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Products of Subgroups in Finite Groups

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In memory of Professor Francesco de Giovanni

Abstract

We give a sufficient condition to ensure that the intersection of the product of two subgroups A and B of a group G with an abelian normal subgroup K of G is a subgroup of G.

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1 Introduction

Throughout this paper, all groups considered are finite. Let G be a group. Suppose that A, B are two subgroups of G and K is an abelian normal subgroup of G. An interesting question arises in [2]: When $AB \cap K$ is a subgroup of G if $AB \cap K$ is closed under inverses, i.e.

$$AB \cap K = BA \cap K.$$

In general, this does not hold, especially for the elementary abelian 2-group K.

For example, let $G = KA$ be the semidirect product of a normal elementary abelian 2-subgroup K and an odd order subgroup A such that A acts nontrivially on K via conjugation. Then there exists some $k \in K$ such that $k \notin C_K(A)$. Take $B = A^k \neq A$ as k does not normalizes A in this case. Then, as $AB \cap K \subseteq K$ consists of the identity element and some involutions,

$$AB \cap K = BA \cap K$$

naturally holds. By Lemma 2.2 below, we will see that

$$|AB \cap K| = |\{[a, k] \mid a \in A\}| > 1$$

dividing $|A : A \cap B|$ is of odd order. Hence $AB \cap K$ can not be a subgroup of K as $|AB \cap K|$ is not a positive power of 2.

In [2, Theorem A], A. Ballester-Bolinches , G. Kaplan and V. Pérez-Calabuig prove that if K is cyclic, or K is of odd order and either $(|K|, |A|) = 1$ or $(|K|, |B|) = 1$ holds, then $AB \cap K = BA \cap K$ implies that $AB \cap K$ is a subgroup of G. Our main result in this paper is a combination of such two cases in [2, Theorem A]. It can be formulated as follows.

Theorem 1.1 *Let* G *be a group with* A, B *two subgroups of* G. *Let* K *be an abelian normal subgroup of* G *and denote by* π_0 *the set of all primes* p *such that* K *has a cyclic Sylow* p*-subgroup. Suppose that the following assumptions hold*:

(1) $2 \in \pi_0$, *that is,* K *has a cyclic Sylow 2-subgroup,*

(2) *either* $(|K|, |A|)$ *or* $(|K|, |B|)$ *is a* π_0*-number.*

Then $AB \cap K$ *is a subgroup of* G *if and only if* $AB \cap K = BA \cap K$.

Note that if K is cyclic, then π_0 is exactly the set of all primes. Thus the hypothesis (1) and (2) in Theorem 1.1 hold naturally. We also guess that Theorem 1.1 could be still true if we remove the hypothesis (2).

2 Proofs

The first result is a generalization of Dedekind Identity (see e.g. [3, 1.1.11]).

Lemma 2.1 *Let* G *be a group and* X, Y *be two subsets (possible not subgroups) of* G. *Let* U *be a subgroup of* G *with* $XU \subseteq X$. *Then*

$$X \cap YU = (X \cap Y)U.$$

Proof — Clearly, $(X \cap Y)U \subseteq YU$ and $(X \cap Y)U \subseteq XU \subseteq X$, hence

$$(X \cap Y)U \subseteq X \cap YU.$$

Conversely, for each $x \in X \cap YU$, we may assume that $x = yu$, where $y \in Y$ and $u \in U$. It implies that $y = xu^{-1} \in XU \subseteq X$, so $y \in X \cap Y$ and $x \in (X \cap Y)U$, as desired. □

Recall that 1-cocycle with respect to the action of A on K is a map $\delta : A \to K$ satisfying $(ab)^\delta = (a^\delta)^b(b^\delta)$, for all $a, b \in A$.

Lemma 2.2 *Let* G *be a group and* K *be a normal subgroup of* G. *Assume that* A, B *are two complements of* K *in* G. *Then there exists a* 1*-cocycle* $\delta : A \to K$ *with respect to the action of* A *on* K *via conjugation such that*

(1) $B = \{aa^\delta : a \in A\}$ *and* $A^\delta = AB \cap K$,

(2) $A^\delta = R^\delta$ *and* $|A^\delta| = |R| = |A : A \cap B|$, *for each right transversal* R *of* $A \cap B$ *in* A.

Proof — See the proof in [2, Proposition 2] or [1, Page 107]. □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1 — Initially, observe that $AB \cap K \leqslant G$ easily implies that $AB \cap K = BA \cap K$. Now we assume that the converse is false and let G be a counterexample with minimal order. Then we can choose two subgroups A, B and an abelian normal subgroup K of G satisfying the hypothesis but $AB \cap K$ is not a subgroup of G. Denote by π_0 the set of all primes p such

that K has a cyclic Sylow p-subgroup. Without loss of generality, we have that $(|K|, |A|)$ is a π_0-number. Now we will derive the contradiction in the following steps:

(1) $G = AK = BK = \langle A, B\rangle$.
Set $H = AK \cap BK, A_1 = A \cap H, B_1 = B \cap H$. Notice that

$$A_1B_1 \cap K = (A \cap H)(B \cap H) \cap K = (A \cap BK)(B \cap AK) \cap K$$
$$= (A \cap BK)B \cap AK \cap K = AB \cap K.$$

Similarly, $BA \cap K = B_1A_1 \cap K$. Hence, $B_1A_1 \cap K = A_1B_1 \cap K$. Since $|A_1|, |B_1|$ divides $|A|, |B|$, respectively, (H, A_1, B_1, K) satisfies the hypothesis of Theorem 1.1, but $A_1B_1 \cap K = AB \cap K$ is not a subgroup. The minimality of G implies that $G = H = AK = BK$.

Set $T = \langle A, B\rangle$ and write π_1 the set of all primes such that $K \cap T$ has a cyclic Sylow p-subgroup. Clearly, $\pi_0 \subseteq \pi_1$, and hence we also see that $(T, A, B, K \cap T)$ satisfies the hypothesis of Theorem 1.1. By the minimality, $G = T = \langle A, B\rangle$, as claimed.

(2) $A \cap K = B \cap K = \mathbb{C}_K(A) = \mathbb{C}_K(B) = 1$.
Set $N = \mathrm{Core}_G(A)$ and assume that $N \neq 1$. In the quotient group $\overline{G} = G/N$, we see

$$\overline{A} = AN/N, \overline{B} = BN/N \leqslant \overline{G} \qquad \text{and} \qquad \overline{K} = KN/N \trianglelefteq \overline{G}.$$

Since $N \leqslant A$, $ABN = ANB = AB$ and $BAN = BA$. By Lemma 2.1, it follows that

$$\overline{AB} \cap \overline{K} = \overline{AB \cap KN} = \overline{(AB \cap K)N} = \overline{(BA \cap K)N}$$
$$= \overline{BA \cap KN} = \overline{BA} \cap \overline{K}.$$

Let π_2 be the set of all primes p such that $\overline{K}$ has a cyclic Sylow p-subgroup. Clearly, $\pi_0 \subseteq \pi_2$, and hence $(\overline{G}, \overline{A}, \overline{B}, \overline{K})$ satisfies the hypothesis of Theorem 1.1. The minimality of G implies that $\overline{AB} \cap \overline{K} \leqslant \overline{G}$. Therefore, $AB \cap KN \leqslant G$ and so

$$AB \cap K = (AB \cap KN) \cap K \leqslant G,$$

which is a contradiction. It follows that $\mathrm{Core}_G(A) = 1$, and also $\mathrm{Core}_G(B) = 1$ with the similar argument. As $G = AK = BK$ and K is abelian, $A \cap K \leqslant \mathbf{C}_A(K)$ are normal in G. Consequently,

$$A \cap K = \mathbf{C}_A(K) = 1$$

as $\mathrm{Core}_G(A) = 1$. Similarly, $B \cap K = \mathbf{C}_B(K) = 1$, as desired.

Now, by (1) and (2), we see that A, B are two complements of K in G. Furthermore, by Lemma 2.2, there exists a 1-cocycle $\delta : A \to K$ with respect to the action of A on K via conjugation such that

$$B = \{a a^\delta : a \in A\},\ A^\delta = AB \cap K \quad \text{and} \quad |A^\delta| = |A : A \cap B|.$$

(3) $K = \langle A^\delta \rangle$.
Let $S = \langle A^\delta \rangle \leqslant K$. We have

$$(b^\delta)^a = (ba)^\delta (a^\delta)^{-1} \in S,$$

for all $a, b \in A$. Hence, S is normalized by A. Moreover, as K is abelian, $S \trianglelefteq G$. Note that $B \subseteq AA^\delta$, by (1), we see that

$$G = \langle A, B \rangle = \langle A, AA^\delta \rangle = \langle A, A^\delta \rangle = AS.$$

It follows from (2) that $K \cap A = 1$ and

$$K = K \cap AS = S(K \cap A) = S,$$

as desired.

(4) For each $x \in A^\delta$, the order of x divides $|A|$.
Let $y \in K$ be the product of all elements in A^δ. Since K is abelian and has a cyclic Sylow 2-subgroup, we have $y = 1$ or the unique involution of K. For both cases, we have $y^a = y$, for all $a \in A$. Let R be a right transversal of $A \cap B$ in A. By Lemma 2.2,

$$A^\delta = \{r^\delta \mid r \in R\} \qquad \text{and} \qquad |A^\delta| = |R| = |A : A \cap B|.$$

For $a \in A$,

$$y = y^a = \prod_{x \in A^\delta} x^a = \prod_{r \in R} (r^\delta)^a = (a^\delta)^{-|R|} \prod_{r \in R} (r^\delta)^a a^\delta$$
$$= (a^\delta)^{-|R|} \prod_{r \in R} (ra)^\delta = (a^\delta)^{-|R|} y.$$

Observe that the last equation holds as Ra is also a right transversal of $A \cap B$ in A. It follows that $(a^\delta)^{|R|} = 1$ and $o(a^\delta)$ divides $|R| = |A : A \cap B|$, as desired.

(5) Final contradiction.
By (4), for each $x \in A^\delta$, we have that $o(x)$ divides $(|K|, |A|)$, which is a π_0-number by hypothesis. Therefore, $A^\delta \subseteq K_{\pi_0}$, where K_{π_0} is the unique Hall π_0-subgroup of K. By (3),

$$K = \langle A^\delta \rangle \subseteq K_{\pi_0}.$$

Thus $K = K_{\pi_0} = \langle A^\delta \rangle$ is cyclic. Consequently, we may assume that $K = \langle x \rangle$, for some $x \in A^\delta$. By (4), $K = o(x)$ divides $|A|$, in particular, $|K| \leqslant |A|$. Since $C_K(A) = 1$, A is isomorphic to a subgroup of $\mathrm{Aut}(K)$. Since $K \neq 1$ is cyclic, it implies that

$$|A| \leqslant |\mathrm{Aut}(K)| < |K|,$$

which is a final contradiction.

Therefore, the statement follows. □

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