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On Quantitative Structure of Symplectic Groups of Dimension Four

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Abstract

The main aim of this article is to study the quantitative structure of projective symplectic groups $PSp_4(q)$ with $q > 2$ even. Indeed, we prove that the groups $PSp_4(q)$ with $q > 2$ even are uniquely determined by their orders and the set of the number of elements of the same order. This result links to the well-known J.G. Thompson's problem (1987) for finite simple groups.

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1 Introduction

For a finite group G and a positive integer n, let G_n consist of all elements x satisfying $x^n = 1$. The *type* of G is defined to be the function whose value at n is the order of G_n. This notion links to the notion of *same-order type* $\tau(G)$ of G, that is to say, *the set of the number of elements of the same order* in G. Indeed, it turns out that if two groups G and H are of the same type, then $\tau(G) = \tau(H)$ and $|G| = |H|$. However, finite groups with the same order and same-order type may not have the same type or even the same *spectrum*, the set of element orders. The smallest examples of such groups that we have found are

$$G := \mathbb{Z}_4 \times (\mathbb{Z}_7 : \mathbb{Z}_3) \qquad \text{and} \qquad H := \mathbb{Z}_3 \times (\mathbb{Z}_7 : \mathbb{Z}_4)$$

of order 84 in which $\tau(G) = \tau(H) = \{1, 2, 6, 12, 14, 28\}$, and G has an element of order 28 while H has no such an element. Moreover, G and H are not of the same type as $|G_3| = 15$ and $|H_3| = 3$.

The main objective of this paper is to investigate the structure of groups with the same-order type as finite simple groups. In this direction, Shao et al. [14] studied finite simple groups whose order is divisible by at most four primes. This problem has been studied for several families of simple groups including small Ree groups and Suzuki groups [1, 2]. In this paper, we study groups with the same order and the same-order type as $PSp_4(q)$ with q even.

Theorem 1.1 *Let* G *be a group such that* $\tau(G) = \tau(S)$ *and* $|G| = |S|$, *where* $S = PSp_4(q)$ *with* $q > 2$ *even. Then* G *is isomorphic to* S.

In 1987, J.G. Thompson [10, Problem 12.37] asked if a group with the same type as a solvable group is still solvable. One may ask this problem for non-solvable groups, in particular, finite simple groups. It is easy to see that two groups with the same type share the same order and and the same spectrum. Hence, the recognition of finite simple groups by their orders and spectra, which is known known in the literature as Shi's problem [10, Problem 12.39], gives a positive answer to Thompson's problem. It is worth noting that Vasil'ev, Grechkoseeva and Mazurov [15] finalized a complete solution to Shi's problem, and consequently Thompson's problem holds for finite simple groups. As noted above, two groups G and H with the same type have the same order and same-order type, and hence Theorem 1.1 also gives a positive answer to Thompson's problem for finite simple groups $PSp_4(q)$ with q even.

Corollary 1.2 *If* G *is a finite group of the same type as* $S := PSp_4(q)$ *with* $q > 2$ *even, then* G *is isomorphic to* S.

Our analysis in proving Theorem 1.1 depends on determining $\tau(S)$, where $S = PSp_4(q)$ with $q > 2$ even, see Proposition 3.3. Despite of the investigation of the groups which have been previously considered, we have generally no isolated vertex in the prime graph of a group G with the same number of elements of the same order as S.

1.1 Definitions and notation

In this section, we give some brief comments on the notation used in this paper. Throughout this article all groups are finite. Our group theoretic notation is standard, and it is consistent with the notation

in [3, 5, 8] with noting that we use Alt_n to denote the alternating group on n letters. In particular, we follow the notation below for the finite simple classical groups:

$\mathrm{PSL}_n(q)$, for $n \geqslant 2$ and $(n,q) \neq (2,2),(2,3)$,

$\mathrm{PSU}_n(q)$, for $n \geqslant 3$ and $(n,q) \neq (3,2)$,

$\mathrm{PSp}_{2m}(q)$, for $m \geqslant 2$ and $(m,q) \neq (2,2)$,

$\mathrm{P\Omega}_{2m+1}(q)$, for $m \geqslant 3$ and q odd,

$\mathrm{P\Omega}^{\pm}_{2m}(q)$, for $m \geqslant 4$.

In this manner, the only repetitions are:

$\mathrm{PSL}_2(4) \simeq \mathrm{PSL}_2(5) \simeq \mathrm{Alt}_5$, $\mathrm{PSL}_2(7) \simeq \mathrm{PSL}_3(2)$, $\mathrm{PSL}_2(9) \simeq \mathrm{Alt}_6$,

$\mathrm{PSL}_4(2) \simeq \mathrm{Alt}_8$ and $\mathrm{PSp}_4(3) \simeq \mathrm{PSU}_4(2)$.

We denote the set of Sylow p-subgroups of G by $\mathrm{Syl}_p(G)$. We also use $n_p(G)$ to denote the number of Sylow p-subgroups of G. For a positive integer n, the set of prime divisors of n is denoted by $\pi(n)$, and if G is a finite group, $\pi(G) := \pi(|G|)$, where $|G|$ is the order of G. Moreover, we denote the set of elements' orders of G by $\omega(G)$ known as *spectrum* of G. The *prime graph* $\Gamma(G)$ of a finite group G is a graph whose vertex set is $\pi(G)$, and two distinct vertices p and q are adjacent if and only if $pq \in \omega(G)$. Assume further that $\Gamma(G)$ has $t(G)$ connected components $\pi_i := \pi_i(G)$, for $i = 1, 2, \ldots, t(G)$. The positive integers n_i with $\pi(n_i) = \pi_i$ are called *order components* of G. Evidently, $|G| = n_1 \ldots n_{t(G)}$. We denote by $\mathrm{oc}(G)$ the set of all order components of G. In the case where G is of even order, we always assume that $2 \in \pi_1$, and π_1 is said to be the even component of G. In this way, π_i and n_i are called odd components and odd order components of G, respectively. Recall that $\tau(G)$ is the set of the number of elements in G with the same order. In other word, $\tau(G)$ consists of the numbers $m_i(G)$ of elements of order i in G, for $i \in \omega(G)$. Here, φ and ψ are the *Euler totient* function and the *Dedekind arithmetic* function, respectively. In the other word, for every positive integer n,

$$\varphi(n) = n \prod_{p \in \pi(n)} \left(1 - \frac{1}{p}\right) \qquad \text{and} \qquad \psi(n) = n \prod_{p \in \pi(n)} \left(1 + \frac{1}{p}\right).$$

We also use $\Phi_n(q)$ to denote the n-*th cyclotomic polynomial* of order n.

2 Preliminaries

In this section, we give some useful results which we frequently use in order to prove our results.

Lemma 2.1 (see [4], Theorem 1) *Let* G *be a Frobenius group of even order with kernel* K *and complement* H*. Then* $\pi(H)$ *and* $\pi(K)$ *are the only connected components of* $\Gamma(G)$*.*

A group G is called 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively.

Lemma 2.2 (see [4], Theorem 2) *Let* G *be a 2-Frobenius group of even order. Then* $\Gamma(G)$ *has exactly two connected components. Moreover, there exists a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that*

(i) $\pi_1(G) = \pi(H) \cup \pi(G/K)$*, and* $\pi_2(G) = \pi(K/H)$*;*

(ii) G/K *and* K/H *are cyclic groups;*

(iii) $|G/K|$ *divides* $|Aut(K/H)|$*, and* $(|G/K|, |K/H|) = 1$*;*

(iv) H *is a nilpotent group and* G *is a solvable group.*

Lemma 2.3 (see [11, 17]) *Let* G *be a finite group with more than one prime graph component. Then one of the following statements holds*:

(i) G *is a Frobenius group;*

(ii) G *is a 2-Frobenius group;*

(iii) G *admits a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that* H *and* G/K *are* $\pi_1(G)$*-groups,* K/H *is a non-abelian simple group,* H *is a nilpotent group,* $|G/K|$ *divides* $|Out(K/H)|$*,* $t(K/H) \geqslant t(G)$*, and for every* $i \in \{2, \ldots, t(G)\}$*, there exists* $j \in \{2, \ldots, t(K/H)\}$ *such that*

$$\pi_i(G) = \pi_j(K/H).$$

Lemma 2.4 (see [13], p. 50) *All subgroups and all quotient groups of any cyclic group are cyclic. If* G *is a cyclic group of finite order* n*, then* G *has just one subgroup* H *of order* m *for each divisor* m *of* n*,* H *is cyclic and* G/H *is cyclic of order* n/m*.*

Lemma 2.5 (see [13], p. 35) *Let n be an integer, $n > 1$, and let the factorization of n as a product of primes be*

$$n = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r},$$

where $r, m_1, \dots, m_r$ are positive integers and $p_1, \dots, p_r$ distinct primes. Then $\mathbb{Z}_n$ is decomposable if $r > 1$, and

$$\mathbb{Z}_n \simeq \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2} \times \dots \times \mathbb{Z}_{q_r},$$

where $q_i = p_i^{m_i}$, for each $i = 1, \dots, r$. Moreover, for each prime p and positive integer m, $_{p^m}$ is indecomposable.

Lemma 2.6 (see [13], p. 198) *Let n be a positive integer, and let G be an abelian group. Also let*

$$G_n = \{x \in G \,:\, x^n = 1\} \qquad \text{and} \qquad G^n = \{x^n \,:\, x \in G\}.$$

Then the following statements hold:

(i) $G_n \leqslant G$, *and* $G/G_n \simeq G^n$;

(ii) *Suppose that G is finite, so that there are elements $x_1, \dots, x_r$ of G such that*

$$G = \langle x_1 \rangle \times \dots \times \langle x_r \rangle.$$

For each $i = 1, \dots, r$, let $n_i = o(x_i)$ and $k_i = n_i/(n_i, n)$. Then

$$G_n = \langle x_1^{k_1} \rangle \times \dots \times \langle x_r^{k_r} \rangle;$$

(iii) *If G is finite, then $G/G^n \simeq G_n$.*

Lemma 2.7 (see [9], Theorem 9.1.2) *Let G be a finite group, and let n be a positive integer dividing $|G|$. Then n divides $|G_n|$.*

The proof of the following result is straightforward by Lemma 2.7.

Lemma 2.8 *Let G be a finite group. Then, for every $i \in \omega(G)$,*

$$m_i(G) = k\varphi(i),$$

where k is the number of cyclic subgroups of order i in G, and i divides

$$\sum_{j|i} m_j(G).$$

Moreover, if $i > 2$, *then* $m_i(G)$ *is even.*

Lemma 2.9 (see [16], Theorem 3) *Let* G *be a finite group. Then the number of elements whose orders are multiples of* n *is either zero, or a multiple of the greatest divisor of* $|G|$ *that is prime to* n.

Lemma 2.10 (see [18], Lemma 2.6) *Let* G *be a finite group, and* $PSp_4(q)$ *be a projective symplectic group, where* $q > 3$. *If* $oc(G) = oc(PSp_4(q))$, *then* $G \simeq PSp_4(q)$.

Lemma 2.11 (see [6]) *Let* p *and* q *be primes such that* $p^m = q^n + 1$, *for some positive integers* m *and* n. *Then one of the following statements holds*:

(i) $(p, q, m, n) = (3, 2, 2, 3)$;

(ii) $p = 2^n + 1$ *is a Fermat prime with* n *a power of 2;*

(iii) $q = 2^m - 1$ *is a Mersenne prime with* m *prime.*

Lemma 2.12 *Let* q *be a power of 2. Then the following statements hold*:

(i) $2q^2 + 3$ *does not divide* $q^4(q^4 - 1)(q^2 - 1)$;

(ii) *If* $q^2 + 2$ *divides* $q^4(q^4 - 1)(q^2 - 1)$, *then* $q = 2, 4$;

(iii) *If* $2q^2 + 1$ *divides* $q^4(q^4 - 1)(q^2 - 1)$, *then* $q = 2$;

(iv) *If* $3q^2 + 2$ *divides* $q^4(q^4 - 1)(q^2 - 1)$, *then* $q = 4$;

(v) $q^4 - 9$ *does not divide* $q^4(q^4 - 1)(q^2 - 1)$.

Proof — We know that $2q^2 + 3$ is coprime to $q^4(q^2 + 1)$. If $2q^2 + 3$ would divide $q^4(q^4 - 1)(q^2 - 1)$, then $2q^2 + 3$ would divide $(q^2 - 1)^2$. Since $4(q^2 - 1)^2 = (2q^2 - 7)(2q^2 + 3) + 25$, it would follow that

$$2q^2 + 3 \mid 25,$$

and so $q^2 \leqslant 11$ which is true when $q = 2$ but in this case $2q^2 + 3 = 11$ does not divide 25. This follows part (i). To prove part (ii), since $q^2 + 2$ is coprime to $q^2 + 1$, if $q^2 + 2$ divides $q^4(q^4 - 1)(q^2 - 1)$, we conclude that $q^2 + 2$ has to divide $q^4(q^2 - 1)^2$. We know that

$$q^4(q^2 - 1)^2 = (q^6 - 4q^4 + 9q^2 - 18)(q^2 + 2) + 36.$$

Then $q^2 + 2$ divides 36, and this is true if and only if $q = 2, 4$, as desired. Parts (iii)–(v) can be proved in a same manner. □

3 Properties of projective symplectic groups

In this section, we state some useful facts about projective symplectic groups of dimension 4 and their element orders in order to obtain the number of the elements of the same order of $\mathrm{PSp}_4(q)$ with $q = 2^f > 2$ in Section 3.2 below.

3.1 Conjugacy classes

Let $\overline{\mathbb{F}}$ be the algebraic closure of the finite field $\mathbb{F}$ of size $q = p^f$. Let also

$$\mathbb{F}_i = \left\{x \in \overline{\mathbb{F}} \ : \ x^{q^i} = x\right\}$$

be the subfield of $\overline{\mathbb{F}}$ with q^i elements, and let $\mathbb{F}_1 := \mathbb{F}$. Let now κ be a fixed generator of the multiplicative group $\mathbb{F}_4^\times$, and set

$$\tau := \kappa^{q^2-1}, \ \theta := \kappa^{q^2+1}, \ \eta := \theta^{q-1} \quad \text{and} \quad \gamma := \theta^{q+1}.$$

Then $\mathbb{F}_2^\times = \langle\theta\rangle$ and $\mathbb{F}^\times = \langle\gamma\rangle$. For $t \in \mathbb{F}$ and $z_i \in \mathbb{F}_4^\times$, define

$$x_a(t) = \begin{bmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & -t \\ & & & 1 \end{bmatrix}, \quad x_{a+b}(t) = \begin{bmatrix} 1 & & t & \\ & 1 & & t \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

$$x_b(t) = \begin{bmatrix} 1 & & & \\ & 1 & t & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad x_{2a+b}(t) = \begin{bmatrix} 1 & & & t \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \tag{3.1}$$

$$\text{and} \quad h(z_1, z_2) = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & z_2^{-1} & \\ & & & z_1^{-1} \end{bmatrix}.$$

Let

$$\omega_\iota := x_\iota(1)' x_\iota(-1) x_\iota(1)$$

for $\iota \in \Delta^+ := \{a, b, a+b, 2a+b\}$, where $x_\iota(1)'$ is the transposed matrix of $x_\iota(1)$. In particular,

$$\omega_a = \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix} \quad \text{and} \quad \omega_b = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix}. \tag{3.2}$$

Therefore, the 4-dimensional projective symplectic group $PSp_4(q)$ over $\mathbb{F}$ is generated by $x_\iota(t)$, $h(z_1, z_2)$, ω_a and ω_b defined as in (3.1) and (3.2). We use the following set of parameters for the conjugacy classes: for $i, j \in \mathbb{Z}$, "$\mathbb{Z} \pmod m$" means that i and j give the same class if $i \equiv j \pmod m$.

$$\begin{aligned}
T_0 &= \{i \in \mathbb{Z} \pmod{q-1}\},\\
T_1 &= \{i \in T_0 : i \not\equiv 0 \pmod{q-1}\},\\
T_2 &= \{i \in \mathbb{Z} \pmod{q+1} : i \not\equiv 0 \pmod{q+1}\},\\
S_0 &= \{(i,j) \in T_0 \times T_0 : i \not\equiv j \pmod{q-1}\},\\
S_1 &= \{(i,j) \in T_1 \times T_1 : i \not\equiv \pm j \pmod{q-1}\},\\
S_2 &= \{(i,j) \in T_2 \times T_2 : i \not\equiv \pm j \pmod{q+1}\},\\
R_1 &= \{i \in \mathbb{Z} \pmod{q^2-1} : i \not\equiv qi \pmod{q^2-1}\},\\
R_2 &= \{i \in \mathbb{Z} \pmod{q^2-1} : i \not\equiv \pm qi \pmod{q^2-1}\}, \text{ and}\\
R_3 &= \{i \in \mathbb{Z} \pmod{q^2+1} : i \not\equiv 0 \pmod{q^2+1}\}.
\end{aligned}$$

Remark 3.1 Let us notice that the detail information of the conjugacy classes of $PSp_4(2^f)$ are recorded in [7]. We summarise required information in Table 1. By [7, p. 92], we also note that:

(i) eight classes $B_t(\pm i, \pm j)$ and $B_t(\pm j, \pm i)$ are the same, for $t = 1, 4$;

(ii) four classes $B_t(\pm i)$, $B_t(\pm qi)$ are the same, for $t = 2, 5$;

(iii) four classes $B_t(\pm i, \pm j)$ are the same, for $t = 3$;

(iv) two classes $C_t(\pm i)$ are the same, for $1 \leqslant t \leqslant 4$;

(v) two classes $D_t(\pm i)$ are the same, for $1 \leqslant t \leqslant 4$.

Table 1: Conjugacy classes of $\mathrm{PSp}_4(q)$ with $q = 2^f$

Name	Conditions	Representative	Order of Reps.	Number of class	Length of class
A_1		$h(1,1)$	1	1	1
A_2		$x_{2a+b}(1)$	2	1	(q^4-1)
A_{31}		$x_{a+b}(1)$	2	1	(q^4-1)
A_{32}		$x_{a+b}(1)x_{2a+b}(1)$	2	1	$(q^2-1)(q^4-1)$
A_{41}		$x_a(1)x_b(1)$	4	1	$\frac{q^2(q^2-1)(q^4-1)}{2}$
A_{42}	$\xi \in \mathbb{F}$	$x_a(1)x_b(1)x_{2a+b}(\xi)$	4	1	$\frac{q^2(q^2-1)(q^4-1)}{2}$
$B_1(i,j)$	$(i,j) \in S_1$	$h(\gamma^i, \gamma^j)$	$\frac{(q-1)}{(q-1,i,j)}$	$\frac{(q-2)(q-4)}{8}$	$q^4(q+1)^2(q^2+1)$
$B_2(i)$	$i \in R_2$	$h(\theta^i, \theta^{qi})$	$\frac{(q^2-1)}{(q^2-1,i)}$	$\frac{q(q-2)}{4}$	$q^4(q^4-1)$
$B_3(i,j)$	$(i,j) \in T_1 \times T_2$	$h(\gamma^i, \eta^j)$	$\frac{(q^2-1)}{(q-1,i)(q+1,j)}$	$\frac{q(q-2)}{4}$	$q^4(q^4-1)$
$B_4(i,j)$	$(i,j) \in S_2$	$h(\eta^i, \eta^j)$	$\frac{q+1}{(q+1,i,j)}$	$\frac{q(q-2)}{8}$	$q^4(q-1)^2(q^2+1)$
$B_5(i)$	$i \in R_3$	$h(\tau^i, \tau^{qi})$	$\frac{q^2+1}{(q^2+1,i)}$	$\frac{q^2}{4}$	$q^4(q^2-1)^2$
$C_1(i)$	$i \in T_1$	$h(1, \gamma^i)$	$\frac{q-1}{(q-1,i)}$	$\frac{(q-2)}{2}$	$q^3(q+1)(q^2+1)$
$C_2(i)$	$i \in T_1$	$h(\gamma^i, \gamma^{-i})$	$\frac{q-1}{(q-1,i)}$	$\frac{(q-2)}{2}$	$q^3(q+1)(q^2+1)$
$C_3(i)$	$i \in T_2$	$h(1, \eta^i)$	$\frac{q+1}{(q+1,i)}$	$\frac{q}{2}$	$q^3(q-1)(q^2+1)$
$C_4(i)$	$i \in T_2$	$h(\eta^i, \eta^{-i})$	$\frac{q+1}{(q+1,i)}$	$\frac{q}{2}$	$q^3(q-1)(q^2+1)$
$D_1(i)$	$i \in T_1$	$h(1, \gamma^i)x_{2a+b}(1)$	$\frac{2(q-1)}{(q-1,i)}$	$\frac{(q-2)}{2}$	$q^3(q+1)(q^4-1)$
$D_2(i)$	$i \in T_1$	$h(\gamma^i, \gamma^{-i})x_{a+b}(1)$	$\frac{2(q-1)}{(q-1,i)}$	$\frac{(q-2)}{2}$	$q^3(q+1)(q^4-1)$
$D_3(i)$	$i \in T_2$	$h(1, \eta^i)x_{2a+b}(1)$	$\frac{2(q+1)}{(q+1,i)}$	$\frac{q}{2}$	$q^3(q-1)(q^4-1)$
$D_4(i)$	$i \in T_2$	$h(\eta^i, \eta^{-i})x_{a+b}(1)$	$\frac{2(q+1)}{(q+1,i)}$	$\frac{q}{2}$	$q^3(q-1)(q^4-1)$

3.2 Elements of the same order

In this section, we determine the number of the elements of the same order of $\mathrm{PSp}_4(q)$ with $q = 2^f > 2$ in Proposition 3.3 below.

Lemma 3.2 (see [12], Lemma 7) *Let* $S := \mathrm{PSp}_4(q)$ *with* $q = 2^f > 2$. *If* n *is an element order of* S, *then* n *divides one of the five numbers*

$$4,\ 2(q-1),\ 2(q+1),\ q^2-1,\ q^2+1.$$

Proof — The representative of the conjugacy classes of the group S is listed as in the third column of Table 1. Note that each such representative is presented in the form of $x_\iota(t)$, $h(z_1, z_2)$ or their products, for some $t \in \mathbb{F}$, $z_i \in \mathbb{F}_4^\times$, and $\iota \in \Delta^+ := \{a, b, a+b, 2a+b\}$. The explicit definition of $x_\iota(t)$ and $h(z_1, z_2)$ are given in (3.1). It is then easy to find the order of each representative of the conjugacy classes of the group S as in the fourth column of Table 1, and hence the proof follows. □

Proposition 3.3 *Let* $S := \mathrm{PSp}_4(q)$ *with* $q = 2^f > 2$. *Then the following statements holds*:

(i) $m_2(S) = (q^2+1)(q^4-1)$;

(ii) $m_4(S) = q^2(q^2-1)(q^4-1)$;

(iii) *if* $1 \neq r \mid q-1$,

$$m_r(S) = \varphi(r)q^3(q^2+1)(q+1)\left(1 - \frac{q(q+1)}{2} + \frac{q(q+1)}{8}\psi(r)\right);$$

(iv) *if* $1 \neq r \mid q+1$,

$$m_r(S) = \varphi(r)q^3(q^2+1)(q-1)\left(1 - \frac{q(q-1)}{2} + \frac{q(q-1)}{8}\psi(r)\right);$$

(v) *if* $1 \neq r \mid q-1$, $m_{2r}(S) = \varphi(r)q^3(q+1)(q^4-1)$;

(vi) *if* $1 \neq r \mid q+1$, $m_{2r}(S) = \varphi(r)q^3(q-1)(q^4-1)$;

(vii) *if* $1 \neq r \mid q-1$ *and* $1 \neq s \mid q+1$, $m_{rs}(S) = \frac{1}{2}\varphi(rs)q^4(q^4-1)$;

(viii) *if* $1 \neq r \mid q^2+1$, $m_r(S) = \frac{1}{4}\varphi(r)q^4(q^2-1)^2$.

Proof — For $r \in \omega(S)$, let $cl_r(S)$ be the set of the conjugacy classes of S containing all elements of order r. Then

$$m_r(S) = \sum_{c \in cl_r(S)} |c|.$$

In what follows, for each $r \in \omega(S)$, we obtain $cl_r(S)$, and then $m_r(S)$.

If $r = 2$, then

$$cl_2(S) = \{A_2, A_{31}, A_{32}\},$$

and so by Table 1, we have that

$$\begin{aligned} m_2(S) &= |A_2| + |A_{31}| + |A_{32}| = (q^4-1) + (q^4-1) + (q^2-1)(q^4-1) \\ &= (q^2+1)(q^4-1). \end{aligned}$$

Similarly,

$$\begin{aligned} m_4(S) &= |A_{41}| + |A_{42}| = q^2(q^2-1)(q^4-1)/2 + q^2(q^2-1)(q^4-1)/2 \\ &= q^2(q^2-1)(q^4-1). \end{aligned}$$

We now prove part (iii). Let $r \neq 1$ be a divisor of $q-1$. By Table 1, the elements of order r are in the classes $B_1(i,j)$, $C_1(i)$ and $C_2(i)$. Thus $\mathrm{cl}_r(S) = B_1 \cup C_1 \cup C_2$, where

$$\begin{aligned} B_1 &= \{B_1(i,j) : (i,j) \in S_1,\ |h(\gamma^i, \gamma^j)| = r\}, \\ C_1 &= \{C_1(i) : i \in T_1,\ |h(1, \gamma^i)| = r\}, \\ C_2 &= \{C_2(i) : i \in T_1,\ |h(\gamma^i, \gamma^{-i})| = r\}. \end{aligned}$$

We now determine $|B_1|$, $|C_1|$ and $|C_2|$.

Let $\mathcal{B} = \{h(\gamma^i, \gamma^j) : (i,j) \in S_1\}$. Then

$$\begin{aligned} \mathcal{B} &= \{h(\gamma^i, \gamma^j) : (i,j) \in S_1\} \\ &= \{h(\gamma^i, \gamma^j) : 0 < i, j < q-1,\ i \not\equiv \pm j \pmod{q-1}\} \\ &= \{h(z_1, z_2) : z_1, z_2 \in \mathbb{F}^\times,\ z_1, z_2 \neq 1,\ z_1 \neq z_2 \text{ and } z_1 \neq z_2^{-1}\}. \end{aligned}$$

Then $\mathcal{B} = \mathcal{B}_0 \setminus (\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4)$, where

$$\mathcal{B}_0 = \{h(z_1, z_2) : z_1, z_2 \in \mathbb{F}^\times\},\ \mathcal{B}_1 = \{h(1, z) : z \in \mathbb{F}^\times\},$$
$$\mathcal{B}_2 = \{h(z, 1) : z \in \mathbb{F}^\times\},\ \mathcal{B}_3 = \{h(z, z) : z \in \mathbb{F}^\times\},$$
$$\text{and} \qquad \mathcal{B}_4 = \{h(z, z^{-1}) : z \in \mathbb{F}^\times\}.$$

Note that

$$\mathcal{B}_0 \simeq \mathbb{Z}_{q-1} \times \mathbb{Z}_{q-1},\ \mathcal{B}_1 \simeq \mathcal{B}_2 \simeq \mathcal{B}_3 \simeq \mathcal{B}_4 \simeq \mathbb{Z}_{q-1},$$
$$\text{and} \qquad \mathcal{B}_i \bigcap \mathcal{B}_j = \{h(1,1)\},$$

for all $1 \leqslant i, j \leqslant 4$ with $i \neq j$. Lemmas 2.5 and 2.6 imply that the number of elements of $\mathcal{B}_0$ of order r is equal to $\varphi(r)\psi(r)$. Moreover, by Lemma 2.4, the number of elements of order r in $\mathcal{B}_i$ is equal to $\varphi(r)$, for all $i = 1, 2, 3, 4$. Therefore, the number $m_r(\mathcal{B})$ of elements of order r in $\mathcal{B}$ is equal to $\varphi(r)\psi(r) - 4\varphi(r)$. Note that, by Remark 3.1,

$$B_1(\pm i, \pm j) = B_1(\pm j, \pm i).$$

Then

$$|B_1| = \frac{1}{8} m_r(\mathcal{B}) = \frac{\varphi(r)}{8}\psi(r) - \frac{\varphi(r)}{2}.$$

We now obtain $|C_1|$. Let $s := (q-1)/r$. By Remark 3.1, note that

$$C_1(i) = C_1(-i),$$

for all $i \in T_1$. Then

$$\begin{aligned} 2|C_1| &= \left|\{i \in T_1 : |h(1, \gamma^i)| = r\}\right| \\ &= \left|\{i \in T_1 : \frac{q-1}{(q-1,i)} = r\}\right| \\ &= \left|\{i \in T_1 : 0 < i < q-1,\ (q-1,i) = s\}\right|. \end{aligned}$$

Observe that $(q-1,i) = s$ if and only if $((q-1)/s, i/s) = 1$, or equivalently, $(r, i/s) = 1$. Thus

$$2|C_1| = |\{i/s : 0 < i/s < r,\ (r, i/s) = 1\}| = \varphi(r).$$

Therefore, $|C_1| = \varphi(r)/2$. Similarly,

$$2|C_2| = |\{i \in T_1 : |h(\gamma^i, \gamma^{-i})| = r\}| = \varphi(r),$$

that is to say, $|C_2| = \varphi(r)/2$. Hence

$$\begin{aligned} m_r(S) &= \sum_{c \in cl_r(S)} |c| \\ &= \sum_{c \in B_1} |c| + \sum_{c \in C_1} |c| + \sum_{c \in C_2} |c| \\ &= |B_1(i,j)| \cdot |B_1| + |C_1(i')| \cdot |C_1| + |C_2(i')| \cdot |C_2| \\ &= q^4(q+1)^2(q^2+1) \cdot \left(\frac{\varphi(r)}{8}\psi(r) - \frac{\varphi(r)}{2}\right) \\ &\qquad + q^3(q+1)(q^2+1) \cdot \frac{\varphi(r)}{2} + q^3(q+1)(q^2+1) \cdot \frac{\varphi(r)}{2} \\ &= \varphi(r)q^3(q^2+1)(q+1)\left(1 - \frac{q(q+1)}{2} + \frac{q(q+1)}{8}\psi(r)\right). \end{aligned}$$

This proves part (iii). By the same argument as in part (iii), part (iv) follows.

In order to prove part (v), suppose that $r \neq 1$ is a divisor of $q-1$.

Then $\mathrm{cl}_{2r}(S) = D_1 \cup D_2$, where

$$D_1 = \{D_1(i) : i \in T_1, |h(1, \gamma^i)x_{2a+b}(1)| = 2r\},$$
$$D_2 = \{D_2(i) : i \in T_1, |h(\gamma^i, \gamma^{-i})x_{a+b}(1)| = 2r\}.$$

It follows from Remark 3.1 that $D_1(i) = D_1(-i)$, for all $i \in T_1$. Therefore, if $s := (q-1)/r$, then

$$\begin{aligned} 2|D_1| &= \big|\{i \in T_1 : |h(1, \gamma^i)x_{2a+b}(1)| = 2r\}\big| \\ &= \big|\{i \in T_1 : \frac{2(q-1)}{(q-1, i)} = 2r\}\big| \\ &= \big|\{i \in T_1 : \frac{q-1}{(q-1, i)} = r\}\big| \\ &= \big|\{i \in T_1 : 0 < i < q-1,\ (q-1, i) = s\}\big|. \end{aligned}$$

As noted above $(q-1, i) = s$ if and only if $(r, i/s) = 1$, and so

$$2|D_1| = |\{i/s : 0 < i/s < r,\ (r, i/s) = 1\}| = \varphi(r).$$

Hence, $|D_1| = \varphi(r)/2$. Similarly, $|D_2| = \varphi(r)/2$. Thus

$$\begin{aligned} m_{2r}(S) &= \sum_{c \in \mathrm{cl}_r(S)} |c| = \sum_{c \in D_1} |c| + \sum_{c \in D_2} |c| \\ &= q^3(q+1)(q^4-1)|D_1| + q^3(q+1)(q^4-1)|D_2| \\ &= q^3(q+1)(q^4-1)\frac{1}{2}\varphi(r) + q^3(q+1)(q^4-1)\frac{1}{2}\varphi(r). \end{aligned}$$

Hence

$$m_{2r}(S) = \varphi(r)q^3(q+1)(q^4-1),$$

and this proves part (v). To conclude the proof, let us notice that parts (vi), (vii) and (viii) follows in the same manner as part (v). □

Lemma 3.4 *Let* $S := \mathrm{PSp}_4(q)$ *with* $q = 2^f > 2$. *If* $r > 1$ *is a divisor of* $q^2 + 1$, *then the order of each Sylow 2-subgroup of* S *divides* $m_r(S)$, *and so* $4 \mid \varphi(r)$.

Proof — Let $r > 1$ be a divisor of $q^2 + 1$. Then by Lemma 3.2, we note that S has no element of order $2r$, and so the Sylow 2-subgroups

of S act fixed point freely (by conjugation) on the set of elements of order r in S. Thus the order of each Sylow 2-subgroup of S divides $m_r(S)$. By Proposition 3.3 (viii), $m_r(S) = \frac{1}{4}\varphi(r)q^4(q^2-1)^2$, and so $q^4 \mid \frac{1}{4}\varphi(r)q^4(q^2-1)^2$. This requires $4 \mid \varphi(r)$. □

4 Proof of the main result

This section is devoted to proving Theorem 1.1.

For a positive integer q, we define the sets $\mathcal{A}_1, \ldots, \mathcal{A}_9$ as below:

$$\mathcal{A}_1 = \{1\},\ \ \mathcal{A}_2 = \big\{(q^2+1)(q^4-1)\big\},\ \ \mathcal{A}_3 = \big\{q^2(q^2-1)(q^4-1)\big\},$$

$$\mathcal{A}_4 = \big\{\varphi(r)q^3(q^2+1)(q+1)\big(1 - \tfrac{q(q+1)}{2} + \tfrac{q(q+1)}{8}\psi(r)\big) : 1 \neq r \mid q-1\big\},$$

$$\mathcal{A}_5 = \big\{\varphi(r)q^3(q^2+1)(q-1)\big(1 - \tfrac{q(q-1)}{2} + \tfrac{q(q-1)}{8}\psi(r)\big) : 1 \neq r \mid q+1\big\},$$

$$\mathcal{A}_6 = \big\{\varphi(r)q^3(q+1)(q^4-1) : 1 \neq r \mid q-1\big\};$$

$$\mathcal{A}_7 = \big\{\varphi(r)q^3(q-1)(q^4-1) : 1 \neq r \mid q+1\big\};$$

$$\mathcal{A}_8 = \big\{\tfrac{1}{2}\varphi(rs)q^4(q^4-1) : 1 \neq r \mid q-1 \text{ and } 1 \neq s \mid q+1\big\};$$

$$\mathcal{A}_9 = \big\{\tfrac{1}{4}\varphi(r)q^4(q^2-1)^2 : 1 \neq r \mid q^2+1\big\}.$$

Proposition 4.1 *Let* $S := \mathrm{PSp}_4(q)$ *with*

$$q = 2^f > 2,$$

and let $\mathcal{A}_i$ *with* $i = 1, \ldots, 9$ *be as above. Suppose that* G *is a group such that* $\tau(G) = \tau(S)$ *and* $|G| = |S|$. *Then*

(i) $m_2(G) \in \mathcal{A}_2$;

(ii) $m_r(G) \in \mathcal{A}_9$ *if* $r > 1$ *is a prime divisor of* q^2+1;

(iii) $m_r(G) \in \mathcal{A}_4 \cup \mathcal{A}_5$ *if* $r > 1$ *is a prime divisor of* q^2-1.

Proof — Since $\tau(G) = \tau(S)$, it follows that $\tau(G)$ consists of the positive numbers recorded in parts (i)-(viii) of Proposition 3.3. Then we observe that 1 and $(q^2+1)(q^4-1)$ are the only odd numbers

of $\tau(G)$, and so Lemma 2.8 implies that $m_2(G) \in \mathcal{A}_2$. In order to prove part (ii), let us suppose that $r > 1$ is a prime divisor of q^2+1. By Lemma 2.8, we have that $(r, m_r(G)) = 1$, and so $m_r(G) \in \mathcal{A}_9$. Similarly, if $r > 1$ is a prime divisor of q^2-1, we apply Lemma 2.8 and conclude that $(r, m_r(G)) = 1$, and hence

$$m_r(G) \in \mathcal{A}_4 \cup \mathcal{A}_5,$$

as desired in part (iii). □

In what follows, for a positive integer i, let $\mathbf{f}(i)$ be the number of elements of order r in G such that r is a multiple of i. Since $m_r(G)$ is the number of elements of order r in G, we conclude that

$$\mathbf{f}(i) = \sum_{i|r} m_r(G).$$

Let also $\mathbf{f}_j(i)$ be the number of elements of order r in G such that r is a multiple of i and $m_r(G) \in \mathcal{A}_j$, for $j = 1, \ldots, 9$, that is to say,

$$\mathbf{f}_j(i) = \sum_{\substack{i|r \\ m_r(G)\in\mathcal{A}_j}} m_r(G).$$

For each possible r, the number $m_r(G)$ of elements of order r in G belongs to exactly one of $\mathcal{A}_j$ with $j = 1, \ldots, 9$, and so

$$\mathbf{f}(i) = \sum_{i|r} m_r(G) = \sum_{j=1}^{9} \sum_{\substack{i|r \\ m_r(G)\in\mathcal{A}_j}} m_r(G).$$

Therefore, $\mathbf{f}(i) = \sum_{j=1}^{9} \mathbf{f}_j(i)$. By Lemma 2.8, if $i \geqslant 3$, then

$$\mathbf{f}(i) = \sum_{j=3}^{9} \mathbf{f}_j(i).$$

Proposition 4.2 *Let $S := PSp_4(q)$ with $q = 2^f > 2$, and let G be a group such that $\tau(G) = \tau(S)$ and $|G| = |S|$. Then $\pi(q^2+1)$ and $\pi(2(q^2-1))$ are not in the same connected component of the prime graph $\Gamma(G)$ of G. In particular, $\Gamma(G)$ has at least two connected components.*

Proof — Suppose, conversely, that $\pi(q^2+1)$ and $\pi(2(q^2-1))$ are in the same connected component of $\Gamma(G)$. Since $\pi(G)$ is disjoint union of $\pi(q^2+1)$ and $\pi(2(q^2-1))$, it follows that there exist

$$p_1 \in \pi(q^2+1) \qquad \text{and} \qquad p_2 \in \pi(2(q^2-1))$$

such that $p_1p_2 \in \omega(G)$. Therefore, by Lemma 2.9, there exists positive integer r such that $\mathbf{f}(p_1) = r|G|/|P_1| \neq 0$, where $P_1 \in \mathrm{Syl}_{p_1}(G)$. Since the elements of $\mathcal{A}_9$ express in the form

$$\frac{1}{4}\varphi(i)q^4(q^2-1)^2$$

where $1 \neq i \mid q^2+1$, and by Lemma 3.4, we obtain that $4 \mid \varphi(i)$. Moreover, Proposition 4.1 implies that there is a positive integer r' such that $\mathbf{f}_9(p_1) = q^4(q^2-1)^2r'$. Thus

$$\mathbf{f}(p_1) - \mathbf{f}_9(p_1) = \sum_{k=3}^{8} \mathbf{f}_k(p_1).$$

If $\sum_{k=3}^{8} \mathbf{f}_k(p_1) \neq 0$, then q^2+1 divides

$$\sum_{k=3}^{8} \mathbf{f}_k(p_1) = q^4(q^2-1)^2[r(q^2+1)/|P_1| - r'],$$

and so $q^2+1 < r(q^2+1)/|P_1|$. This implies that $|G| < \mathbf{f}(p_1)$, which is impossible. Therefore,

$$\sum_{k=3}^{8} \mathbf{f}_k(p_1) = 0,$$

or equivalently, $\mathbf{f}(p_1) = \mathbf{f}_9(p_1)$. Thus, $m_{p_1p_2}(G) \in \mathcal{A}_9$, and hence

$$\mathbf{f}_9(p_2) = q^4(q^2-1)^2s \neq 0,$$

for some positive integer s.

Since p_2 is a prime divisor of $2(q^2-1)$ and q is a power of 2, it follows that $p_2 = 2$ or $p_2 \in \pi(q^2-1)$. We now discuss these two possible cases:

(1) Let $p_2 = 2$. Then Proposition 4.1 implies that $m_{p_2}(G) \in \mathcal{A}_2$, and so $\mathbf{f}(p_2) \neq 0$. By Lemma 2.9, there exists a positive integer t such that $\mathbf{f}(p_2) = (q^4-1)(q^2-1)t$. Thus

$$\mathbf{f}(p_2) = \sum_{k=2}^{8} \mathbf{f}_k(p_2) + \mathbf{f}_9(p_2).$$

Since $\sum_{k=2}^{8} \mathbf{f}_k(p_2) = (q^2+1)u$, for some positive integer u, we have that

$$(q^2+1)\big((q^2-1)^2t - u\big) = q^4(q^2-1)^2s.$$

Thus

$$q^4(q^2-1)^2 \mid (q^2-1)^2t - u.$$

Therefore,

$$|G| = q^4(q^2-1)^2(q^2+1) < (q^2+1)(q^2-1)^2t \leqslant \mathbf{f}(p_2),$$

and hence $|G| < \mathbf{f}(p_2)$, which is a contradiction.

(2) Let $p_2 \in \pi(q^2-1)$. Hence, by Proposition 4.1, $m_{p_2}(G) \in \mathcal{A}_4 \cup \mathcal{A}_5$. Then $\mathbf{f}_4(p_2) + \mathbf{f}_5(p_2) \neq 0$. It follows from Lemma 2.9 that

$$\mathbf{f}(p_2) = t|G|/|P_2| \neq 0,$$

for some positive integer t and Sylow p_2-subgroup P_2 of G. Thus

$$\mathbf{f}(p_2) = \sum_{k=3}^{8} \mathbf{f}_k(p_2) + \mathbf{f}_9(p_2).$$

Note that $\sum_{k=3}^{8} \mathbf{f}_k(p_2) = (q^2+1)u$, for some positive integer u. Then

$$(q^2+1)\big(\tfrac{q^4(q^2-1)^2}{|P_2|}t - u\big) = q^4(q^2-1)^2s,$$

for some positive integer s. Therefore, $(q^2+1) \mid s$, and hence

$$|G| = q^4(q^2-1)^2(q^2+1) \leqslant q^4(q^2-1)^2s = \mathbf{f}_9(p_2) < \mathbf{f}(p_2),$$

that is to say, $|G| < \mathbf{f}(p_2)$, which is a contradiction. □

Proposition 4.3 *Let* $S := PSp_4(q)$ *with* $q = 2^f > 2$. *If* G *is a group such that* $\tau(G) = \tau(S)$ *and* $|G| = |S|$, *then we have that* G *has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ *such that* H *and* G/K *are* $\pi_1(G)$*-groups,* K/H *is a non-abelian simple group,* H *is a nilpotent group,* $|G/K|$ *divides* $|Out(K/H)|$, *and* $t(K/H) \geqslant t(G)$. *Moreover, any odd component of* G *is also an odd component of* K/H.

Proof — By Lemma 2.3 and Proposition 4.2, it suffices to show G is neither a Frobenius group, nor a 2-Frobenius group. Let first G be a Frobenius group with kernel K and complement H. Then Lemma 2.1 implies that $\pi(H)$ and $\pi(K)$ are the connected components of $\Gamma(G)$. By Proposition 4.2, $\pi(q^2+1)$ and $\pi(2(q^2-1)$ must be the connected components of $\Gamma(G)$, and so

$$|H| = q^2 + 1 \qquad \text{and} \qquad |K| = q^4(q^2-1)^2,$$

or

$$|H| = q^4(q^2-1)^2 \qquad \text{and} \qquad |K| = q^2 + 1.$$

However, both cases cannot occur as $|H|$ does not divide $|K|-1$. Let now G be a 2-Frobenius group. Then by Lemma 2.2, there exists normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H, respectively. Moreover,

$$\pi_1(G) = \pi(H) \cup \pi(G/K) \qquad \text{and} \qquad \pi_2(G) = \pi(K/H).$$

Applying Proposition 4.2, we conclude that $|H|$ divides $q^4(q^2-1)^2$ and $|K/H| = q^2+1$. Since G/H is a Frobenius group with kernel K/H, it follows that $|G/K|$ divides $|K/H|-1$, and so $|G/K|$ divides q^2. Hence, we obtain that $q^2(q^2-1)^2$ is a divisor of $|H|$. This implies that the prime graph of $\Gamma(H)$ is complete because H is nilpotent, and so by Proposition 4.2, $\pi(q^2+1)$ and $\pi(2(q^2-1))$ are connected components of the graph $\Gamma(G)$. Thus $oc(G) = oc(S)$. We apply Lemma 2.10 and conclude that $G \simeq S$, and hence S is a 2-Frobenius group which is a contradiction. This completes the proof. □

4.1 Proof of Theorem 1.1

Let $S := PSp_4(q)$ be a projective symplectic simple group, with $q = 2^f$. If G is a group with $\tau(G) = \tau(S)$ and $|G| = |S|$, then Proposition 4.3

implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group, $|G/K|$ divides $|\mathrm{Out}(K/H)|$, and $t(K/H) \geqslant t(G)$. Moreover, any odd component of G is also an odd component of K/H. It also follows from Proposition 4.2 that q^2+1 is an odd order component of G.

Since $t(K/H) \geqslant t(G) \geqslant 2$, if K/H has two connected components, then G has exactly two connected components, and hence q^2+1 is the odd order component of K/H. If K/H has more than two connected components, then by Proposition 4.2, we know that q^2+1 is an odd order component of G, and so it is equal to an odd order component of K/H or a product of some of odd order components of K/H. Since K/H is a non-abelian simple group with $t(K/H) \geqslant t(G) \geqslant 2$, the group K/H is isomorphic to one of the simple groups listed in [11, 17]. For convenience, we record the odd order components of some non-abelian finite simple groups in Table 2.

Table 2: The odd order components of some non-abelian finite simple groups with at least two components.

Group	Conditions	Odd order components
Alt_n	$n = p, p+1, p+2$, p odd prime	$p, p-2$
$\mathrm{PSL}_p(q')$	p odd prime, $(p,q') \neq (3,2),(3,4)$	$(q'^p-1)/[(q'-1)(p,q'-1)]$
$\mathrm{PSL}_{p+1}(q')$	p odd prime, $q'-1 \mid p \pm 1$	$(q'^p-1)/(q'-1)$
$\mathrm{PSL}_3(2)$		$3,7$
$\mathrm{PSL}_3(4)$		$3^2,5,7$
$\mathrm{PSL}_2(q')$	q' even	$q'\pm 1$
$\mathrm{PSL}_2(q')$	$q' \equiv \pm 1 \pmod 4$	$q', (q'\mp 1)/2$
$\mathrm{PSU}_p(q')$	p odd prime	$(q'^p+1)/[(q'+1)(p,q'+1))]$
$\mathrm{PSU}_{p+1}(q')$	p odd prime, $q'+1 \mid p+1$, $(p,q') \neq (3,2),(5,2)$	$(q'^p+1)/(q'+1)$
$\mathrm{PSU}_4(2)$		5
$\mathrm{PSU}_6(2)$		7, 11
$\mathrm{PSp}_{2m}(q')$	$m = p$ prime, $q' = 2,3$	$(q'^p-1)/(2,q'-1)$
$\mathrm{PSp}_{2m}(q')$	$m = 2^t \geqslant 2$	$(q'^m+1)/(2,q'-1)$
$\mathrm{P}\Omega_{2m+1}(q')$	$m = 2^t \geqslant 4$	$(q'^m+1)/2$
$\mathrm{P}\Omega_{2m+1}(3)$	m odd prime	$(3^m-1)/2$
$\mathrm{P}\Omega^+_{2m}(q')$	$m \geqslant 5$ odd prime, $q' = 2,3,5$	$(q'^m-1)/(q'-1)$
$\mathrm{P}\Omega^+_{2m+2}(3)$	m odd prime	$(3^m-1)/2$
$\mathrm{P}\Omega^-_{2m}(q')$	$m = 2^t \geqslant 4$	$(q'^m+1)/(2,q'-1)$
$\mathrm{P}\Omega^-_{2m}(3)$	$2^t+1 \neq m \geqslant 5$ odd	$(3^m+1)/4$
$\mathrm{P}\Omega^-_{2m}(3)$	$5 \leqslant m = 2^t+1$ not prime	$(3^{m-1}+1)/2$
$\mathrm{P}\Omega^-_{2m}(3)$	$5 \leqslant m = 2^t+1$ odd prime	$(3^{m-1}+1)/2, (3^m+1)/4$
$\mathrm{P}\Omega^-_{2m+2}(2)$	$m \neq 2^t-1$ odd prime	2^m-1
$\mathrm{P}\Omega^-_{2m+2}(2)$	$m = 2^t-1$ odd prime	$2^{m-1}+1, 2^m+1.$
${}^2\mathrm{B}_2(q')$	$q' = 2^{2t+1} \geqslant 8$ even	$q'-1, q' \pm \sqrt{2q'}+1$
$\mathrm{F}_4(q')$		$q'^4+1, q'^4-q'^2+1$

In what follows, we discuss the possibilities for the factor group K/H

and the proof of Theorem 1.1 follows immediately from the following lemmas.

Lemma 4.4 *The group* K/H *cannot be isomorphic to an alternating group, the Tits group* ${}^2F_4(2)'$ *or a sporadic simple group.*

Proof — If K/H is isomorphic to Alt_n with $n \geqslant 7$, then by Table 2, we have that $n = p$, $p+1$ or $p+2$. As noted above, q^2+1 is an odd order component or a product of some odd order components of K/H. Thus q^2+1 is p, $p-2$ or $p(p-2)$. If $q^2+1 = p$ or $p-2$, then

$$q^2 - 3 = p - 4 \qquad \text{or} \qquad q^2 + 3 = p,$$

respectively. So $q^2 \mp 3$ divides $|G|$ and this contradicts Lemma 2.12(v). If $q^2+1 = p(p-2)$, then

$$(p-1-q)(p-1+q) = 2,$$

which is impossible. If $K/H \simeq \mathrm{Alt}_n$, where $n = 5, 6$, then $q^2+1 = 3, 5$ or 15 which leads to no suitable solution for q. Therefore, K/H cannot be isomorphic to an alternating group. Similarly, the case where K/H is the Tits group or a sporadic simple group can be ruled out. □

Lemma 4.5 *The group* K/H *cannot be isomorphic to a finite simple exceptional group.*

Proof — By Table 2, the group K/H is isomorphic to one of the simple groups ${}^2B_2(q')$ with

$$q' = 2^{2t+1} \geqslant 8$$

the groups ${}^3D_4(q')$, $G_2(q')$, ${}^2G_2(q')$ with

$$q' = 3^{2t+1} \geqslant 27,$$

the groups $F_4(q')$, ${}^2F_4(q')$ with

$$q' = 2^{2t+1} \geqslant 8,$$

and $E_6^{\pm}(q')$, $E_6^{-}(2)$, $E_7(2)$, $E_7(3)$ and $E_8(q')$. Recall that q^2+1 is an odd order component or a product of some order components of K/H.

We easily see that K/H cannot be isomorphic to $E_6^-(2)$, $E_7(2)$ or $E_7(3)$ as if q^2+1 has to be equal to one of the following numbers:

$$\begin{aligned} E_6^-(2): &\quad 13, 17, 19, 13\cdot 17, 13\cdot 19, 17\cdot 19, 13\cdot 17\cdot 19,\\ E_7(2): &\quad 73, 127, 73\cdot 127,\\ E_7(3): &\quad 757, 1093, 757\cdot 1093. \end{aligned}$$

Thus, the only case that can occur is the one in which $K/H \simeq E_6^-(2)$ when $q = 4$. However, in this case, $|E_6^-(2)|$ must divide $|PSp_4(4)|$, which is a contradiction.

Suppose that K/H is isomorphic to one of the groups listed in the first column of Table 3. Then q^2+1 is equal to one of the polynomials recorded in the third column of the same table. But such an equality contradicts the fact that q is a power of 2.

Suppose that K/H is isomorphic to a Suzuki group ${}^2B_2(q')$. Then the odd order components of K/H are

$$q'-1 \qquad \text{or} \qquad q' \pm \sqrt{2q'}+1$$

by Table 2. Therefore, q^2+1 is equal to

$$\begin{gathered} q'-1,\ q' \pm \sqrt{2q'}+1,\ {q'}^2+1,\ (q'-1)(q' \pm \sqrt{2q'}+1)\\ \text{or} \qquad (q'-1)({q'}^2+1). \end{gathered}$$

If $q^2+1 \neq {q'}^2+1$, then we conclude that q is not a power of 2, which is a contradiction. Thus $q^2+1 = {q'}^2+1$, and so $q = q'$. Clearly, every two vertices r and s of $\Gamma(K/H)$ with

$$r \in \pi(q' - \sqrt{2q'}+1) \qquad \text{and} \qquad s \in \pi(q' + \sqrt{2q'}+1)$$

are disconnect, and hence they are also disconnected in $\Gamma(G)$. Therefore the Sylow s-subgroup of G acts fixed point freely on $m_r(G)$. This implies that $q' + \sqrt{2q'}+1 \mid m_r(G)$. On the other hand, it follows from Lemma 4.1 that $m_r(G) \in \mathcal{A}_9$. Therefore, $q' + \sqrt{2q'}+1$ divides

$$\varphi(r)q^4(q^2-1)^2/4 = \varphi(r){q'}^4({q'}^2-1)^2/4$$

and hence $q' + \sqrt{2q'}+1$ divides $\varphi(r)$, where $r \in \pi(q' - \sqrt{2q'}+1)$, which is a contradiction.

Suppose that K/H is isomorphic to $F_4(q')$. Then by Table 2, the odd order components of K/H is

$$q'^4+1 \qquad \text{or} \qquad q'^4-q'^2+1,$$

and so q^2+1 is

$$q'^4+1,\ q'^4-q'^2+1 \qquad \text{or} \qquad q'^8-q'^6+2q'^4-q'^2+1.$$

If $q^2+1=q'^4-q'^2+1$ or $q'^8-q'^6+2q'^4-q'^2+1$, then

$$q^2=q'^4-q'^2 \qquad \text{or} \qquad q'^8-q'^6+2q'^4-q'^2,$$

which is a contradiction as q is a power 2. Thus $q^2+1=q'^4+1$, and so $q^{12}=q'^{24}$. Since q'^{24} divides $|K/H|$, so does $|G|$, and hence the 2-part of $|G|$ is at least q^{12}, which is a contradiction.

Suppose that K/H is isomorphic to $E_6^{\pm}(q')$ with $q'\equiv\pm1 \pmod 3$. Then $q^2+1=(q'^6\pm q'^3+1)/3$, and so $3q^2+2=q'^6\pm q'^3$. Note that $q'^6\pm q'^3$ divides $|K/H|$, so does $|G|$. Thus $3q^2+2$ divides $|G|$ which is impossible by Lemma 2.12. □

Table 3: Some polynomials in Lemma 4.5.

Group	Conditions	Polynomials
$G_2(q')$		$\Phi_3(q')$, $\Phi_6(q')$, $\Phi_3(q'^2)$
$E_6(q')=E_6^+(q')$	$q'\overset{3}{\equiv}0,-1$	$\Phi_9(q')$
$E_8(q')$		$\Phi_{15}(q')$, $\Phi_{20}(q')$, $\Phi_{24}(q')$, $\Phi_{30}(q')$, $\Phi_{15}(q')\Phi_{20}(q')$, $\Phi_{15}(q')\Phi_{24}(q')$, $\Phi_{15}(q')\Phi_{30}(q')$, $\Phi_{20}(q')\Phi_{24}(q')$, $\Phi_{20}(q')\Phi_{30}(q')$, $\Phi_{24}(q')\Phi_{30}(q')$, $\Phi_{15}(q')\Phi_{20}(q')\Phi_{24}(q')$, $\Phi_{15}(q')\Phi_{20}(q')\Phi_{30}(q')$, $\Phi_{15}(q')\Phi_{24}(q')\Phi_{30}(q')$, $\Phi_{20}(q')\Phi_{24}(q')\Phi_{30}(q')$, $\Phi_{15}(q')\Phi_{20}(q')\Phi_{24}(q')\Phi_{30}(q')$
${}^3D_4(q')$		$\Phi_{12}(q')$
${}^2G_2(q')$		$\Phi_6^+(q')$, $\Phi_6^-(q')$, $\Phi_6(q')$
${}^2F_4(q')$		$\Phi_{12}^+(q')$,$\Phi_{12}^-(q')$,$\Phi_{12}(q')$
${}^2E_6(q')=E_6^-(q')$	$q'\overset{3}{\equiv}0,1$	$\Phi_{18}(q')$

Note: $\Phi_6^{\pm}(q'):=q'\pm\sqrt{3q'}+1$ and $\Phi_{12}^{\pm}(q'):=q'^2\pm q'\sqrt{2q'}+q'\pm\sqrt{2q'}+1$.

Lemma 4.6 *If* K/H *is isomorphic to* $PSL_n(q')$ *with* $(n,q')\neq(2,2),(2,3)$, *then* $(n,q')=(2,q^2)$ *with* $q=2^f$, *and hence* $G\simeq S$.

Proof — By Table 2, we have to consider the following possibilities:

(1) $n \geqslant 5$ is prime. It follows from Table 2 that

$$q^2+1 = (q'^n-1)/[(q'-1)(n, q'-1)].$$

Suppose that q' is even. If $q' = 2$, then $q^2+1 = q'^n-1$, and so $2^{2f-1}+1 = 2^{n-1}$, which is a contradiction. Thus $q' > 2$, and hence

$$q'^n-1 = (q'-1)(n, q'-1)(q^2+1) > q^2+1.$$

Then $q'^{2n} > q^4$. Let us observe that the 2-part of $|K/H|$ is $q'^{\frac{n(n-1)}{2}}$ and the 2-part of G is q^4. Since $|K/H|$ divides $|G|$, we have that

$$n(n-1)/2 < 2n$$

which is valid only for $n < 5$, which is a contradiction. If q' is odd, then $2q^2 < q'^n-1$, and so $q'^{\frac{n(n-1)}{2}}$ divides $(q-1)^2$ or $(q+1)^2$ as $|K/H|$ is a divisor of G. Thus

$$2q^2 < q'^n-1 \leqslant q'^{\frac{n(n-1)}{2}} \leqslant (q+1)^2,$$

and so $2q^2 < (q+1)^2$. Therefore, $q < 3$, which is a contradiction.

(2) $n = p+1$, where p is an odd prime and $q'-1 \mid p \pm 1$. In this case,

$$q^2+1 = (q'^p-1)/(q'-1),$$

that is to say,

$$q^2 = q'^{p-1}+q'^{p-2}+\ldots+q' = q'(q'^{p-1}-1)/(q'-1).$$

Thus $p = 2$ and $q' = q^2$ which is a contradiction.

(3) $n = 3$. If $q' = 2, 4$, then by Table 2, q^2+1 is equal to one of

$$15,\ 21,\ 27,\ 35,\ 45,\ 63,\ 105,\ 135,\ 189,\ 315 \text{ and } 945$$

which leads to no possible solution for q. Therefore, $q' \neq 2, 4$, and hence $q^2+1 = (q'^2+q'+1)/(3, q'-1)$. Thus

$$q'(q'+1) = q^2 \qquad \text{or} \qquad 3q^2+2.$$

Since q is a power of 2, we deduce that $q'(q'+1) = 3q^2+2$. More-

over, $3q^2+2$ divides $|K/H|$, and so does $|G|$, and this contradicts Lemma 2.12.

(4) $n=2$ and q' is odd and $q' \equiv \pm 1 \pmod 4$. In this case, we must have

$$q^2+1 = q', \ (q' \mp 1)/2 \qquad \text{or} \qquad q'(q' \mp 1)/2.$$

If $q^2+1=q'$, then $q^2+2 = q'+1$ which is a divisor of $|K/H|$. This implies that q^2+2 divides $|G|$, and so by Lemma 2.12-(ii) and the fact that $q>2$, we conclude that $q=4$, and hence

$$q' = q^2+1 = 17.$$

It follows that $K/H \simeq PSL_2(17)$ and $S = PSp_4(4)$. Since $|G/K|$ divides $|Out(K/H)| = 2$, we deduce that $|H| = 2^a \cdot 5^2$, where $a = 3,4$. Since H is nilpotent, it follows that the Sylow 5-subgroup of H is normal in G, and so $m_5(H) = m_5(G)$. Therefore $m_5(G) = m_5(H)$ divides $|H|$, but $|H| \mid 400$, and this contradicts Proposition 4.1.

If $q^2+1 = (q' \mp 1)/2$, then

$$q' = 2q^2+3 \qquad \text{or} \qquad 2q^2+1.$$

Thus $2q^2+3$ or $2q^2+1$ is a divisor of $|G|$, which is impossible by Lemma 2.12 as $q \neq 2$. Similarly, if $q^2+1 = q'(q' \mp 1)/2$, then

$$2q^2 = (q' \pm 1)(q' \mp 2)$$

which is a contradiction because q is even and $q' \mp 2$ is odd.

(5) $n=2$ and q' is even. Then q^2+1 is equal to $q'-1, q'+1, q'^2-1$. We first observe that q^2+1 cannot be equal to $q'-1$ or q'^2-1 as if not $q^2+2 = q'$ or q'^2, respectively, which is impossible. Thus

$$q^2+1 = q'+1,$$

and hence $q^2 = q'$. Note that $\pi(q^2+1)$ is a connected component of $\Gamma(K/H)$. Then Proposition 4.2, $\pi(q^2+1)$ is a connected component of $\Gamma(G)$. We show that $\pi(2(q^2-1))$ is the other connected component of $\Gamma(G)$. If $|H|$ is coprime to q^2-1, then q^2-1 divides $|G/K|$ which is a divisor of

$$|Out(K/H)| = \log_2 q' = 2\log_2 q,$$

and so $q^2-1 \mid 2\log_2 q$, which is impossible. It follows that there is

a prime $p \in \pi(q^2-1)$ such that $p \mid |H|$. On the other hand, we have that $2 \in \pi(H)$. Since H is a nilpotent group, we have $2p \in \omega(H)$, and this implies that $2p \in \omega(G)$. Therefore, $\pi(2(q^2-1))$ is the other connected component of $\Gamma(G)$. This, together with the fact that $|G| = |S|$, implies that $oc(G) = oc(S)$, and hence by Lemma 2.10, we conclude that $G \simeq S$. □

Lemma 4.7 *The group* K/H *cannot be isomorphic to* $PSU_n(q')$.

Proof — According to Table 2, we have the following possibilities:
(1) $n = 4, 6$ and $q' = 2$. Then q^2+1 is equal to one of 5, 7, 11, and 77 which leads to no possible solution for q.
(2) $n = p+1$, or $n = p$ and $(q'+1, p) = 1$. Then

$$q^2+1 = (q'^p+1)/(q'+1)$$

and so

$$q^2 = q'^{p-1} - q'^{p-2} + \ldots - q',$$

which is a contradiction as q is a power 2.
(3) $n = p$ and $(q'+1, p) = p$. Then

$$q^2+1 = \frac{q'^p+1}{(q'+1)\cdot p}.$$

If q' is even, then we have that the 2-part $|K/H|_2 = q'^{\frac{p(p-1)}{2}}$ of K/H divides $|G|_2 = q^4$. It follows from $q^2+1 = \frac{q'^p+1}{(q'+1)\cdot p}$ that $q^4 < q'^{2p}$, and so

$$q'^{\frac{p(p-1)}{2}} \leqslant q^4 < q'^{2p}.$$

Then $\frac{p(p-1)}{2} < 2p$. This inequality is true only for $p = 3$ in which case we have that $3q^2 = (q'+1)(q'-2)$, which is impossible. Suppose now that q' is odd. Since $|K/H|$ is a divisor of $|G|$, it follows that $q'^{\frac{p(p-1)}{2}} \mid (q \pm 1)^2$. On the other hand, $q^2+1 = \frac{q'^p+1}{(q'+1)\cdot p}$ implies that $2q^2 < q'^p + 1$. If $p > 3$, then

$$2q^2 < q'^p + 1 \leqslant q'^{\frac{p(p-1)}{2}} < (q \pm 1)^2,$$

that is to say, $2q^2 < (q \pm 1)^2$ which implies that $q < 3$, which is a contradiction. If $p = 3$, then $q^2+1 = \frac{q'^3+1}{3(q'+1)}$, and so $3q^2+2$ divides $|G|$,

and so Lemma 2.12-(iv) implies that $q = 4$. Thus,

$$\frac{q'^3 + 1}{3(q' + 1)} = q^2 + 1 = 17,$$

and so $q'^2 - q' - 50 = 0$, which has no possible solution. □

Lemma 4.8 *If* K/H *is isomorphic to* $PSp_{2m}(q')$*, then* $(m, q') = (2, q)$ *with* $q = 2^f$*, and hence* $G \simeq S$*.*

Proof — By Table 2, we have the following possibilities:
(1) $m = p$ is prime, and $q' = 2, 3$. Then $q^2 + 1 = (q'^p - 1)/(2, q' - 1)$, and so $2q^2 = 3^p - 3$ or $q^2 + 2 = 2^p$, which is a contradiction because q is a power 2.
(2) $m = 2^t \geqslant 2$. Then

$$q^2 + 1 = (q'^m + 1)/(2, q' - 1).$$

If q' is odd, then $2q^2 + 1 = q'^m$, and hence, by Lemma 2.11, we have $m = 1$, which is a contradiction. Therefore, q' is even, and hence $q^2 + 1 = q'^m + 1$ which implies that $q^{2m} = q'^{m^2}$. Since $|K/H|$ divides $|G|$, we have that $q'^{m^2} \mid q^4$, and so $q^{2m} \mid q^4$. Thus $m = 2$, and hence $q = q'$. Therefore $K/H \simeq PSp_4(q)$, and since $|K/H| = |G|$, we conclude that $H = 1$, that is to say, $G = K \simeq PSp_4(q)$. □

Lemma 4.9 *The group* K/H *cannot be isomorphic to*

$$P\Omega_n(q') \qquad \textit{and} \qquad P\Omega_n^{\pm}(q').$$

Proof — Assume to the contrary that K/H is isomorphic to $P\Omega_n(q')$ or $P\Omega_n^{\pm}(q')$. We have one of the following possibilities by Table 2:
(1) K/H is isomorphic to $P\Omega_{2m+1}(q')$ with $m = 2^t$ for $t \geqslant 2$. Then

$$q^2 + 1 = (q'^m + 1)/2,$$

and so $2q^2 + 1 = q'^m$, which is impossible by Lemma 2.11.
(2) K/H is isomorphic to $P\Omega_{2m+1}(3)$ with m prime. Then

$$q^2 + 1 = (3^m - 1)/2$$

and so $2q^2 = 3^m - 3$ which is a contradiction because q is a power of 2.

(3) K/H is isomorphic to $P\Omega^+_{2m}(q')$, where $m \geqslant 5$ is an odd prime and $q' = 2, 3, 5$. Then

$$q^2 + 1 = (q'^m - 1)/(q' - 1).$$

Thus

$$q^2 = q'^{m-1} + q'^{m-2} + \ldots + 1,$$

which is a contradiction.

(4) K/H is isomorphic to $P\Omega^+_{2m+2}(q')$, where m is an odd prime and $q' = 3$. Then $q^2 + 1 = (3^m - 1)/2$. Thus $2q^2 = 3^m - 3$, which is a contradiction.

(5) K/H is isomorphic to $P\Omega^-_{2m}(q')$, where $m = 2^t \geqslant 4$. If q' is odd, then

$$q^2 + 1 = (q'^m + 1)/2$$

and so $2q^2 + 1 = q'^m$. Thus $2q^2 + 1$ divides $|K/H|$, and hence $2q^2 + 1$ divides $|G|$, which is a contradiction. If q' is even, then

$$q^2 + 1 = q'^m + 1,$$

and so $q^{2(m-1)} = q'^{m(m-1)}$. Since $|K/H|$ divides $|G|$, we deduce that $q^{2(m-1)} \mid q^4$. This implies that $m - 1 \mid 2$, and so $m = 2$ or 3 which contradicts the fact that $m = 2^t \geqslant 4$.

(6) K/H is isomorphic to $P\Omega^-_{2m}(3)$, where $2^t + 1 \neq m \geqslant 5$ is odd. Then $q^2 + 1 = (3^m + 1)/4$, and so $4q^2 = 3^m - 3$, which is a contradiction as q is a power 2.

(7) K/H is isomorphic to $P\Omega^-_{2m}(3)$, where $5 \leqslant m = 2^t + 1$ is not prime. Then

$$q^2 + 1 = (3^{m-1} + 1)/2$$

and so $2q^2 + 1 = 3^{m-1}$. By Lemma 2.11, $m = 3$, which is impossible as $m \geqslant 5$.

(8) K/H is isomorphic to $P\Omega^-_{2m}(3)$, where $5 \leqslant m = 2^t + 1$ is an odd prime. Then $q^2 + 1$ equals to

$$(3^{m-1} + 1)/2, \ (3^m + 1)/4 \qquad \text{or} \qquad (3^{m-1} + 1)(3^m + 1)/8.$$

If $q^2+1=(3^{m-1}+1)/2$, then

$$2q^2+1=3^{m-1},$$

and since $3^{m-1} \mid |K/H|$, it follows that $2q^2+1 \mid |G|$, and so by Lemma 2.12, we conclude that $q=2$, which is a contradiction. Now, if $q^2+1=(3^m+1)/4$, then

$$4q^2=3^m-3,$$

which is a contradiction because q is a power 2. If

$$q^2+1=\Big(\frac{3^{m-1}+1}{2}\Big)\Big(\frac{3^m+1}{4}\Big).$$

Since $3^{m(m-1)}$ divides $|K/H|$, it follows that $3^{m(m-1)} \mid |G|$, and so

$$3^{m(m-1)} \mid (q\pm 1)^2.$$

Then $3^{m(m-1)/2} \leqslant q+1$, and so

$$3^{m(m-1)/2} \leqslant q+1 < q^2+1 \leqslant \Big(\frac{3^{m-1}+1}{2}\Big)\Big(\frac{3^m+1}{4}\Big) \leqslant 3^{2m},$$

that is to say, $m(m-1)/2 \leqslant 2m$. This is true only if $m=5$ in which case

$$q^2+1=\Big(\frac{3^4+1}{2}\Big)\Big(\frac{3^5+1}{4}\Big)=2501,$$

and so $q=50$, which is a contradiction.

(9) K/H is isomorphic to $P\Omega^-_{2m+2}(2)$, where $m \neq 2^t-1$ prime. Then $q^2+1=2^m-1$, which is a contradiction as q is a power of 2.

(10) K/H is isomorphic to $P\Omega^-_{2m}(2)$, where $m=p+1$ with p odd prime. Then

$$q^2+1=2^p+1,\ 2^{p+1}+1 \qquad \text{or} \qquad (2^p+1)(2^{p+1}+1).$$

If $q^2+1=2^p+1$, then $p=2f$, which is a contradiction because p is odd prime. If $q^2+1=2^{p+1}+1$, then

$$2^{4f}=2^{2p+2}.$$

Since $|K/H|$ divides $|G|$ and the 2-part of $|G|$ is $q^4 = 2^{4f}$, we deduce that $2^{p(p+1)} \mid 2^{4f}$. This implies that $p \leqslant 2$, which is a contradiction. If $q^2 + 1 = (2^p + 1)(2^{p+1} + 1)$, then

$$q^2 = 2^{2p+1} + 2^p + 2^{p+1},$$

which is a contradiction because q is a power 2. □

Proof of Theorem 1.1 — The proof immediately follows from Proposition 4.3 and Lemmas 4.4-4.9. □

Proof of Corollary 1.2 — Since G and S are of the same type, it follows that $\tau(G) = \tau(S)$ and $|G| = |S|$, and hence the result immediately follows from Theorem 1.1. □

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REFERENCES

[1] S.H. Alavi – A. Daneshkhah – H. Parvizi Mosaed: "On quantitative structure of small Ree groups", *Comm. Algebra* 45 (2017), 4099–4108.

[2] S.H. Alavi – A. Daneshkhah – H. Parvizi Mosaed: "Finite groups of the same type as Suzuki groups", *Int. J. Group Theory* 8 (2019), 35–42.

[3] R.W. Carter: "Simple Groups of Lie Type", *Wiley*, London (1972).

[4] G. Chen: "On structure of Frobenius and 2-Frobenius group", *J. Southwest China Normal Univ.* 20 (1995), 485–487.

[5] J.H. Conway – R.T. Curtis – S.P. Norton – R.A. Parker – R.A. Wilson: "ATLAS of Finite Groups", *Oxford University Press*, Eynsham (1985).

[6] P. Crescenzo: "A Diophantine equation which arises in the theory of finite groups", *Adv. Math.* 17 (1975), 25–29.

[7] H. Enomoto: "The characters of the finite symplectic group $Sp(4, q)$, $q = 2^f$", *Osaka Math. J.* 9 (1972), 75–94.

[8] D. Gorenstein: "Finite Groups", *Chelsea*, New York (1980).

[9] M. Hall: "The Theory of Groups", *Macmillan*, New York (1959).

[10] E.I. Khukhro – V.D. Mazurov: "The Kourovka Notebook: Unsolved Problems in Group Theory" (2014); *ArXiv*:1401.0300.

[11] A.S. Kondrat′ev: "On prime graph components of finite simple groups", *Mat. Sb.* 180 (1989), 787–797.

[12] V.D. Mazurov – M.C. Su – C.P. Chao: "Recognition of the finite simple groups $L_3(2^m)$ and $U_3(2^m)$ from the orders of their elements", *Algebra Log.* 39 (2000), 567–585.

[13] J.S. Rose: "A Course on Group Theory", *Dover*, New York (1994).

[14] C. Shao – W. Shi – Q. Jiang: "Characterization of simple K_4-groups", *Front. Math. China* 3 (2008), 355–370.

[15] A.V. Vasil′ev – M.A. Grechkoseeva – V.D. Mazurov: "Characterization of finite simple groups by spectrum and order", *Algebra Logika* 48 (2009), 685–728.

[16] L. Weisner: "On the number of elements of a group which have a power in a given conjugate set", *Bull. Amer. Math. Soc.* 31 (1925), 492–496.

[17] J.S. Williams: "Prime graph components of finite groups", *J. Algebra* 69 (1981), 487–513.

[18] L. Zhang – W. Shi: "A new characterization of the group $S_4(q)$ using a graph of noncommutativity", *Sibirsk. Mat. Zh.* 50 (2009), 669–679.

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