq Q_8$.

The next example shows that $\widetilde{B}B \neq B\widetilde{B}$ in general.

Example 6.5 Let us consider the exact factorization $D_{16} = HL$ for $H = \{e, s\}$, $L = \{e, r^2, r^4, r^6, rs, r^3s, r^5s, r^7s\}$. We define B

on D_{16} in such a manner that $H = \mathrm{Ker}(\widetilde{B})$, $\mathrm{Ker}(B) = \langle r^4, rs\rangle$, and

$$B(r^2) = B(r^6) = B(r^3s) = B(r^7s) = r^4.$$

Hence, B is a homomorphism from L onto $\langle r^4\rangle \simeq \mathbb{Z}_2$. In addition, $\mathrm{Im}(B)|_L$ normalizes $\mathrm{Ker}(\widetilde{B})$. Thus, we get an RB-operator on D_{16}. Note that

$$\begin{aligned}(B\widetilde{B})(r) &= B(\widetilde{B}(r)) = B(\widetilde{B}(sr)) = B(\widetilde{B}(r^7s))\\ &= B(r^7sB(r^7s)) = B(r^3s) = r^4,\end{aligned}$$

and

$$\begin{aligned}(\widetilde{B}B)(r) &= \widetilde{B}(B(r)) = \widetilde{B}(r^7\widetilde{B}(r^7)) = \widetilde{B}(r^7\widetilde{B}(sr^7))\\ &= \widetilde{B}(r^7\widetilde{B}(rs)) = \widetilde{B}(r^7\cdot rs) = \widetilde{B}(s) = e.\end{aligned}$$

The following example shows that the center of a group may not be preserved by an RB-operator.

Example 6.6 Define B on D_{2n} in such a way that

$$\widetilde{B}(r^{2k}) = e,\ \widetilde{B}(r^{2k+1}) = r^{n/2}s,\ \widetilde{B}(r^{2k}s) = r^{n/2},\ \widetilde{B}(r^{2k+1}s) = s,$$

with $0\leqslant k < n/2$. Then $\widetilde{B}$ is a homomorphism from D_{2n} onto the group $\mathbb{Z}_2\times\mathbb{Z}_2$ and B is a Rota–Baxter operator on D_{2n} due to the trivial factorization $D_{2n} = \mathrm{Ker}(B)\cdot\mathrm{Im}(B) = 1\cdot D_{2n}$. If $4\nmid n$, we have $B(r^{n/2}) = r^{n/2}\widetilde{B}(r^{n/2}) = s$.

Example 6.7 Consider the exact factorization

$$G = D_{60} = HL = \langle r^{10}\rangle\cdot\langle r^3, s\rangle.$$

We define an RB-operator on $L\simeq D_{20}$ as in Example 6.6. Further, we extend such B on the whole G, putting $\mathrm{Ker}(B) = H$. Surely, the group H is normalized by $\mathrm{Im}(\widetilde{B})$, since H is normal in G.

Remark 6.8 The descendent group $D_{2n}^{(\circ)}$ for odd prime n is discussed in [7, Corollary 6.5].

Recall that a generalized quaternion group Q_{4n} of order $4n$ is defined as follows:

$$Q_{4n} = \langle r, s \mid r^{2n} = e,\ s^2 = r^n,\ r^s = r^{-1} \rangle.$$

Corollary 6.9 *Let* B *be a non-splitting RB-operator on* Q_{4n}*, where* n *is odd. Then* B *is defined by* Remark 5.4.

Proof — It is known that $Z(Q_{4n}) = \{e, r^n\}$. Since r^n is a unique element of order 2 in Q_{4n}, we conclude, by Lemma 2.14, that the normal subgroup $K = \langle r^n \rangle$ is B-invariant: either $B(r^n) = r^n$ or $B(r^n) = e$. Therefore, B is an RB-operator on $Q_{4n}/K \simeq D_{2n}$. By Theorem 6.1 (a), we know that all RB-operators on D_{2n} are splitting. Hence, $|\mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B})| \leqslant 2$. Since B is non-splitting, we have $|R| = 2$. □

Let us show that the case $\mathrm{Im}(B) \cap \mathrm{Im}(\widetilde{B}) \simeq \mathbb{Z}_2$ is realizable on Q_{4n}.

Example 6.10 Consider the exact factorization

$$Q_{60} = \langle r^{10} \rangle \cdot \langle r^3, s \rangle = H \cdot L$$

and define $B\colon Q_{60} \to Q_{60}$ as follows, $\mathrm{Ker}(B) = H$ and $\widetilde{B}\colon L \to L$ acts by the formula

$$\widetilde{B}(r^{3k}s^i) = \begin{cases} r^{15}, & i = 1, \\ e, & i = 0. \end{cases}$$

It is an RB-operator on Q_{60}, since $\widetilde{B}$ is an endomorphism of L and $\mathrm{Im}(\widetilde{B}|_L)$ normalizes H.

Remark 6.11 *In [8], all braces with dihedral or generalized quaternion multiplicative group were described.*

7 Rota–Baxter operators on alternating groups

This section is devoted to the study of Rota–Baxter operators on simple groups A_n, $n \geqslant 5$. For completeness, let us deal with A_n for small n. Since $A_3 \simeq \mathbb{Z}_3$ is abelian, an RB-operator on it is nothing more than a homomorphism.

Computations show that, up to equivalence, we have the following non-trivial RB-operators on A_4:

(B1) Splitting one coming from the exact factorization $A_4 = HL$, where

$$H = \langle (234) \rangle \simeq \mathbb{Z}_3 \quad \text{and} \quad L = V_4$$

is the Klein four-group.

(B2) Non-splitting one coming from the exact factorization of the type $A_4 = HL$, where

$$H = \{e\} \quad \text{and} \quad L = A_4,$$

and an RB-operator C on L is a natural homomorphism from $L = \mathbb{Z}_3 \rtimes V_4$ onto $\mathbb{Z}_3$. Therefore, $\mathrm{Im}(B\widetilde{B}) \simeq \mathbb{Z}_3$ and Lemma 5.1 is again valid.

The fact that $A_4^{(\circ)}$ is isomorphic either to A_4 (for trivial RB-operators) or to $\mathbb{Z}_6 \times \mathbb{Z}_2$ may be derived from [13].

The study of splitting Rota–Baxter operators is the same as the study of exact factorizations, which are described in [34]. Therefore, we will focus on non-splitting RB-operators on A_n.

Below, we will apply the following conditions on the numbers m, q:

(i) q is a prime-power.

(ii) Each prime divisor of m divides $q - 1$.

(iii) $q \equiv 3 \pmod 4$ and $m \equiv 2 \pmod 4$.

Theorem 7.1 *The following statements hold*:

(1) *A non-splitting Rota–Baxter operator on a simple alternating group* A_n *exists if and only if* $n \in \{q^m,\ q^m+1\}$ *and* m, q *satisfy the conditions* (i)–(iii).

(2) *If a non-splitting Rota–Baxter operator on* A_n *exists, then there are* s *non-equivalent non-splitting Rota–Baxter operators on* A_n, *where*:

(a) $s = 3$ *for* $n \in \{7^2,\ 23^2\}$.

(b) $s = 2$ *for* $n = q^m$, $n \notin \{3^2,\ 7^2,\ 23^2\}$.

(c) $s = 1$ *for* $n \in \{3^2,\ q^m+1\}$.

Proof — Let B be a non-splitting RB-operator on $G = A_n$. According to Proposition 2.4, we have

$$G = \mathrm{Im}(B)\cdot\mathrm{Im}(\widetilde{B}) = H\cdot L,\ \mathrm{Ker}(\widetilde{B}) \trianglelefteq \mathrm{Im}(B),\ \text{and}\ \mathrm{Ker}(B) \trianglelefteq \mathrm{Im}(\widetilde{B}).$$

If $\mathrm{Ker}(B) = 1$ then B is a splitting RB-operator [4, Theorem 50]. If $\mathrm{Ker}(B) = \mathrm{Im}(\widetilde{B})$ then B is also a splitting RB-operator by Proposition 2.10. The same is true for $\mathrm{Ker}(\widetilde{B})$. Therefore, we consider only the following case:

$$1 < \mathrm{Ker}(\widetilde{B}) < \mathrm{Im}(B) \qquad \text{and} \qquad 1 < \mathrm{Ker}(B) < \mathrm{Im}(\widetilde{B}).$$

Define $R = H \cap L$. By Lemma 5.2 (a) it follows that either

$$R \not\leqslant \mathrm{Ker}(B) \qquad \text{or} \qquad R \not\leqslant \mathrm{Ker}(\widetilde{B}).$$

Since $\mathrm{Ker}(\widetilde{B}) > 1$ and $\mathrm{Ker}(B) > 1$, we obtain that $G = HL$, where H and L are proper subgroups of G. Then, according to [27, Theorem D], L is k-homogeneous ($1 \leqslant k \leqslant 5$), and

$$A_{n-k} \trianglelefteq H \leqslant (S_{n-k} \times S_k) \cap A_n.$$

We consider all the possibilities for k. In order to avoid non-simple groups A_n, we assume that $n \geqslant 10$. The cases $5 \leqslant n \leqslant 9$

can be checked manually using the information from [12] or with the help of GAP.

$k = 1$. In this case, $H = A_{n-1}$ is a simple group and it follows from the proof of [4, Theorem 53] that B must be splitting.

$k = 5$. By [25, Theorem 1], there is no 5-homogeneous but not 5-transitive finite group. Therefore, L is 5-transitive, and, by [9, Theorem 5.3], L must be one of the Mathieu groups M_{12} or M_{24}. Mathieu groups are simple, and we do not have a non-splitting RB-operator.

$k = 4$. Observe that the Mathieu groups M_{11} and M_{23} are the only finite 4-transitive groups, which are not 5-transitive and so have not been considered. Since these two groups are simple, we need to consider 4-homogeneous but not 4-transitive groups. By [25, Theorem 1], we have $L \in \{PSL_2(8), P\Gamma L_2(8), P\Gamma L_2(32)\}$.

Since $n \geqslant 10$, we need to consider only the case $L = P\Gamma L_2(32)$. By [25], we have that $L = P\Gamma L_2(32)$ acts sharply 3-transitively on $\Omega = \{1, 2, \ldots, 33\}$. Since the only proper normal subgroup of L is $PSL_2(32)$, we have

$$\mathrm{Ker}(B) = PSL_2(32) \qquad \text{and} \qquad |P\Gamma L_2(32) : PSL_2(32)| = 5.$$

Then,

$$R \simeq \mathbb{Z}_5 \qquad \text{and} \qquad R \cdot \mathrm{Ker}(B) = \mathrm{Im}(\widetilde{B}),$$

since $(|R|, |\mathrm{Ker}(B)|) = 1$. Therefore,

$$G = \mathrm{Im}(B) \cdot \mathrm{Ker}(B) = H \cdot PSL_2(32)$$

is the exact factorization. Since $PSL_2(32)$ acts sharply 3-transitively on Ω, we get that $H = A_{30}$, see [34, Theorem A(I)]. Thus, H is simple, and we do not have a non-splitting RB-operator.

$k = 2$. In this case, $A_{n-2} \trianglelefteq H \leqslant (S_{n-2} \times S_2) \cap A_n$. Since H is not simple, we have

$$\mathrm{Im}(B) = H = (S_{n-2} \times S_2) \cap A_n \simeq S_{n-2}.$$

The only proper normal subgroup of S_{n-2} is A_{n-2}. Therefore

$$\mathrm{Ker}(\widetilde{B}) = A_{n-2} \qquad \text{and} \qquad |R| = |S_{n-2} : A_{n-2}| = 2.$$

(a) Assume that $R \not\leqslant \mathrm{Ker}(B)$. By Lemma 5.2 (b) we have the exact factorization

$$G = \mathrm{Im}(B)\,\mathrm{Ker}(B) = S_{n-2} \cdot K.$$

According to [34, Theorem A(II)], we have that $K = ASL(1, q)$, where $q \equiv 3 \pmod 4$. Since

$$\mathrm{Ker}(B) \trianglelefteq \mathrm{Im}(\widetilde{B}) \qquad \text{and} \qquad |\mathrm{Im}(\widetilde{B}) : \mathrm{Ker}(B)| = |R| = 2,$$

it follows that $\mathrm{Im}(\widetilde{B}) = AGL(1, q)$. However, $AGL(1, q)$ contains odd permutations; a contradiction.

(b) Assume that $R \not\leqslant \mathrm{Ker}(\widetilde{B})$. By Lemma 5.2 (b) we have the exact factorization

$$G = \mathrm{Im}(\widetilde{B})\,\mathrm{Ker}(\widetilde{B}) = L \cdot A_{n-2}.$$

According to [34, Theorem A(I)], L is a sharply 2-transitive group. Since $|\mathrm{Im}(\widetilde{B}) : \mathrm{Ker}(B)| = 2$, L must contain a normal subgroup of index 2 and we may apply Corollary 3.3.

At first, consider the case (1) of Corollary 3.3, where

$$L = L(m, q, t) = FN.$$

Then we have the following conditions:

(1) Each prime divisor of m divides $q - 1$.

(2) If 4 divides m, then 4 divides $q - 1$.

(3) $q \equiv 3 \pmod 4$.

Since $q \equiv 3 \pmod 4$, then condition (2) is equivalent to the condition $m \equiv 2 \pmod 4$ and we have the conditions (i)–(iii) of the theorem. By Corollary 3.3(1), $L = FN$ has two non-isomorphic normal subgroups FS_1 and FS_2.

Thus, we apply Remark 5.4 and get two non-equivalent RB-operators on G (see Examples 7.2 and 7.3 below).

The cases (2) and (3) of Corollary 3.3 provide two other sharply 2-transitive groups $L = FN$ of degree $n = 7^2$ and $n = 23^2$, respectively. Each of these groups has only one normal subgroup of index 2, and again by Remark 5.4 we get one RB-operator in each case. Note that, if $n = 7^2$, we have two non-isomorphic sharply 2-transitive groups

$$L_1 = L(m, q, t) = L(2, 7, 1) \quad \text{and} \quad L_2 = FN$$

with $N \simeq SL_2(3).2$. As it was mentioned above, L_1 provides two non-equivalent RB-operators on G and L_2 provides the third one. Since $L = \mathrm{Im}(\widetilde{B})$ and $L_1 \not\simeq L_2$, we have three non-equivalent RB-operators on $G = A_{7^2}$. Similarly, we get three non-equivalent RB-operators on $G = A_{23^2}$.

k = 3. In this case, we have that $A_{n-3} \trianglelefteq H \leqslant (S_{n-3} \times S_3) \cap A_n$ and L is 3-homogeneous. Since H is not simple, we have

$$H \in \big\{(S_{n-3} \times S_3) \cap A_n,\ (S_{n-3} \times S_2) \cap A_n,\ A_{n-3} \times A_3\big\}.$$

(a) At first, consider the case $H = (S_{n-3} \times S_2) \cap A_n \simeq S_{n-3}$. The only proper normal subgroup of H is A_{n-3}. Therefore

$$\mathrm{Ker}(\widetilde{B}) = A_{n-3},\ |R| = |H : \mathrm{Ker}(\widetilde{B})| = 2,$$

and

$$|\mathrm{Im}(\widetilde{B})| = |L| = \frac{|G| \cdot |R|}{|H|} = \frac{(n!/2) \cdot 2}{(n-3)!} = n(n-1)(n-2).$$

Then L is a sharply 3-transitive group. Such groups have degree $q + 1 = p^r + 1$, for some prime p. It follows that, if $p = 2$, by [23, Theorem 2.1], $L \simeq PSL_2(q)$ is simple, which is not the case.

Let p be an odd prime. By [23, Theorem 2.6], we have either $L = PGL_2(q)$, or r is even and $L = M(p^r) \simeq PSL_2(q).2$. In

both cases, $\mathrm{PSL}_2(q)$ is a subgroup of index 2 in L. Since $\mathrm{PGL}_2(q)$ contains an element g of order $q+1$, we have $g \notin \mathrm{A}_{q+1} = \mathrm{A}_n$ and $\mathrm{L} \neq \mathrm{PGL}_2(q)$.

The group $\mathrm{L} = \mathrm{M}(p^r)$ acts on the projective line

$$\Omega = \mathrm{GF}(p^r) \cup \{\infty\}$$

(see [23, Example 1.3]). The stabilizer L_∞ is a sharply 2-transitive Frobenius group of odd degree q. This case was considered above, and we have the conditions (i)–(iii) on the numbers m and q.

Note that, under these conditions, we have the exact factorization $\mathrm{G} = \mathrm{A}_{n-3} \cdot \mathrm{L}$, where $\mathrm{Ker}(\widetilde{\mathrm{B}}) = \mathrm{A}_{n-3} \trianglelefteq \mathrm{H} = (\mathrm{S}_{n-3} \times \mathrm{S}_2) \cap \mathrm{A}_n$ and $\mathrm{Ker}(\mathrm{B}) = \mathrm{PSL}_2(q) \trianglelefteq \mathrm{L} = \mathrm{M}(q^m)$. Now we can apply Remark 5.4 to get one RB-operator in each case.

(b) $\mathrm{H} = (\mathrm{S}_{n-3} \times \mathrm{S}_3) \cap \mathrm{A}_n \simeq (\mathrm{A}_{n-3} \times \mathrm{A}_3) : \mathbb{Z}_2$. We have

$$|\mathrm{R}| = \frac{|\mathrm{H}| \cdot |\mathrm{L}|}{|\mathrm{G}|} = \frac{(n-3)! \cdot 3 \cdot |\mathrm{L}|}{(n!/2)} = 3 \cdot \frac{2|\mathrm{L}|}{(n-2)(n-1)n}.$$

If L is 3-transitive, then $n(n-1)(n-2)$ divides L. If L is not 3-transitive, then $|\mathrm{L}|$ is divisible by $\mathrm{L} = A\Gamma\mathrm{L}_1(32)$ or

$$|\mathrm{PSL}_2(q)| = (q-1)q(q+1)/2 = (n-2)(n-1)n/2$$

(see [25, Theorem 1 (ii), (iii)]). Since $|A\Gamma\mathrm{L}_1(32)| = 32 \cdot 31 \cdot 5$, we have $|\mathrm{R}| = 30/33$ and this case is impossible. In two other cases, we obtain that 3 divides $|\mathrm{R}| = |\mathrm{Im}(\mathrm{B}) : \mathrm{Ker}(\widetilde{\mathrm{B}})|$.

On the other hand, $\mathrm{Im}(\mathrm{B}) = \mathrm{H} = (\mathrm{A}_{n-3} \times \mathrm{A}_3) : \mathbb{Z}_2$ has only three proper normal subgroups: $\mathrm{A}_{n-3} \times \mathrm{A}_3$, A_{n-3}, and A_3.

Since $|\mathrm{H} : (\mathrm{A}_{n-3} \times \mathrm{A}_3)| = 2$, the case $\mathrm{Ker}(\widetilde{\mathrm{B}}) = \mathrm{A}_{n-3} \times \mathrm{A}_3$ is impossible.

If $\mathrm{Ker}(\widetilde{\mathrm{B}}) = \mathrm{A}_3$ then $|\mathrm{R}| = |\mathrm{Im}(\mathrm{B})|/3 = (n-3)!$, a contradiction with $\mathrm{R} \leqslant \mathrm{L}$.

Assume that $\mathrm{Ker}(\widetilde{\mathrm{B}}) = \mathrm{A}_{n-3}$. Then $\mathrm{R}| = |\mathrm{Im}(\mathrm{B}) : \mathrm{Ker}(\widetilde{\mathrm{B}})| = 6$

and

$$|L| = \frac{|G| \cdot |R|}{|H|} = \frac{(n!/2) \cdot 6}{((n-3)!/2) \cdot 6} = n(n-1)(n-2).$$

Then L is a sharply 3-transitive group. According to case (a) we have $L \simeq PSL_2(q).2$. Therefore, L has proper normal subgroups only of index 2. It is a contradiction with

$$|R| = |\mathrm{Im}(\widetilde{B}) : \mathrm{Ker}(B)| = |L : \mathrm{Ker}(B)| = 6,$$

(c) $H = A_{n-3} \times A_3$. The only proper normal subgroups of H are A_{n-3} and A_3. If $\mathrm{Ker}(\widetilde{B}) = A_3$ then

$$|R| = \frac{|H|}{|A_3|} = \frac{(n-3)!}{2},$$

a contradiction with $R \leqslant L$.

Therefore, $\mathrm{Ker}(\widetilde{B}) = A_{n-3}$ and $|R| = |H : \mathrm{Ker}(\widetilde{B})| = 3$. We have

$$|\mathrm{Im}(\widetilde{B})| = |L| = \frac{|G| \cdot |R|}{|H|} = \frac{(n!/2) \cdot 3}{((n-3)!/2) \cdot 3} = n(n-1)(n-2).$$

Thus, L is a sharply 3-transitive group. According to case (a) we have $L \simeq PSL_2(q).2$. Therefore, L has proper normal subgroups only of index 2. It is a contradiction with

$$|R| = |\mathrm{Im}(\widetilde{B}) : \mathrm{Ker}(B)| = |L : \mathrm{Ker}(B)| = 3,$$

which completes the proof. □

Example 7.2 We start with an exact factorization

$$A_9 = \mathrm{Ker}(B) \cdot \mathrm{Im}(B) = A_7 \cdot L,$$

where $A_7 = \langle (1234567), (123) \rangle$ and

$$L = \langle (1456)(2789), (1683)(2579) \rangle \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3) : Q_8.$$

Then

$$H = \mathrm{Im}(\widetilde{B}) = (S_7 \times S_2) \cap A_9 = \langle A_7, r \rangle,$$

where $r = (17)(25)(34)(89)$. Note that $R = H \cap L = \langle r \rangle$. The group L has three isomorphic subgroups of index 2, and we choose one of them,

$$\mathrm{Ker}(\widetilde{B}) = S = \langle (167)(238)(459), (1683)(2579) \rangle \simeq (\mathbb{Z}_3 \times \mathbb{Z}_3) : \mathbb{Z}_4.$$

Then the RB-operator B on A_9 is defined by Remark 5.4. Namely, each $x \in A_9$ has the form

$$x = kl = kt^{\delta}y,$$

where $k \in \mathrm{Ker}(B)$, $y \in \mathrm{Ker}(\widetilde{B})$, and $\delta \in \{0, 1\}$. The operator B acts as follows:

$$B(x) = B(kt^{\delta}y) = l^{-1}r^{\delta}.$$

The following example provides an infinite series of non-splitting Rota–Baxter operators on A_n.

Example 7.3 Let $m = 2$, $q = p$ be a prime, and $p \equiv 3 \pmod 4$. There is an infinite number of such primes p and the conditions (i)–(iii) of Theorem 7.1 hold. For every such p we get a Rota–Baxter operator on A_{p^2} as in Example 7.2.

We start with an exact factorization

$$A_n = \mathrm{Ker}(B) \cdot \mathrm{Im}(B) = A_{n-2} \cdot L,$$

where L is a sharply 2-transitive group. Then

$$H = \mathrm{Im}(\widetilde{B}) = (S_{n-2} \times S_2) \cap A_n = \langle A_{n-2}, r \rangle,$$

where $r \in R = H \cap L$. The group L has two non-isomorphic subgroups of index 2 and we choose one of them as $\mathrm{Ker}(\widetilde{B})$. Denote $S = \mathrm{Ker}(\widetilde{B})$.

Observe that each $x \in A_n$ has the form $x = kl = kt^{\delta}y$, where $k \in \mathrm{Ker}(B)$, $y \in \mathrm{Ker}(\widetilde{B})$, and $\delta \in \{0, 1\}$. The operator B

again acts by the formula (5.2). Let us clarify the structure of the descendent group $A_n^{(\circ)}$ (2.2). We have

$$A_n^{(\circ)} = A_{n-2} \rtimes L = (A_{n-2} \times S^{op}).2,$$

where for all $h \in A_{n-2}$, $l \in L$ one computes

$$l \circ h = lB(l)hB(l)^{-1} = r^{\delta} h r^{\delta} l = h^{r^{\delta}} l = h^{r^{\delta}} \circ l;$$

the product $\circ$ on A_{n-2} coincides with $\cdot$; the product $\circ$ on the set S is antiisomorphic to $\cdot$, i.e. $s \circ s' = s's$.

The authors can make the GAP code used in this study available upon request.

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