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Classifying Character Degree Graphs with Seven Vertices

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Abstract

We study here the graphs with seven vertices in an effort to classify which of them appear as the prime character degree graphs of finite solvable groups. This classification is complete for the disconnected graphs. Of the 853 non-isomorphic connected graphs, we were able to demonstrate that twenty-two occur as prime character degree graphs. Two are of diameter three, while the remaining are constructed as direct products. Forty-four graphs remain unclassified.

Mathematics Subject Classification (2020): 20C15, 05C25, 20D10

Keywords: character degree graphs; finite solvable groups

1 Introduction

In this paper we fix G to be a finite solvable group. We write $\mathrm{Irr}(G)$ to denote the set of irreducible characters associated to G, and then by convention

$$\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}.$$

One can then consider the set $\rho(G)$, which is defined to be all the prime numbers that divide some character in $\mathrm{cd}(G)$. The prime character degree graph of G, denoted $\Delta(G)$, is therefore the simple graph whose vertex set is $\rho(G)$. Thus, in this context, the words prime and vertex become synonymous. There is then an edge between two vertices p and q in $\Delta(G)$ if there exists some character $a \in \mathrm{cd}(G)$ such

that pq divides a. Prime character degree graphs have been studied quite extensively, both historically and recently (see [11] for background or [16] for a comprehensive overview).

Prime character degree graphs can be considered for any group, but here we focus specifically on solvable groups, allowing the use of Pálfy's condition. This landmark result from [19] essentially mandates that any prime character degree graph that corresponds to a solvable group cannot have a triangle in its complement graph. This was generalized more recently in [1] from a triangle to any odd cycle. We will make frequent use of these results to fulfill the goal of this paper, which is to study graphs that have seven vertices.

The main question we wish to answer is to determine which graphs with seven vertices can occur as the prime character degree graph of a finite solvable group. Classifications of this nature have been studied previously. Some of the first graphs to be considered were proven to be character degree graphs in [24], but one can see [10] where Huppert lists all of the graphs with four vertices or fewer that can arise as $\Delta(G)$ for some solvable group G. Lewis classifies all graphs with five vertices in [15] with one notable exception, and Bissler et al. then classify graphs with six vertices in [3]. They had nine graphs that went unclassified in their paper.

Following [18], one can list a staggering 853 pairwise non-isomorphic connected graphs with seven vertices. Using known results from the literature (like from [1]), we can reduce this number to only eighty-five eligible graphs that can potentially occur as the prime character degree graph of some solvable group. After going through these remaining graphs, we determine that twenty-two of the connected graphs with seven vertices occur as $\Delta(G)$, as well as two disconnected graphs. After eliminating nineteen graphs, we are left with forty-four graphs that are still under consideration. Our results can be collated into the following.

Main Theorem *The graphs with seven vertices that arise as* $\Delta(G)$ *for some solvable group* G *are precisely those graphs in* Figures 2, 6, 7, 9, 10, *and* 13, *and possibly those in* Figures 12, 18, *and* 19.

2 Preliminaries

Many of the results we employ throughout this article are well-known. In the case where we use certain work more frequently, we put it in

this section, so as to stay as self-contained as possible while simultaneously making it easier on the reader. We shall first go through traditional results like Pálfy's condition, and then review work that aids in classifications: results on disconnected graphs, results concerning graphs with diameter three, and results in relation to admissible vertices. We then round out the section with setting notation and conventions used throughout the paper.

2.1 Families of graphs

Pálfy's condition, stated fully below and given in [19], gives a great starting point for how we can show that some graphs do not occur as the prime character degree graph of any solvable group.

Lemma 1 *Let* G *be a solvable group and let* π *be a set of primes contained in* $\Delta(G)$*. If* $|\pi| = 3$*, then there exists an irreducible character of* G *with degree divisible by at least two primes from* π (*in other words, any three vertices of the prime character degree graph of a solvable group span at least one edge*).

Equivalently, Pálfy's condition says that there must not be a 3-cycle in the complement graph, denoted $\overline{\Delta}(G)$. Much more recently, in [1], the authors prove a generalization of Pály's condition, extending from a 3-cycle to any odd-cycle.

Theorem 2 (see [1], Theorem A) *Let* G *be a finite solvable group. Then the graph* $\overline{\Delta}(G)$ *does not contain any cycle of odd length.*

A powerful consequence is stated below, one that we will rely upon heavily in Section 4. First, however, it is important to understand the notion of a *clique*, which is nothing more than a complete subgraph of a given graph.

Corollary 3 (see [1], Corollary B) *Let* G *be a finite solvable group. Then* $\rho(G)$ *is covered by two subsets, each inducing a clique in* $\Delta(G)$*.*

Next we recall the classification of the family of graphs $\{\Gamma_{k,t}\}$ given in [2]. These graphs come up frequently in our classification of graphs with seven vertices, and therefore stating the construction and results seem worthwhile.

For $k \geqslant t \geqslant 1$, the graph $\Gamma_{k,t}$ has two complete graphs A and B on k and t vertices, respectively. Letting

$$\rho(A) = \{a_1, a_2, \ldots, a_k\} \qquad \text{and} \qquad \rho(B) = \{b_1, b_2, \ldots, b_t\}$$

(with the restriction that $\rho(A) \cap \rho(B) = \varnothing$) we have that a_i is adjacent to b_i, for all $1 \leqslant i \leqslant t$, while the remaining $k - t$ vertices in A are adjacent to no vertices in B. These graphs were fully classified, the statement of the result of which is below.

Theorem 4 (see [2]) *The graph $\Gamma_{k,t}$ occurs as the prime character degree graph of a solvable group precisely when $t = 1$ or $k = t = 2$. Otherwise, $\Gamma_{k,t}$ does not occur as the prime character degree graph of any solvable group.*

2.2 Disconnected graphs

Disconnected graphs often arise. Beyond using the classification done in [13] and one of the main results therein, we will also frequently use a result from Pálfy. For completion's sake, we state both frequently used results here.

Theorem 5 (see [13], Theorem 5.5) *Let G be a solvable group and suppose that $\Delta(G)$ has two connected components. Then there is precisely one prime p so that the Sylow p-subgroup of the Fitting subgroup of G is not central in G.*

The following result contains Pálfy's inequality stated in [20].

Theorem 6 *Let G be a solvable group and $\Delta(G)$ its prime character degree graph. Suppose that $\Delta(G)$ is disconnected with two components having size a and b, where $a \leqslant b$. Then $b \geqslant 2^a - 1$.*

2.3 Diameter three graphs

It is not uncommon that we will encounter a graph with seven vertices that has diameter three. These diameter three graphs were substantially tamed in [21], the groups investigated in [6], and an example ultimately constructed in [14], and then done in more general in [8].

In particular to the specific result we will use, we recall that having a diameter of three in $\Delta(G)$ imposes a partition of $\rho(\Gamma)$ into four nonempty disjoint sets:

$$\rho(\Gamma) = \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4.$$

One can do this in the following way: find vertices p and q where the distance between them is three. Let ρ_4 be the set of all vertices that are distance three from the vertex p, which will include the vertex q.

Let ρ_3 be the set of all vertices that are distance two from the vertex p. Let ρ_2 be the set of all vertices that are adjacent to the vertex p and some vertex in ρ_3. Finally, let ρ_1 consist of p and all vertices adjacent to p that are not adjacent to anything in ρ_3. This notation is not unique, and depends on the vertices p and q. Using the above notation, one can always arrange the four disjoint sets to have the following result.

Proposition 7 (see [21]) *Let* G *be a solvable group where* $\Delta(G)$ *has diameter three. Then the following statements are satisfied*:

(i) $|\rho_3| \geqslant 3$,

(ii) $|\rho_1 \cup \rho_2| \leqslant |\rho_3 \cup \rho_4|$,

(iii) *if* $|\rho_1 \cup \rho_2| = n$, *then* $|\rho_3 \cup \rho_4| \geqslant 2^n$,

(iv) G *has a normal Sylow* p*-subgroup for exactly one prime* $p \in \rho_3$.

2.4 Admissible vertices

In [4], Bissler et al. construct a family of graphs that do not occur as the prime character degree graph of any solvable group, where in their arguments they make use of labeling certain vertices as admissible. This has far-reaching consequences and is an incredible aid with the journey to proving that a certain graph does not occur as the prime character degree graph of any solvable group.

Definition 8 (see [4]) Let Γ be a graph and p a vertex of Γ. Consider the following three conditions:

(i) the subgraph of Γ obtained by removing p and all edges incident to p is non-occurring,

(ii) all of the subgraphs of Γ obtained by removing one or more of the edges incident to p are non-occurring,

(iii) all of the subgraphs of Γ obtained by removing p, the edges incident to p, and one or more of the edges between two adjacent vertices of p are non-occurring.

If p satisfies conditions (i) and (ii), then p is said to be an *admissible* vertex. If p satisfies all three conditions, then p is said to be a *strongly admissible* vertex of Γ.

Several immediate consequences follow, where in the latter two, it is to conclude that no normal nonabelian Sylow subgroup exists corresponding to a particular vertex. This argument is used heavily in Section 7.

Lemma 9 (see [4]) *Let* G *be a solvable group, and suppose* p *is an admissible vertex of* $\Delta(G)$*. For every proper normal subgroup* N *of* G*, suppose that* $\Delta(N)$ *is a proper subgraph of* $\Delta(G)$*. Then* $O^p(G) = G$*.*

Lemma 10 (see [4]) *Let* G *be a solvable group and assume that* p *is a prime whose vertex is a strongly admissible vertex of* $\Delta(G)$*. For every proper normal subgroup* N *of* G*, suppose that* $\Delta(G/N)$ *is a proper subgraph of* $\Delta(G)$*. Then a Sylow* p*-subgroup of* G *is not normal.*

Lemma 11 (see [4]) *Let* Γ *be a graph satisfying Pálfy's condition. Let* q *be a vertex of* Γ*, and denote* π *to be the set of vertices of* Γ *adjacent to* q*, and* ρ *to be the set of vertices of* Γ *not adjacent to* q*. Assume that* π *is the disjoint union of nonempty sets* π_1 *and* π_2*, and assume that no vertex in* π_1 *is adjacent in* Γ *to any vertex in* π_2*. Let* ν *be a vertex in* π_2 *adjacent to an admissible vertex* s *in* ρ*. Furthermore, assume there exists another vertex* w *in* ρ *that is not adjacent to* ν*.*

Let G *be a solvable group such that* $\Delta(G) = \Gamma$*, and assume that for every proper normal subgroup* N *of* G*,* $\Delta(N)$ *is a proper subgraph of* $\Delta(G)$*. Then a Sylow* q*-subgroup of* G *for the prime associated to* q *is not normal.*

Lastly, we state the result which will often lead us to a final contradiction. It once again makes use of admissible vertices.

Lemma 12 (see [4]) *Let* Γ *be a graph with* $n \geqslant 5$ *vertices that satisfies Pálfy's condition. Also, assume there exist distinct vertices* a *and* b *of* Γ *such that* a *is adjacent to an admissible vertex* c*,* b *is not adjacent to* c*, and* a *is not adjacent to an admissible vertex* d*.*

Let G *be a solvable group and suppose for all proper normal subgroups* N *of* G *we have that* $\Delta(N)$ *and* $\Delta(G/N)$ *are proper subgraphs of* Γ*. Let* F *be the Fitting subgroup of* G *and suppose that* F *is minimal normal in* G*. Then* Γ *is not the prime character degree graph of any solvable group.*

2.5 Notation and conventions

To simplify language throughout this paper, we will adopt the shorthand of referring to a graph Γ as *occurring* if $\Gamma = \Delta(G)$ for some finite solvable group G, and *non-occurring* otherwise.

Moreover, following [7], we shall make use of the notation $\Gamma[p]$ to denote the subgraph of Γ obtained by deleting p and its incident

edges from Γ. Furthermore, we will use the expression $\epsilon(p, q)$ to refer to the edge joining p and q, and therefore $\Gamma[\epsilon(p, q)]$ denotes the subgraph of Γ obtained by deleting the edge between p and q, but not the vertices themselves. We can then combine these notations, where for example $\Gamma[p_1, \epsilon(p_2, p_3)]$ would be the subgraph of Γ obtained by deleting p_1 and its incident edges, along with the edge connecting p_2 and p_3. Making use of this notation is helpful when referring to subgraphs, which is done quite frequently when showing that a certain vertex is admissible. Specifically, we will make use of this often in Section 7.

Finally, when referring to a complete graph on n vertices, we will use the common notation of K_n. It is well-known that K_n occurs as the prime character degree graph for some solvable group, for all $n \in \mathbb{N}$. This will be useful in Section 5 where we often use K_n to construct a new group via a direct product. For completion's sake, one can recall that the direct product between two groups, denoted $H \times K$, results in their vertex set as

$$\rho(H \times K) = \rho(H) \cup \rho(K).$$

Since there are infinitely many primes for which these groups have their respective graphs, it is no problem requiring that $\rho(H)$ and $\rho(K)$ be disjoint. There is then an edge between p and q in $\Delta(H \times K)$ if either there was an edge between p and q in $\Delta(H)$ or $\Delta(K)$, or if

$$p \in \rho(H) \qquad \text{and} \qquad q \in \rho(K).$$

Direct products are a great way to build new groups and graphs from old ones. In fact, the singleton K_1 can be easily constructed as $\Delta(G)$ for some finite solvable group G, and then an induction argument yields $K_n = K_1 \times K_{n-1}$.

3 Disconnected graphs

In this section, we fully classify the disconnected graphs with seven vertices. Due to Pálfy's condition, if a prime character degree graph is disconnected, then there can only be two components. Furthermore, each component must be complete. This leaves us with three cases concerning the sizes of the two complete components: (1) one and

six, (2) two and five, and (3) three and four.

We will show, by construction, that only the disconnected graphs in Figure 2 occur as the prime character degree graph of a solvable group.

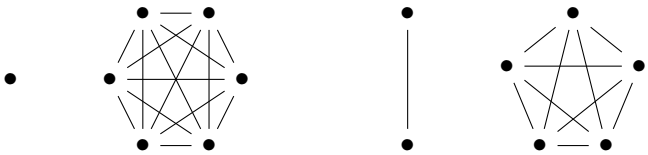


Figure 2: Disconnected graphs that occur as the prime character degree graph of a solvable group.

3.1 Constructing case (1)

Following the constructions seen in [3] or [15], we begin by considering the field of order 2^{81} acted on by its full multiplication group and then its Galois group. Set the resulting solvable group as G_1. Noting that $2^{81}-1$ is the product of six distinct prime numbers, namely

$$2^{81}-1 = 7 \cdot 73 \cdot 2593 \cdot 71119 \cdot 262657 \cdot 97685839,$$

we get that

$$\mathrm{cd}(G_1) = \{1, 3, 9, 27, 81, 2^{81}-1\}.$$

It is easy to see that $\Delta(G_1)$ is the disconnected graph with two complete components, as seen in Figure 3. For ease of notation, let us set $p = 71119$, $q = 262657$, and $r = 97685839$.

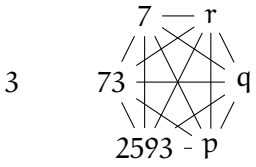


Figure 3: The graph $\Delta(G_1)$

3.2 Constructing case (2)

Similarly, now consider the field of order 2^{51} acted on by its full multiplication group and then its Galois group. Let this solvable group

be G_2. Observe that $2^{51}-1$ is the product of five distinct prime numbers: in particular, $2^{51}-1 = 7 \cdot 103 \cdot 2143 \cdot 11119 \cdot 131071$. One can compute that

$$\mathrm{cd}(G_2) = \{1, 3, 17, 51, 2^{51}-1\}.$$

Setting $p = 11119$ and $q = 131071$, one can see the resulting prime character degree graph in Figure 4.

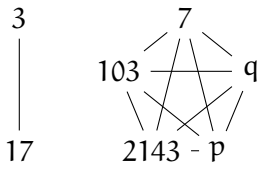


Figure 4: The graph $\Delta(G_2)$

3.3 Eliminating case (3)

It is clear that the disconnected graph in Figure 5 does not occur as the prime character degree graph of any solvable group. This is due to Pálfy's inequality (see Theorem 6) by taking $a = 3$ and $b = 4$. Notice then that $4 = b \geqslant 2^a - 1 = 2^3 - 1 = 7$, a contradiction.

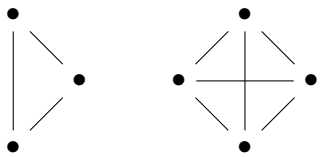


Figure 5: Disconnected graphs that do not occur as the prime character degree graph of a solvable group.

4 Connected graphs

There are 853 non-isomorphic simple connected graphs with seven vertices, as per [18]. Due to the powerful landmark result from [1] (see Corollary 3), we know that if a graph Γ is going to occur as the prime character degree graph of some solvable group, then the vertex set $\rho(\Gamma)$ must be covered by two subsets, each forming a complete

subgraph (a clique). Following the above, we are able to immediately reduce the possibilities to eighty-five eligible graphs. Much like the disconnected cases from Section 3, we too have three options here for the connected case. A connected graph with seven vertices, if it is to occur, must therefore be covered by cliques of one and six, two and five, or three and four.

We then organize these remaining eighty-five graphs into one of the three Appendices 8, 9, or 10. Each graph is placed into the appendix of the largest possible clique. For example, the graph A_1 is covered by cliques of one and six, but also covered by cliques of two and five. But since we want to place it into exactly one category, we choose to place it into the highest available clique: one and six, instead of two and five.

These remaining eighty-five graphs, the complete lists of which are organized by edge count within each appendix, will then be discussed in Sections 5, 6, and 7, respectively. In particular, Appendix 8 has six graphs, Appendix 9 houses twenty-six graphs, while Appendix 10 contains fifty-three graphs, totalling the aforementioned eighty-five. As we shall see in the sections that follow, of these eighty-five, we conclude that twenty-two of these graphs do indeed occur as $\Delta(G)$ for some finite solvable group G, whereas nineteen graphs do not occur as the prime character degree graph of any solvable group. This leaves us with forty-four graphs that are yet-to-be classified, and therefore possibly occur.

5 Cliques of one and six

There are six connected graphs with cliques of one and six, all of which we fully classify. One can see Appendix 8 for the complete list. In fact, each graph is a direct product of graphs that have been shown to occur in previous papers, or graphs that have already been studied.

It is not difficult to see that A_1 is the graph $\Gamma_{6,1}$ from [2] (see Theorem 4). Next, the graph A_i can be realized as the direct product between $\Gamma_{7-i,1}$ and K_{i-1}, for all $2 \leqslant i \leqslant 5$. Finally, the graph A_6 is exactly K_7. Hence, all the graphs in Figure 6 occur as the prime character degree graph of a solvable group.

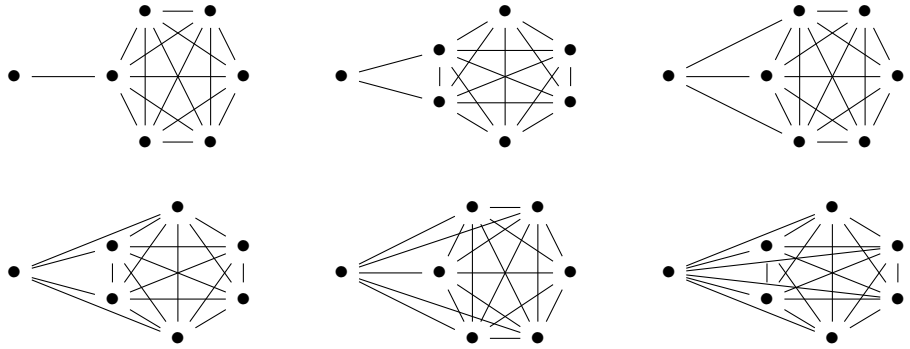


Figure 6: The graphs A_i for $1 \leqslant i \leqslant 6$

6 Cliques of two and five

There are twenty-six connected graphs whose vertex sets are covered by cliques of two and five, where one can see Appendix 9 for the complete list. In this section we will classify fifteen of these graphs, leaving eleven unknown.

Thus, the goal of this section is to show that all the graphs in Figures 7, 9, and 10 occur, and that all the graphs in Figure 12 possibly occur. Otherwise, as we shall see, the graphs in Figure 11 do not occur as the prime character degree graph of any solvable group.

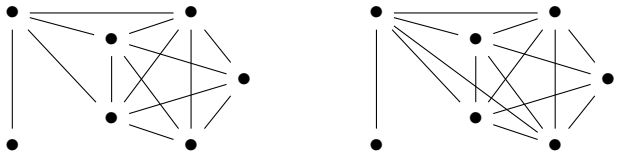


Figure 7: The graphs B_3 and B_4

The graphs in Figure 7 will be constructed below, but earn special attention as they both have diameter three, and are not merely constructed as direct products.

6.1 Constructing the graphs B_3 and B_4

The graphs B_3 and B_4 result from a direct application of the techniques found in [8], which expand on the construction found in [14]. These works explicitly construct solvable groups for which the character degree graph has diameter three. Of particular note in [8] is that

the resulting character degree graph is easy to calculate directly. We begin by choosing primes p, q, and r that satisfy certain divisibility conditions.

Specifically, we allow p to be any prime. From here, we require that q and r are chosen so that $q < r$ and the quantities $\Phi_q(p)$, $\Phi_r(p)$, and $\Phi_{qr}(p)$ are pairwise coprime, with $\Phi_n(x)$ being the nth cyclotomic polynomial. As illustrated in [8], the computation of character degrees is substantially more complicated when one chooses q to be a prime larger than 3. Fortunately, the case $q = 3$ is sufficient to construct both B_3 and B_4.

To construct G, we begin with a field F of order p^{3r}. We then consider $F\{X\}$, the skew polynomial ring, in which

$$X\alpha = \alpha^p X,$$

for all $\alpha \in F$. Next, consider the quotient ring, R, of $F\{X\}$ and the ideal generated by X^4. Let J be the Jacobson radical of R, and note that elements of J all have the form $\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$, where x is the image of the indeterminate X in R. Then, define $P = 1 + J$, which is a group of order p^{9r}. We next let C be the subgroup of $F^\times$ that has order $\frac{p^{3r}-1}{p-1}$, and define

$$T = P \rtimes_\varphi C,$$

where $c \in C$ acts on P via the automorphism defined by:

$$\varphi_c\left(1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3\right) = 1 + \alpha_1 c x + \alpha_2 c^{p+1} x^2 + \alpha_3 c^{p^2+p+1} x^3.$$

Finally, G is the semidirect product of T and $\mathcal{G}$, with $\mathcal{G}$ being the Galois group of F over its prime subfield. One can see that

$$|G| = 3rp^{9r}\left(\frac{p^{3r}-1}{p-1}\right),$$

so the constructed group is indeed finite.

To construct B_3 and B_4, we impose the additional condition that the quantity

$$\frac{p^{3r}-1}{p-1} = stuv,$$

with s, t, u, and v being prime and distinct from p, q, and r; this helps

to ensure a seven-vertex character degree graph. The argument in [8] yields the following set of character degrees:

$$\begin{aligned}\mathrm{cd}(G) = \Big\{ & 1, 3, r, 3r, stuv, p^{\frac{3r-1}{2}} stuv, 3p^{3r}\left(\frac{stuv}{p^2+p+1}\right), \\ & p^{3r-3}\left(\frac{stuv}{p^2+p+1}\right), 3p^{3r-3}\left(\frac{stuv}{p^2+p+1}\right), p^{3r-3}stuv, \\ & p^{3r-2}stuv\Big\}.\end{aligned}$$

Let G_3 correspond to the choices $p = 23$, $q = 3$ and $r = 5$, which gives $s = 7$, $t = 79$, $u = 292561$, and $v = 74912328481$. In this case, we get

$$\begin{aligned}\mathrm{cd}(G_3) = \Big\{ & 1, 3, 5, 15, stuv, 23^7 stuv, 3 \cdot 23^{15} uv, \\ & 23^{12} uv, 3 \cdot 23^{12} uv, 23^{12} stuv, 23^{13} stuv\Big\}.\end{aligned}$$

Similarly, let G_4 correspond to the choices $p = 2$, $q = 3$, and $r = 11$, giving $s = 7$, $t = 23$, $u = 89$, and $v = 599479$. This yields

$$\begin{aligned}\mathrm{cd}(G_4) = \Big\{ & 1, 3, 11, 33, stuv, 2^{16} stuv, 3 \cdot 2^{33} tuv, \\ & 2^{30} tuv, 3 \cdot 2^{30} tuv, 2^{30} stuv, 2^{31} stuv\Big\}.\end{aligned}$$

We can see $\Delta(G_3)$ and $\Delta(G_4)$ in Figure 8, which are clearly B_3 and B_4, respectively.

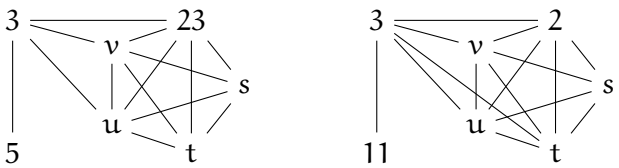


Figure 8: The graphs $\Delta(G_3) = B_3$ and $\Delta(G_4) = B_4$

6.2 Direct products

There are ten graphs in Appendix 9 that are direct products of other graphs that are known to occur. These graphs are B_{15}, shown in Fig-

ure 9, and B_i, for $i \in \mathcal{I}$, where

$$\mathcal{I} := \{6, 13, 14, 16, 19, 21, 23, 24, 26\},$$

seen in Figure 10.

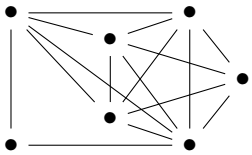


Figure 9: The graph B_{15}

The graph B_{15} deserves some special attention. It can be constructed as the direct product between the singleton K_1 and the diameter three graph with six vertices, which was constructed in [14] and classified in [3].

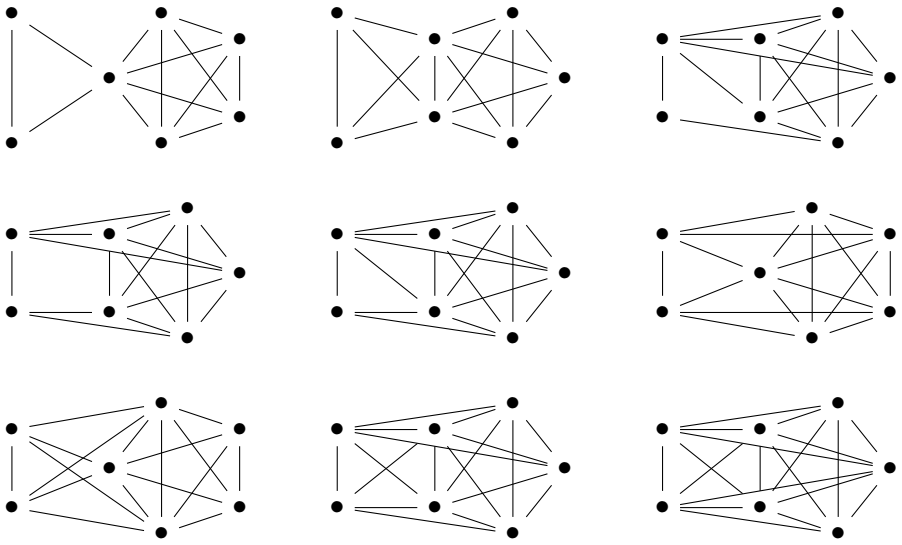


Figure 10: The graphs B_i for $i \in \mathcal{I}$

Next we'll consider the graphs in Figure 10. The graph B_6 is again the direct product of the singleton K_1 and the disconnected graph with two complete components of sizes two and four (shown to occur in [3]). As for B_{14}, it can be seen as the direct product between two disconnected graphs: one having complete components of sizes

one and one, and the other having complete components of sizes one and four (shown to occur in [15]). Next, the graph B_{16} is also the direct product between two disconnected graphs known to occur as $\Delta(G)$ for some solvable group G; this time it is between one having complete components of sizes one and two, whereas the other has complete components of sizes one and three.

The remaining six graphs in Figure 10 are the direct product between the singleton K_1 and one of the graphs shown to occur on page 503 in [3].

6.3 Eliminations

In this section, we will eliminate the graphs in Figure 11. The first two graphs, B_1 and B_2, clearly have diameter three, and the last graph, B_5, has actually been previously studied.

For B_1, notice that it has two cut-vertices. This contradicts the main result from [17] which requires prime character degree graphs to have at most one cut vertex. Hence, B_1 cannot be the prime character degree graph of any solvable group.

For B_2, notice that it has diameter three, as previously mentioned. In particular, following the labeling in Section 2.3, we see that

$$|\rho_3| = 2 < 3,$$

which contradicts Proposition 7 (7). Thus, B_2 is not $\Delta(G)$ for any finite solvable group G.

Finally, one can see that the graph B_5 is exactly the graph $\Gamma_{5,2}$ from [2]. Using Theorem 4, we can conclude that $B_5 = \Gamma_{5,2}$ does not occur as the prime character degree graph of any solvable group.

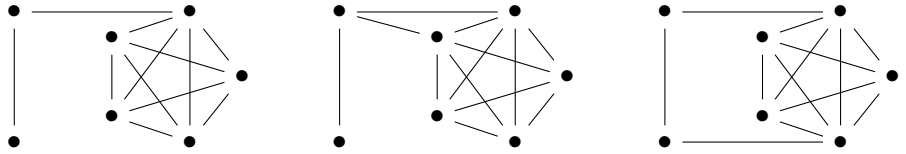


Figure 11: The graphs B_1, B_2, and B_5

6.4 Unclassified graphs

There are eleven connected graphs that are currently unknown whose vertex sets are covered by cliques of two and five (see Figure 12). Set-

ting

$$\mathcal{J} := \{7, 8, 9, 10, 11, 12, 17, 18, 20, 22, 25\},$$

we have the graphs B_j, for all $j \in \mathcal{J}$.

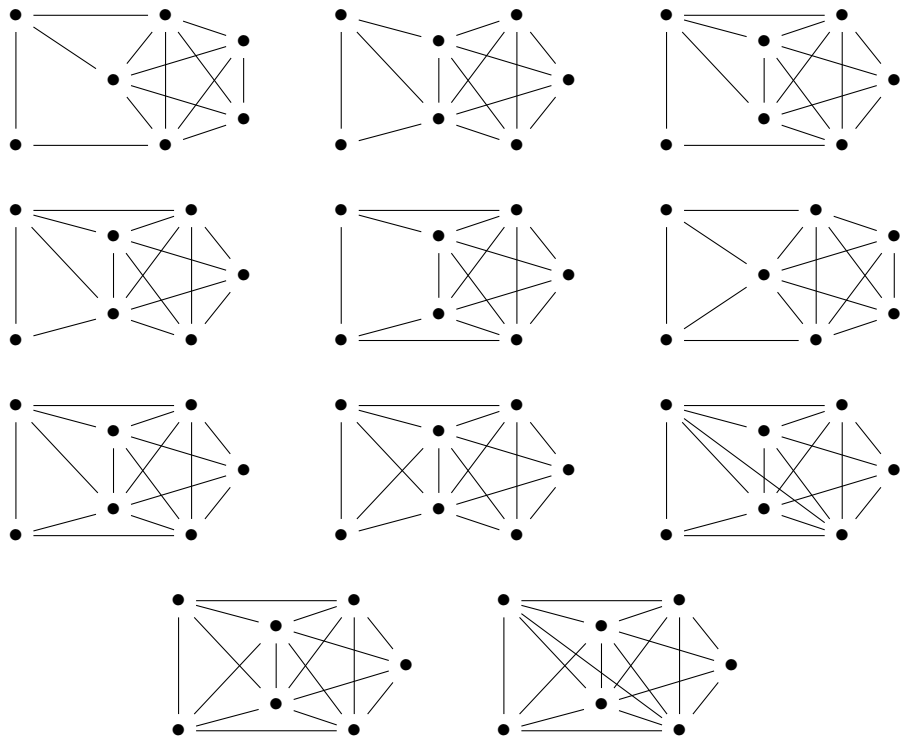


Figure 12: The graphs B_j for $j \in \mathcal{J}$

Part of the main reason that these graphs went unclassified is that they rely upon subgraphs with six vertices that are currently unclassified from the work done in [3]. The graph B_7 is an example of a graph whose classification has been halted by this issue.

The next main issue we faced with these graphs in Figure 12 is that they have the disconnected graph with component sizes of two and five that has been shown to occur (see Section 3.2). This issue can be overcome, but our current argument to eliminate graphs forbids the possibility of a connected subgraph occurring. For example, the graph B_8 has B_6 as a subgraph, and B_6 was shown to occur in Section 6.2. We hit a similar issue with the graph B_9, for instance, because it has B_3 as a subgraph, shown to occur in Section 6.1.

7 Cliques of three and four

There are fifty-three connected graphs whose vertex sets are covered by cliques of three and four (see Appendix 10), of which we classify twenty. Of these, we conclude that four of them do indeed occur, whereas sixteen of them do not. The remaining thirty-three are unknown and require further discussion and investigation, along with (presumably) new methods.

Therefore, the goal of this section is to show that all the graphs in Figure 13 occur, and that all the graphs in Figures 18 and 19 possibly occur. The graphs in Figures 14, 15, 16, and 17 do not occur as the prime character degree graph of any solvable group, as shown in detail below.

7.1 Direct products

There are four graphs in Appendix 10 that can be constructed as direct products of other prime character degree graphs known to occur. These graphs, shown in Figure 13, are C_{26}, C_{50}, C_{51}, and C_{53}.

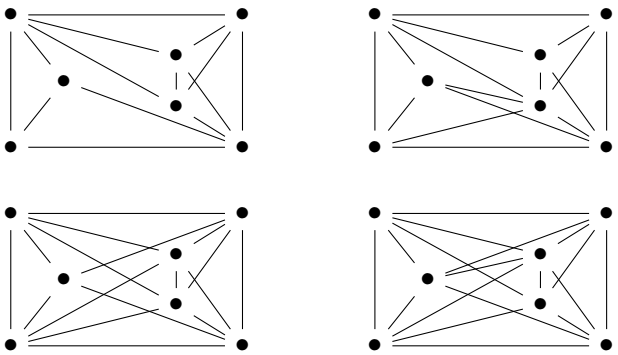


Figure 13: The graphs C_{26}, C_{50}, C_{51}, and C_{53}

For the interested reader, one can see that C_{26} is the direct product between the disconnected graph where one component is K_2 and the other K_3 and the disconnected graph with two total vertices. Next, the graph C_{50} can be realized as the direct product between the bowtie graph constructed in [15] and the disconnected graph with two total vertices. The graph C_{51} is again the direct product of the disconnected graph with two total vertices and another graph which occurs from [15]. Finally, the graph C_{53} is the direct product of the regular graph with six vertices where each vertex is of degree four

(see [3], [23], or [26]) and the singleton. Hence, the graphs in Figure 13 occur as $\Delta(G)$ for some finite solvable group G.

7.2 Previously classified graphs

In this section, we eliminate the graphs in Figure 14. Both of these graphs have been previously studied.

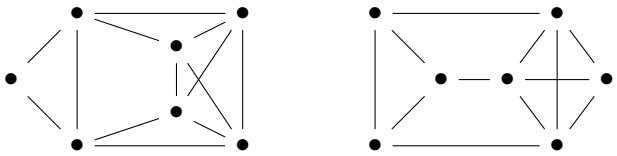


Figure 14: The graphs C_{10} and C_{17}

The first graph, C_{10}, was studied in [3]. It was also classified in a family that was studied in [12], and in that context was denoted by $\Sigma^{L}_{2,2}$. Following either result, we can conclude that $\Sigma^{L}_{2,2} = C_{10}$ does not occur as $\Delta(G)$, for any solvable group G.

Furthermore, the second graph, C_{17}, is exactly the graph $\Gamma_{4,3}$, which was classified in [2]. In that paper, the authors stated that $\Gamma_{4,3} = C_{17}$ also does not occur as the prime character degree graph of any solvable group (see Theorem 4).

7.3 Diameter three graphs

There are twelve graphs that are covered by cliques of three and four that have diameter three. Specifically, the graph C_i has diameter three, for all $i \in \{1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14, 16\}$. For ease of notation, define

$$\mathcal{I} := \{1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14, 16\}.$$

One can see Figure 15 for the complete list. We claim that C_i does not occur as the prime character degree graph for any solvable group, for all $i \in \mathcal{I}$.

One can use the main result in [17] to eliminate the graph C_1, because that graph has two cut-vertices. However, a simple method to eliminate C_i, for all $i \in \mathcal{I}$, is to employ the results in [21] (see Proposition 7 (7)). Using the prescribed notation set in Section 2.3, one can see that for each $i \in \mathcal{I}$ we have for C_i that

$$|\rho_1 \cup \rho_2| = 3 \qquad \text{and} \qquad |\rho_3 \cup \rho_4| = 4,$$

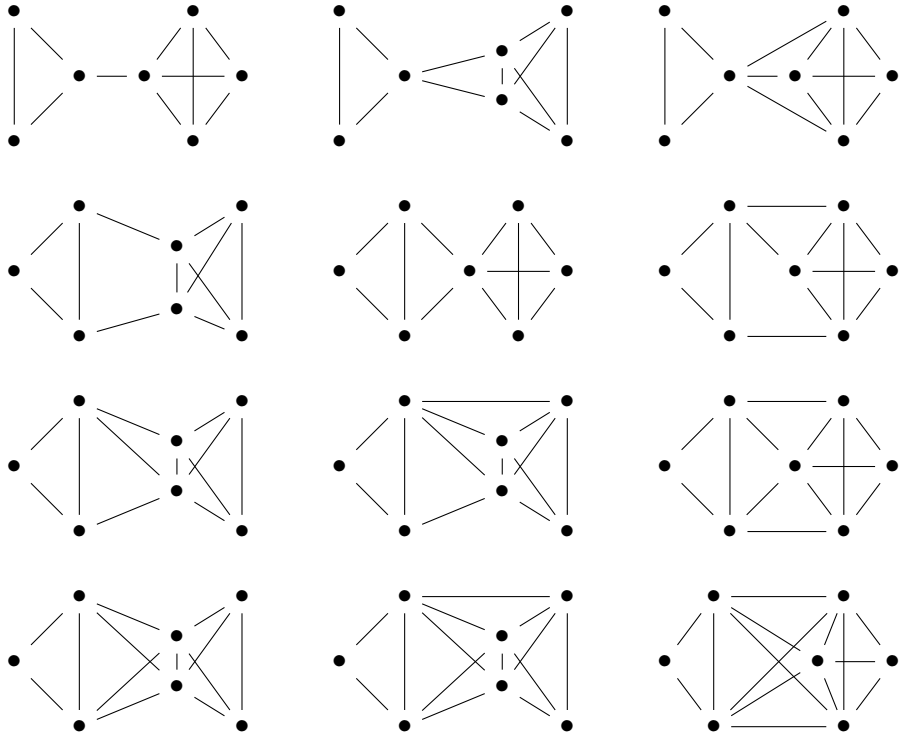


Figure 15: The graphs C_i for $i \in \mathcal{I}$

yet

$$4 = |\rho_3 \cup \rho_4| \geqslant 2^{|\rho_1 \cup \rho_2|} = 2^3 = 8,$$

a contradiction.

7.4 Eliminating the graph C_{18}

Our goal in this section is to eliminate the graph C_{18} (see Figure 16). In other words, we will conclude that C_{18} does not occur as the prime character degree graph for any solvable group. We follow a similar strategy that is employed in [15], or even [3], [7], or [12].

Lemma 13 *Assume that* $C_{18} = \Delta(G)$ *for some finite solvable group* G*, where* $|G|$ *is minimal. Then* G *does not have any normal nonabelian Sylow* p_i*-subgroup for* $i \in \{1, 3, 4, 5, 6, 7\}$*. In particular, each* p_i *is a strongly admissible vertex.*

Proof — We begin by checking for admissibility of the vertex p_1. If we remove the vertex p_1 and all incident edges, we are left with

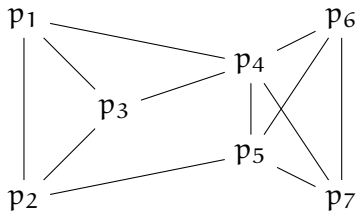


Figure 16: The graph C_{18}

the graph $C_{18}[p_1] = \Gamma_{4,2}$, which is non-occurring from [2] (see Theorem 4). Next, the edges $\epsilon(p_1, p_2)$ and $\epsilon(p_1, p_3)$ must remain, for losing either or both would violate Pálfy's condition. Additionally, removing the edge between p_1 and p_4 leaves us with a diameter three graph

$$C_{18}[\epsilon(p_1, p_4)] = C_4,$$

which violates the main result from [21] and has been shown to not occur in Section 7.3. Therefore, p_1 is admissible.

For strong admissibility, we consider the removal of p_1, all incident edges, and one or more of the edges between two vertices adjacent to p_1. It is easy to see that $C_{18}[p_1, \epsilon(p_2, p_3)]$ violates Pálfy's condition, and so we cannot lose the edge between p_2 and p_3. Now, note that $C_{18}[p_1, \epsilon(p_3, p_4)]$ has diameter three, which again does not occur by [21]. Thus, p_1 is strongly admissible. By symmetry, we also have that p_3 is strongly admissible.

Next, we consider the vertex p_4. The graph $C_{18}[p_4]$ has diameter three and is not the unique six-vertex diameter three graph which occurs in [14], so it is therefore non-occurring. The graph $C_{18}[\epsilon(p_4, p_5)]$ contains a five-cycle in its complement, which means it violates the generalized Pálfy's condition from [1]. The vertices p_6 and p_7 are symmetric, and thus the edges $\epsilon(p_4, p_6)$ and $\epsilon(p_4, p_7)$ are symmetric. Losing either of the aforementioned edges would result in violations of Pálfy's condition. It has already been established that we cannot remove the edge $\epsilon(p_1, p_4)$. Symmetry demands that we also cannot remove the edge $\epsilon(p_3, p_4)$. Thus, if we lose any combination of edges from p_4, the resulting graphs will all be non-occurring due to either Pálfy's condition or having diameter three. Hence, p_4 is admissible.

For strong admissibility, we consider the removal of p_4, all incident edges, and one or more of the edges between two vertices adjacent to p_4. Observe that $C_{18}[p_4]$ has diameter three. From here, removing

any number of edges between two vertices adjacent to p_4 will either maintain our diameter three violation or create a non-occurring disconnected graph with six vertices classified in [3]. Therefore, p_4 is strongly admissible.

Consider the vertex p_5. Notice that $C_{18}[p_5]$ has diameter three and is non-occurring by [21]. Next, we established previously that we cannot lose $\epsilon(p_4, p_5)$. Similarly, the symmetry between p_6 and p_7 makes $\epsilon(p_5, p_6)$ and $\epsilon(p_5, p_7)$ symmetric as well, and the removal of either edge would give us a graph which violates Pálfy's condition, so these graphs again do not occur. Removing the edge between p_2 and p_5 creates the diameter three graph $C_{18}[\epsilon(p_2, p_5)] = C_5$, which violates the main result from [21] and has also been shown not to occur in Section 7.3. Thus, losing any combination of edges incident to p_5 results in a graph that does not occur. Therefore, p_5 is admissible.

For strong admissibility, we consider the removal of p_5, all incident edges, and one or more of the edges between two vertices adjacent to p_5. Notice that the graph $C_{18}[p_5]$ has diameter three and does not occur. All edges between two vertices adjacent to p_5 lie in the complete subgraph on vertices p_4, p_6, and p_7, and so the graphs obtained by removing any one or more of these edges will create either a diameter three graph or a non-occurring disconnected graph on six vertices. Thus, p_5 is strongly admissible.

We now check for admissibility of p_6. The graph $C_{18}[p_6]$ is simply the graph $\Sigma^L_{2,1}$ which was shown not to occur in [12]. It is straightforward to verify that removing one or more of the edges incident to p_6 will violate Pálfy's condition. Thus, p_6 is admissible.

For strong admissibility, we consider the removal of p_6, all incident edges, and one or more of the edges between two vertices adjacent to p_6. The graph $C_{18}[p_6, \epsilon(p_4, p_5)]$ has a five-cycle in its complement, thus violating the generalized Pálfy's condition from [1]. Similarly, removing the edges $\epsilon(p_4, p_7)$ or $\epsilon(p_5, p_7)$ will also violate this generalized Pálfy's condition. Hence, p_6 is strongly admissible. Again by symmetry, we also have that p_7 is strongly admissible.

Since p_i is strongly admissible, for each $i \in \{1, 3, 4, 5, 6, 7\}$, then there is no normal nonabelian Sylow p_i-subgroup for each i by way of Lemma 10. □

Lemma 14 *Assume that* $C_{18} = \Delta(G)$ *for some finite solvable group* G, *where* $|G|$ *is minimal. There is no normal nonabelian Sylow* p_2*-subgroup.*

Proof — We will follow the notation from Lemma 11. For $q = p_2$, we have

$$\pi = \{p_1, p_3, p_5\} \qquad \text{and} \qquad \rho = \{p_4, p_6, p_7\}.$$

Setting $\pi_1 = \{p_5\}$ and $\pi_2 = \{p_1, p_3\}$, we choose $\nu = p_1$, which is adjacent to $s = p_4 \in \rho$, shown to be admissible in Lemma 13. We also note that $w = p_6 \in \rho$ is not adjacent to ν, and hence the result follows from Lemma 11. □

Lemma 15 *No proper subgraph with the same vertex set as C_{18} occurs as the prime character degree graph of any solvable group.*

Proof — We have established already that we cannot lose any edges in the complete graphs on $\{p_1, p_2, p_3\}$ and $\{p_4, p_5, p_6, p_7\}$. Thus, we are left to consider the three edges $\epsilon(p_1, p_4)$, $\epsilon(p_3, p_4)$, and $\epsilon(p_2, p_5)$. We will check all subgraphs for which we remove one or more of those edges.

First, recall that p_1 and p_3 are symmetric. This means that $\epsilon(p_1, p_4)$ and $\epsilon(p_3, p_4)$ are symmetric as well. Notice that

$$C_{18}[\epsilon(p_1, p_4)] = C_{18}[\epsilon(p_3, p_4)] = C_4$$

and $C_{18}[\epsilon(p_2, p_5)] = C_5$ have both been shown not to occur. Next, considering losing two of the aforementioned three edges, we have that

$$\begin{aligned} C_{18}[\epsilon(p_1, p_4), \epsilon(p_3, p_4)] &= C_{18}[\epsilon(p_1, p_4), \epsilon(p_2, p_5)] \\ &= C_{18}[\epsilon(p_3, p_4), \epsilon(p_2, p_5)] = C_1, \end{aligned}$$

which does not occur (shown in Section 7.3).

Lastly, removing all three edges will give us the disconnected graph in Figure 5, which was shown not to occur in Section 3.3.

Hence, no proper subgraph of C_{18} with the same vertex set occurs as the prime character degree graph of any solvable group. □

Assuming that there exists some solvable group G with $|G|$ minimal such that $\Delta(G) = C_{18}$, we now have the conclusion that there are no normal nonabelian Sylow p-subgroups, for all $p \in \rho(G)$ (using Lemmas 13 and 14). With F as the Fitting subgroup of G, we then note that $\rho(G) = \pi(|G : F|)$. Therefore, $\rho(G) = \rho(G/\Phi(G))$. Here, as is convention, $\Phi(G)$ denotes the Frattini subgroup of G. Lemma 15 above verifies there is no proper subgraph of C_{18} with the same ver-

tex set which occurs as the prime character degree graph of any solvable group, and therefore, since $|G|$ is minimal, we are forced to conclude that $\Phi(G) = 1$. We can then apply [9, Hilfssatz III.4.4] to obtain the existence of a subgroup H of G such that $G = HF$ and $H \cap F = 1$.

Lemma 16 *Assume that* $C_{18} = \Delta(G)$ *for some finite solvable group* G, *where* $|G|$ *is minimal. Then the Fitting subgroup* F *of* G *is minimal normal in* G.

Proof — Following the assumptions, we then can define H as above. Furthermore, set E to be the Fitting subgroup of H.

Next, we proceed by contradiction. Thus, we suppose that there exists some normal subgroup N of G such that $1 < N < F$. We aim to show that no such N can exist.

By [9, Satz III.4.5], notice that we know there exists some normal subgroup M of G such that $F = N \times M$, and since both N and M are nontrivial, we have that

$$\rho(G/N) \subset \rho(G) \qquad \text{and} \qquad \rho(G/M) \subset \rho(G).$$

For any prime $q \in \rho(G) \setminus \rho(G/N)$, it is known that G/N must have a normal nonabelian Sylow q-subgroup whose class is a formation. Therefore, it is required that $q \in \rho(G/M)$. Hence, we get that

$$\rho(G) = \rho(G/N) \cup \rho(G/M).$$

Hence, for this $q \in \rho(G) \setminus \rho(G/N)$, notice that $q \notin \rho(G/F) = \rho(H)$, and so E must contain the Sylow q-subgroup of H. Since $q \in \rho(G)$, however, we must have that q divides $|H|$, and so q divides $|E|$ too. Then, since $\mathrm{cd}(G)$ contains a degree divisible by all the prime divisors of $|EF : F| = |E|$, we get that $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M))$ lies in a complete subgraph of $\Delta(G)$. Hence, $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M))$ must lie in one of the following sets:

$$A := \{p_1, p_2, p_3\}, \quad B := \{p_1, p_3, p_4\}, \quad C := \{p_2, p_5\},$$
$$\text{or} \quad D := \{p_4, p_5, p_6, p_7\}$$

Assume first that $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M)) \subseteq A$. Following [15], we see that E has a Hall A-subgroupof H. One can then conclude that E is in fact the Hall A-subgroup since $\mathrm{cd}(G)$ has a degree that is divisible by all the primes dividing $|E|$ and $|E|$ is divisible by no other primes. Next, we can find a character $\chi \in \mathrm{Irr}(G)$ with $p_1p_2p_3$

dividing $\chi(1)$. Letting θ be an irreducible constituent of χ_{FE}, one can see that $\chi(1)/\theta(1)$ divides $|G : FE|$ and that $\chi(1)$ is relatively prime to $|G : FE|$. This forces $\chi_{FE} = \theta$. Since p_1, p_2, and p_3 all divide $\theta(1)$, and the only possible primes that could divide a character in $cd(G/FE)$ are p_4, p_5, p_6, or p_7, we get that $cd(G/FE) = \{1\}$ by Gallagher's Theorem. Hence, G/FE is abelian. For instance, this then yields $O^{p_4}(G) < G$, a contradiction as p_4 is admissible.

Now, suppose that $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M)) \subseteq B$. This implies that

$$\{p_2, p_5, p_6, p_7\} \subseteq \rho(G/N) \cap \rho(G/M).$$

Since $p_1 \in \rho(G) = \rho(G/N) \cup \rho(G/M)$, then we know that

$$p_1 \in \rho(G/N) \qquad \text{or} \qquad p_1 \in \rho(G/M).$$

Without loss of generality, we can suppose $p_1 \in \rho(G/N)$. Since $\rho(G/N)$ is proper in $\rho(G)$, we have three options for $\rho(G/N)$:

(i) $\{p_1, p_2, p_5, p_6, p_7\}$,

(ii) $\{p_1, p_2, p_3, p_5, p_6, p_7\}$,

(iii) $\{p_1, p_2, p_4, p_5, p_6, p_7\}$.

For (i), we have that

$$\rho(G/N) = \{p_1, p_2, p_5, p_6, p_7\},$$

and one can see that no connected graph with this vertex set occurs as the prime character degree graph of any solvable group. The only possibility is for the disconnected graph to occur, with complete components $\{p_1, p_2\}$ and $\{p_5, p_6, p_7\}$. Then by [13, Theorem 5.5] we know that G/N has a central Sylow p_3-subgroup or a central Sylow p_4-subgroup, in which case $O^{p_3}(G) < G$ or $O^{p_4}(G) < G$, respectively. This is our contradiction, since both p_3 and p_4 have been shown to be admissible, and so $O^{p_3}(G) = G$ and $O^{p_4}(G) = G$. For (ii), we have

$$\rho(G/N) = \{p_1, p_2, p_3, p_5, p_6, p_7\}.$$

However, in this scenario, no graph, connected or disconnected, can occur with that vertex set. This leaves us with (iii), giving

$$\rho(G/N) = \{p_1, p_2, p_4, p_5, p_6, p_7\}.$$

It is not hard to see that the only possible graph arising is the disconnected graph with complete components $\{p_1, p_2\}$ and D. Hence, again by [13, Theorem 5.5], this implies that G/N has a central Sylow p_3-subgroup, which means that $G = O^{p_3}(G) < G$, a contradiction since p_3 is admissible.

Suppose that $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M)) \subseteq C$. This implies that

$$\{p_1, p_3, p_4, p_6, p_7\} \subseteq \rho(G/N) \cap \rho(G/M).$$

Since $\rho(G/N)$ and $\rho(G/M)$ are both proper in $\rho(G)$, and also

$$\rho(G) = \rho(G/N) \cup \rho(G/M),$$

then without loss of generality, we must have that

$$\rho(G/N) = \{p_1, p_2, p_3, p_4, p_6, p_7\}$$

and

$$\rho(G/M) = \{p_1, p_3, p_4, p_5, p_6, p_7\}.$$

However, this is a contradiction because no graph can occur with the vertex set given to $\rho(G/N)$ as stipulated above, regardless if it is connected or disconnected.

Suppose that $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M)) \subseteq D$. This implies that

$$A \subseteq \rho(G/N) \cap \rho(G/M).$$

Since $p_7 \in \rho(G) = \rho(G/N) \cup \rho(G/M)$, then we know that

$$p_7 \in \rho(G/N) \qquad \text{or} \qquad p_7 \in \rho(G/M).$$

Without loss of generality, we can assume $p_7 \in \rho(G/N)$. Since $\rho(G/N)$ is proper in $\rho(G)$, we then have seven options as to what $\rho(G/N)$ can be one of the following:

(i) $\{p_1, p_2, p_3, p_7\}$,

(ii) $\{p_1, p_2, p_3, p_4, p_7\}$,

(iii) $\{p_1, p_2, p_3, p_5, p_7\}$,

(iv) $\{p_1, p_2, p_3, p_6, p_7\}$,

(v) $\{p_1, p_2, p_3, p_4, p_5, p_7\}$,

(vi) $\{p_1, p_2, p_3, p_4, p_6, p_7\}$,

(vii) $\{p_1, p_2, p_3, p_5, p_6, p_7\}$.

For (i), we have that $\rho(G/N) = \{p_1, p_2, p_3, p_7\}$, and one can see that the only graph that arises here is the disconnected graph with complete components A and $\{p_7\}$. Then, by [13, Theorem 5.5], we have that G/N has a central Sylow p_4-subgroup, central Sylow p_5-subgroup, or central Sylow p_6-subgroup. Therefore,

$$O^{p_4}(G) < G, \quad O^{p_5}(G) < G \qquad \text{or} \qquad O^{p_6}(G) < G,$$

respectively. Regardless of the situation, we have a contradiction, as $O^{p_4}(G) = O^{p_5}(G) = O^{p_6}(G) = G$ since p_4, p_5, and p_6 are all admissible vertices. For (ii), (iii), and (iv), one can check that the only graph that can arise for G/N is the disconnected graph with complete component A and the other component having two vertices, one of which is p_7. In each case, we can again apply [13, Theorem 5.5] to conclude that G/N has a central Sylow p_i-subgroup for some $4 \leqslant i \leqslant 6$, which is a contradiction since p_i is admissible for each of these i. For (v), (vi), and (vii), notice that there is no graph, connected or disconnected, that can occur with each respective vertex set.

Hence, no such N can occur, and we get our desired conclusion that the Fitting subgroup F is minimal normal in G. □

Proposition 17 *The graph C_{18} does not occur as the prime character degree graph for any solvable group.*

Proof — For the sake of contradiction, suppose that $C_{18} = \Delta(G)$, for some finite solvable group G, where $|G|$ is minimal. Observe that the Fitting subgroup F of G is minimal normal in G by Lemma 16. Next, consider the vertices $a = p_1$, $b = p_2$, $c = p_4$, and $d = p_5$, where we observe that p_4 and p_5 are admissible by Lemma 13. Applying Lemma 12 yields our contradiction, and therefore the graph C_{18} is not the prime character degree graph for any solvable group. □

7.5 Eliminating the graph C_{20}

Here our goal is to show that C_{20} does not occur as the prime character degree graph for any solvable group (see Figure 17). Our strategy will mimic what was done in Section 7.4.

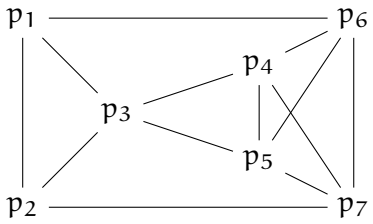


Figure 17: The graph C_{20}

Lemma 18 *Assume that* $C_{20} = \Delta(G)$ *for some finite solvable group* G*, where* $|G|$ *is minimal. Then the group* G *does not have a normal nonabelian Sylow* p_i*-subgroup, for* $i \in \{3, 4, 5, 6, 7\}$*. In particular, each* p_i *is a strongly admissible vertex.*

Proof — Let us begin with the vertex p_3. Removing p_3 and all incident edges leaves us with the graph

$$C_{20}[p_3] = \Gamma_{4,2},$$

which is non-occurring by [2] (see Theorem 4). We cannot lose either of the edges $\epsilon(p_1, p_3)$ or $\epsilon(p_2, p_3)$, for doing so would violate Pálfy's condition. Therefore, neither of these edges can be lost. Vertices p_4 and p_5 are symmetric, thus making edges $\epsilon(p_3, p_4)$ and $\epsilon(p_3, p_5)$ symmetric as well. Removing either of those edges would create the non-occurring graph $\Gamma_{4,3}$, and removing both edges would create the graph

$$C_{20}[\epsilon(p_3, p_4), \epsilon(p_3, p_5)] = C_4.$$

Neither of these graphs occur by Theorem 4 and Section 7.3, respectively. Thus, p_3 is admissible.

For strong admissibility, we consider the removal of p_3, all incident edges, and one or more of the edges between two vertices adjacent to p_3. There are two edges in consideration: $\epsilon(p_4, p_5)$ and $\epsilon(p_1, p_2)$. If we lose the edge $\epsilon(p_4, p_5)$, our new graph violates Pálfy's condition and hence that edge cannot be lost. Next, removing the edge $\epsilon(p_1, p_2)$ would create the graph $C_{20}[p_3, \epsilon(p_1, p_2)]$, and this graph does not occur by [3]. Hence, p_3 is strongly admissible.

Next, we consider the vertex p_4. Removing the vertex p_4 gives us the graph $C_{20}[p_4] = \Gamma_{3,3}$, which is non-occurring by [2] (again, see Theorem 4). Note that we cannot remove any combination of the edges $\epsilon(p_4, p_5)$, $\epsilon(p_4, p_6)$, and $\epsilon(p_4, p_7)$, as doing so would vi-

olate Pálfy's condition. This leaves us to check the edge $\epsilon(p_3, p_4)$, but we established previously that losing this edge would create the graph $\Gamma_{4,3}$. Therefore, p_4 is admissible.

For strong admissibility, we consider the removal of p_4, all incident edges, and one or more of the edges between two vertices adjacent to p_4. It is easy to see that the removal of one or more of the edges $\epsilon(p_5, p_6)$, $\epsilon(p_5, p_7)$, and $\epsilon(p_6, p_7)$ would violate Pálfy's condition, and therefore cannot be lost. The only other edge to consider is between p_3 and p_5. However, the graph $C_{20}[p_4, \epsilon(p_3, p_5)]$ does not occur by [3]. Thus, p_4 is strongly admissible. By symmetry, we also have that p_5 is strongly admissible.

We now check for admissibility of p_6. Removal of the vertex p_6 gives us the graph $C_{20}[p_6] = \Sigma^L_{2,1}$, which is non-occurring by [12] (or see [3]). We cannot lose one or more of the edges between p_6 and any of the vertices p_4, p_5, and p_7, as doing so would create graphs that violate Pálfy's condition. If we remove the edge $\epsilon(p_1, p_6)$, we are left with the graph C_6 which has been shown to not occur in Section 7.3. Hence, p_6 is admissible.

For strong admissibility, we consider the removal of p_6, all incident edges, and one or more of the edges between two vertices adjacent to p_6. Similar to the case for p_4, it is easy to see that removing one or more of the edges $\epsilon(p_4, p_5)$, $\epsilon(p_4, p_7)$, or $\epsilon(p_5, p_7)$ from $C_{20}[p_6]$ would result in graphs that violate Pálfy's condition. Therefore, p_6 is strongly admissible. Again by symmetry, we also have that p_7 is strongly admissible.

Since p_i is strongly admissible, for each $i \in \{3, 4, 5, 6, 7\}$, then there is no normal nonabelian Sylow p_i-subgroup, for each i by way of [4] (see Lemma 10). □

Lemma 19 *Assume that* $C_{20} = \Delta(G)$ *for some finite solvable group* G, *where* $|G|$ *is minimal. Then there exists no normal nonabelian Sylow* p_j*-subgroup, for* $j \in \{1, 2\}$.

Proof — First, we will begin by considering the vertex p_1. We will follow the notation from Lemma 11. For $q = p_1$, we have

$$\pi = \{p_2, p_3, p_6\} \qquad \text{and} \qquad \rho = \{p_4, p_5, p_7\}.$$

Setting $\pi_1 = \{p_6\}$ and $\pi_2 = \{p_2, p_3\}$, we choose $\nu = p_3$, which is adjacent to $s = p_4 \in \rho$, shown to be admissible in Lemma 18. We also note that $w = p_7 \in \rho$ is not adjacent to ν, and hence the result

follows from Lemma 11. By symmetry, we can conclude that there is no normal nonabelian Sylow p_2-subgroup. □

Lemma 20 *No proper subgraph with the same vertex set as* C_{20} *occurs as the prime character degree graph of any solvable group.*

Proof — We have established previously that we cannot lose any edges in the complete graphs on $\{p_1, p_2, p_3\}$ and $\{p_4, p_5, p_6, p_7\}$. Thus, we are left with four edges to consider

$$\epsilon(p_1, p_6),\ \epsilon(p_2, p_7),\ \epsilon(p_3, p_4),\ \text{textand}\ \epsilon(p_3, p_5).$$

We will check all subgraphs for which we remove one or more of those edges.

First, recall that p_1 and p_2 are symmetric, and p_6 and p_7 are also symmetric. Thus, it follows that the edges $\epsilon(p_1, p_6)$ and $\epsilon(p_2, p_7)$ are symmetric. Similarly, recall that p_4 and p_5 are symmetric. This means that $\epsilon(p_3, p_4)$ and $\epsilon(p_3, p_5)$ are symmetric as well. Notice that

$$C_{20}[\epsilon(p_1, p_6)] = C_{20}[\epsilon(p_2, p_7)] = C_6$$

and

$$C_{20}[\epsilon(p_3, p_4)] = C_{20}[\epsilon(p_3, p_5)] = \Gamma_{4,3}$$

have both been shown not to occur.

Removing both of the edges $\epsilon(p_1, p_6)$ and $\epsilon(p_2, p_7)$ gives us the graph

$$C_{20}[\epsilon(p_1, p_6), \epsilon(p_2, p_7)] = C_2$$

which does not occur (see Section 7.3). In a similar fashion, losing both edges $\epsilon(p_3, p_4)$ and $\epsilon(p_3, p_5)$ gives us the graph

$$C_{20}[\epsilon(p_3, p_4), \epsilon(p_3, p_5)] = C_4,$$

which again does not occur. Without loss of generality (due to symmetry), we have

$$C_{20}[\epsilon(p_1, p_6), \epsilon(p_3, p_4)] = C_4$$

as well.

If we remove any combination of three of our four viable edges, we are left with the graph C_1, which has been shown not to occur

in Section 7.3. Finally, removing all four edges will give us the disconnected graph in Figure 5, which was also shown not to occur.

Hence, no proper subgraph of C_{20} with the same vertex set occurs as the prime character degree graph of any solvable group. □

Just as before, assuming that there exists some solvable group G with $|G|$ minimal such that $\Delta(G) = C_{20}$, by Lemmas 18 and 19, we again have the conclusion that there are no normal nonabelian Sylow p-subgroups, for all $p \in \rho(G)$. With F as the Fitting subgroup of G, we then note that $\rho(G) = \pi(|G : F|)$. Therefore, we obtain that $\rho(G) = \rho(G/\Phi(G))$. Lemma 20 verifies there is no proper subgraph of C_{20} with the same vertex set which occurs as the prime character degree graph of any solvable group, and therefore, as $|G|$ is minimal, we can conclude that $\Phi(G)$ is trivial. Hence, we can apply [9, Hilfssatz III.4.4] to obtain the existence of a subgroup H of G such that $G = HF$ and $H \cap F = 1$.

Lemma 21 *Assume that* $C_{20} = \Delta(G)$ *for some finite solvable group* G, *where* $|G|$ *is minimal. Then the Fitting subgroup* F *of* G *is minimal normal in* G.

Proof — We follow an argument almost identical to that which was done in Lemma 16. As before, we can follow the assumptions and then can define H as above. Also, set E to be the Fitting subgroup of H.

Next, we proceed by contradiction. Thus, we suppose that there exists some normal subgroup N of G such that $1 < N < F$. We aim to show that no such N can exist.

Notice once more by [9, Satz III.4.5], particularly, that we know there exists some normal subgroup M of G such that $F = N \times M$, and since both N and M are nontrivial, we have that

$$\rho(G/N) \subset \rho(G) \qquad \text{and} \qquad \rho(G/M) \subset \rho(G).$$

For any prime $q \in \rho(G) \setminus \rho(G/N)$, it is known that G/N must have a normal nonabelian Sylow q-subgroup whose class is a formation. Thus, it is required that $q \in \rho(G/M)$. Hence, we get that

$$\rho(G) = \rho(G/N) \cup \rho(G/M).$$

Hence, for this $q \in \rho(G) \setminus \rho(G/N)$, notice that $q \notin \rho(G/F) = \rho(H)$, and so E must contain the Sylow q-subgroup of H. Since $q \in \rho(G)$, however, we must have that q divides $|H|$, and so q divides $|E|$ too.

Then since $\mathrm{cd}(G)$ contains a degree divisible by all the prime divisors of $|EF : F| = |E|$, we get that

$$\rho(G) \setminus (\rho(G/N) \cap \rho(G/M))$$

lies in a complete subgraph of $\Delta(G)$. So, $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M))$ must lie in one of the following sets:

$$A := \{p_1, p_2, p_3\}, \quad B := \{p_1, p_6\}, \quad C := \{p_2, p_7\},$$
$$D := \{p_3, p_4, p_5\}, \quad \text{or} \quad E := \{p_4, p_5, p_6, p_7\}.$$

Assume first that $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M)) \subseteq A$. Following [15], we see that E has a Hall A-subgroup of H. One can then conclude that E is in fact the Hall A-subgroupsince $\mathrm{cd}(G)$ has a degree that is divisible by all the primes dividing $|E|$ and $|E|$ is divisible by no other primes. Next, we can find a character $\chi \in \mathrm{Irr}(G)$ with $p_1p_2p_3$ dividing $\chi(1)$. Letting θ be an irreducible constituent of χ_{FE}, one can see that $\chi(1)/\theta(1)$ divides $|G : FE|$ and that $\chi(1)$ is relatively prime to $|G : FE|$. This forces $\chi_{FE} = \theta$. Since p_1, p_2, and p_3 all divide $\theta(1)$, and the only possible primes that could divide a character in $\mathrm{cd}(G/FE)$ are p_4, p_5, p_6, or p_7, we get that $\mathrm{cd}(G/FE) = \{1\}$ by way of Gallagher's Theorem. Hence, G/FE is abelian. For instance, this then yields $O^{p_4}(G) < G$, a contradiction as p_4 is admissible.

Now, suppose that $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M)) \subseteq B$. This implies that

$$\{p_2, p_3, p_4, p_5, p_7\} \subseteq \rho(G/N) \cap \rho(G/M).$$

Furthermore, since $\rho(G/N)$ and $\rho(G/M)$ are both proper in $\rho(G)$, and $\rho(G) = \rho(G/N) \cup \rho(G/M)$, then without loss of generality, we must have that

$$\rho(G/N) = \{p_1, p_2, p_3, p_4, p_5, p_7\}$$

and

$$\rho(G/M) = \{p_2, p_3, p_4, p_5, p_6, p_7\}.$$

However, this is a contradiction because no graph can occur with the vertex set given to $\rho(G/N)$ as defined above, regardless if it is connected or disconnected. By symmetry, no graph can occur when

$$\rho(G) \setminus (\rho(G/N) \cap \rho(G/M)) \subseteq C,$$

thus eliminating the third case.

Suppose that $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M)) \subseteq D$. This implies that

$$\{p_1, p_2, p_6, p_7\} \subseteq \rho(G/N) \cap \rho(G/M).$$

Since $p_3 \in \rho(G) = \rho(G/N) \cup \rho(G/M)$, then we know that

$$p_3 \in \rho(G/N) \qquad \text{or} \qquad p_3 \in \rho(G/M).$$

Without loss of generality, we can suppose that $p_3 \in \rho(G/N)$. Observe that, since $\rho(G/N)$ is proper in $\rho(G)$, we then have three options for $\rho(G/N)$:

(i) $\{p_1, p_2, p_3, p_6, p_7\}$,

(ii) $\{p_1, p_2, p_3, p_4, p_6, p_7\}$,

(iii) $\{p_1, p_2, p_3, p_5, p_6, p_7\}$.

For (i), we have that

$$\rho(G/N) = \{p_1, p_2, p_3, p_6, p_7\},$$

and one can see that no connected graph with this vertex set occurs as the prime character degree graph of any solvable group. The only possibility is for the disconnected graph to occur, with complete components $\{p_1, p_2, p_3\}$ and $\{p_6, p_7\}$. Then, by [13, Theorem 5.5], we know that G/N has a central Sylow p_4-subgroup or a central Sylow p_5-subgroup, in which case

$$O^{p_4}(G) < G \qquad \text{or} \qquad O^{p_5}(G) < G,$$

respectively. This is our contradiction, since both p_4 and p_5 have been shown to be admissible in Lemma 18, and so $O^{p_4}(G) = G$ and $O^{p_5}(G) = G$. For (ii), we have

$$\rho(G/N) = \{p_1, p_2, p_3, p_4, p_6, p_7\}.$$

No graph can occur with the vertex set given to $\rho(G/N)$ as defined above, regardless if it is connected or disconnected. Recall that the vertices p_4 and p_5 are symmetric. Then for (iii), this argument follows identically to (ii).

Finally, suppose that $\rho(G) \setminus (\rho(G/N) \cap \rho(G/M)) \subseteq E$. This implies that $A \subseteq \rho(G/N) \cap \rho(G/M)$. Similar to the reasoning above concerning the set D, since $p_7 \in \rho(G) = \rho(G/N) \cup \rho(G/M)$, then we know that $p_7 \in \rho(G/N)$ or $p_7 \in \rho(G/M)$. Without loss of generality, we can suppose that $p_7 \in \rho(G/N)$. Since $\rho(G/N)$ is proper in $\rho(G)$, we then have seven options as to what $\rho(G/N)$ can be:

(i) $\{p_1, p_2, p_3, p_7\}$,

(ii) $\{p_1, p_2, p_3, p_4, p_7\}$,

(iii) $\{p_1, p_2, p_3, p_5, p_7\}$,

(iv) $\{p_1, p_2, p_3, p_6, p_7\}$,

(v) $\{p_1, p_2, p_3, p_4, p_5, p_7\}$,

(vi) $\{p_1, p_2, p_3, p_4, p_6, p_7\}$,

(vii) $\{p_1, p_2, p_3, p_5, p_6, p_7\}$.

For (i), we have that $\rho(G/N) = \{p_1, p_2, p_3, p_7\}$, and so

$$\rho(G/M) = \{p_1, p_2, p_3, p_4, p_5, p_6\}.$$

Notice that neither the connected nor disconnected graph with vertex set $\rho(G/M)$ occurs as the prime character degree graph of any solvable group. For (ii), notice that

$$\rho(G/N) = \{p_1, p_2, p_3, p_4, p_7\},$$

and the only graph that can arise is the disconnected graph with complete components A and $\{p_4, p_7\}$. Applying [13, Theorem 5.5], we then get that G/N has a central Sylow p_5-subgroup or a central Sylow p_6-subgroup. This implies that

$$O^{p_5}(G) < G \qquad \text{or} \qquad O^{p_6}(G) < G,$$

which is a contradiction since both p_5 and p_6 are admissible, in which case $O^{p_5}(G) = O^{p_6}(G) = G$. For (iii), the argument follows (ii) by symmetry. For (iv),

$$\rho(G/N) = \{p_1, p_2, p_3, p_6, p_7\}.$$

Once again, the only graph that can occur is the disconnected graph with complete components $\{p_1, p_2, p_3\}$ and $\{p_6, p_7\}$. Therefore, we apply [13, Theorem 5.5] and get that G/N has a central Sylow p_4-subgroup or a central Sylow p_5-subgroup. This gives

$$O^{p_4}(G) < G \qquad \text{and} \qquad O^{p_5}(G) < G,$$

which is a contradiction since both p_4 and p_5 are admissible, meaning that

$$O^{p_4}(G) = O^{p_5}(G) = G.$$

For (v), (vi), and (vii), notice that there is no graph, connected or disconnected, that can occur with each respective vertex set.

Hence, no such N can occur, and we get our desired conclusion that the Fitting subgroup F is minimal normal in G. □

Proposition 22 *The graph C_{20} does not occur as the prime character degree graph for any solvable group.*

Proof — For the sake of contradiction, suppose that $C_{20} = \Delta(G)$ for some finite solvable group G, where $|G|$ is minimal. Notice that the Fitting subgroup F of G is minimal normal in G by Lemma 21. Next, take the vertices

$$a = p_1, \quad b = p_2, \quad c = p_4, \quad \text{and} \quad d = p_5,$$

where we observe that p_4 and p_5 are admissible by Lemma 18. Employing Lemma 12 grants our contradiction, and hence the graph C_{20} is not the prime character degree graph for any solvable group, as desired. □

7.6 Unclassified graphs

As mentioned, there are thirty-three graphs here that are unclassified (see Figures 18 and 19). Setting

$$\mathcal{J} := \{8, 13, 15, 21, 22, 23, 24, 25, x, 52 \mid 27 \leqslant x \leqslant 49\},$$

we then have the graphs C_{19} (Figure 18) and C_j with $j \in \mathcal{J}$ (Figure 19).

The graph C_{19} is worth looking at on its own because, in general, it has a structure that is eerily similar to the bowtie graph that Lewis constructs in [15]. One wonders if C_{19} can be constructed in a similar way.

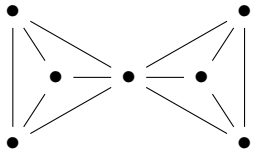


Figure 18: The graph C_{19}

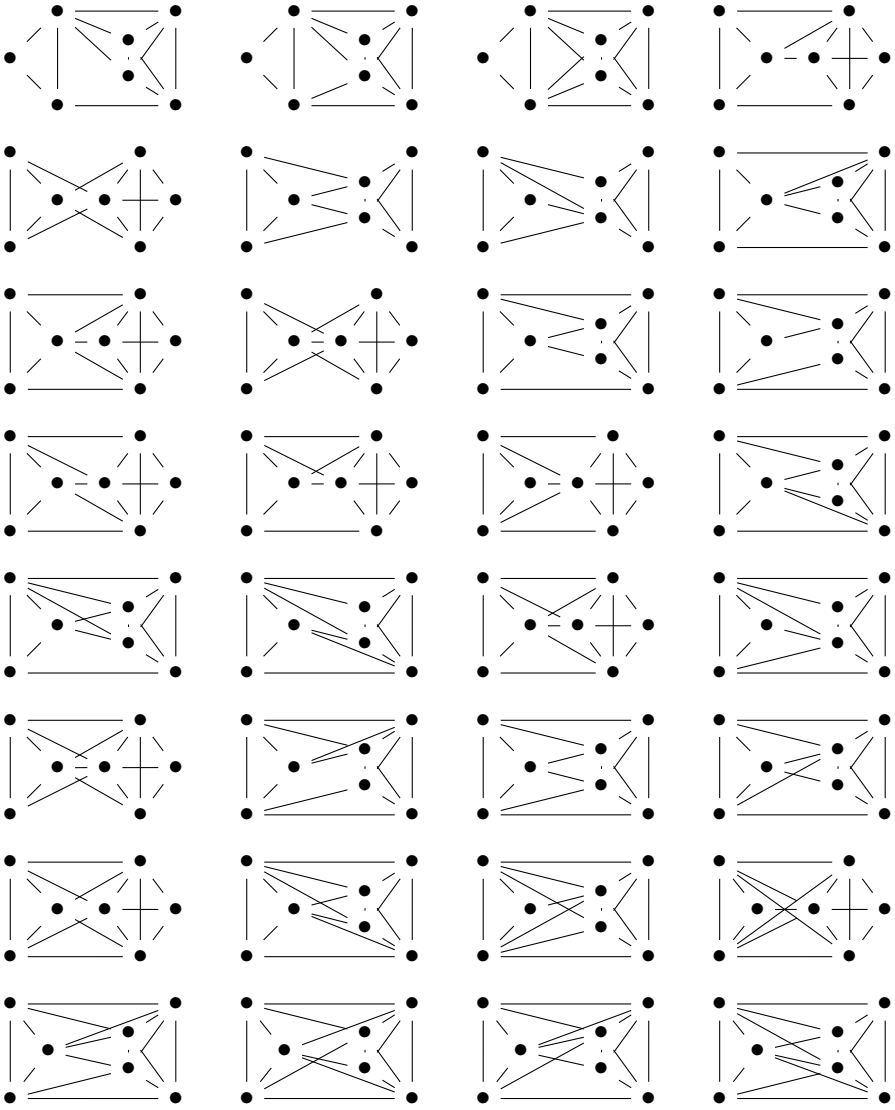
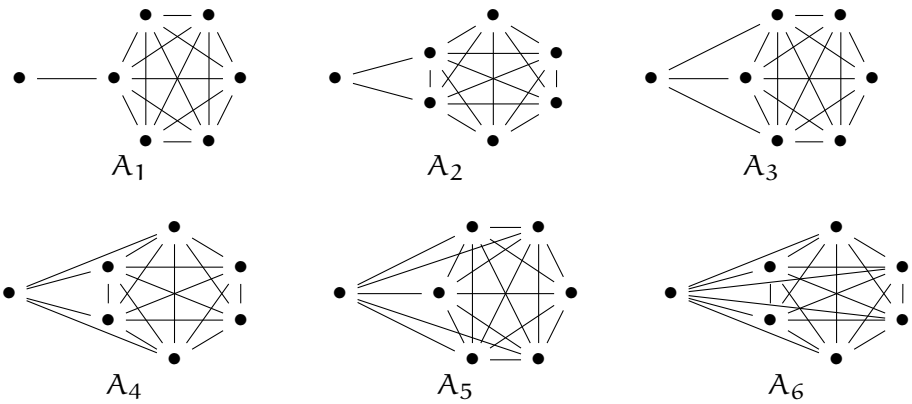


Figure 19: The graphs C_j for $j \in \mathcal{J}$

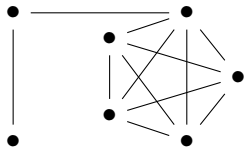
One of the main reasons these graphs went unclassified is that they have a subgraph with six vertices that occurs. For example, the graph C_8 has the lone diameter three graph constructed in [14] as a subgraph. Having subgraphs that occur often leads to issues with our method because we rely on vertices being admissible. This problem then compounds rapidly, because once we know C_8 cannot be tamed, then any graph that has C_8 as a subgraph can also not be tamed (like C_{13} for instance). More work is constantly being done in this area, so one could hope that more graphs can be classified in the future. Papers like [22] however only cooperate with graphs that have an even number of vertices, so their results could not be applied to our remaining graphs. On the other hand, a paper like [25] does relate to some of our remaining graphs, but could not be applied based on our current approach.

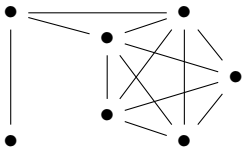
We have also seen that many of these graphs that went unclassified rely upon the nine graphs with six vertices that were not classified in [3]. In particular, we found that one of these graphs with six vertices, denoted $\Sigma_{2,1}^{R}$ from [7], has far-reaching consequences. Hypothetically, if one could conclude that $\Sigma_{2,1}^{R}$ does not occur, then this will imply that C_{21}, C_{22}, and C_{32} all do not occur. One of the issues with this graph $\Sigma_{2,1}^{R}$ is that it does not quite satisfy the technical hypothesis from [5]; further methods will need to be explored.

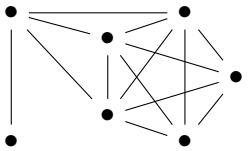
8 Connected graphs with cliques of one and six

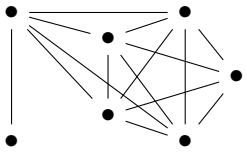


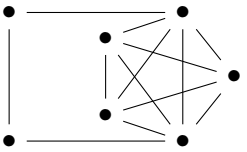
9 Connected graphs with cliques of two and five

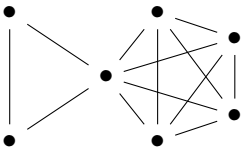

B_1

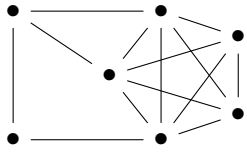

B_2

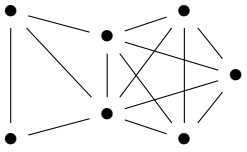

B_3

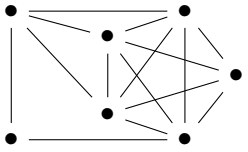

B_4

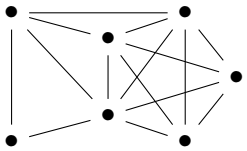

B_5

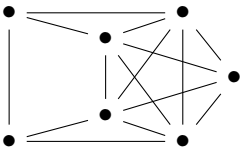

B_6

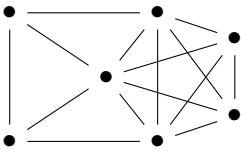

B_7

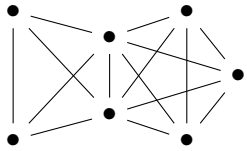

B_8

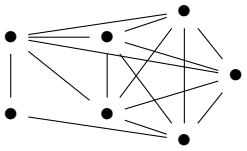

B_9

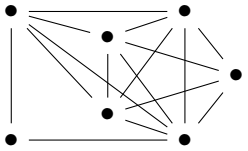

B_{10}

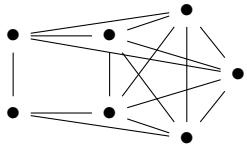

B_{11}

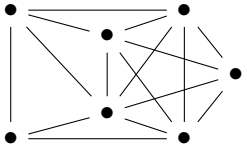

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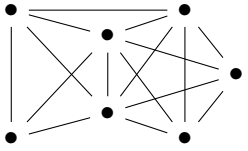

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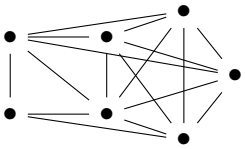

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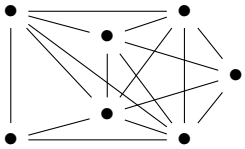

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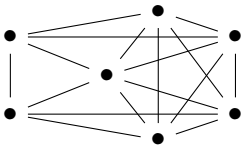

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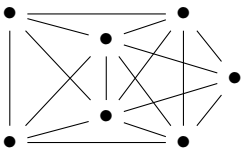

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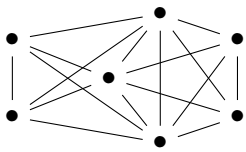

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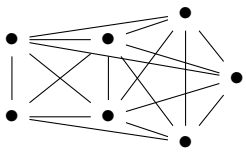

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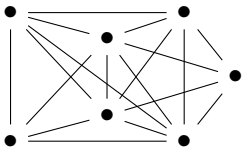

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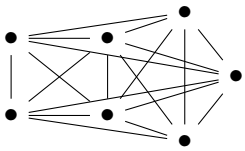

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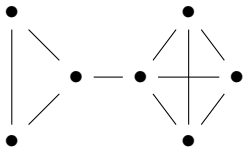

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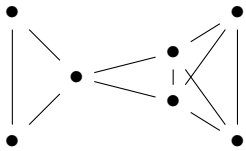

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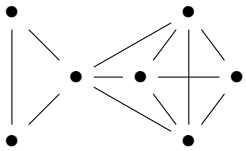

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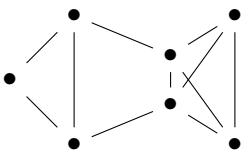

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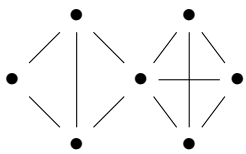
10 Connected graphs with cliques of three and four

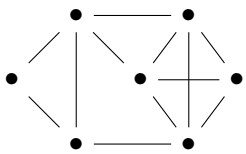

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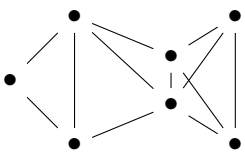

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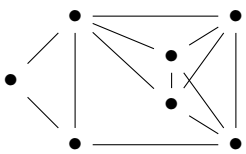

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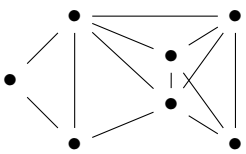

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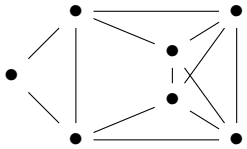

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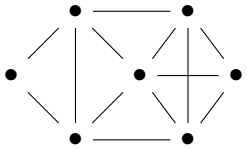

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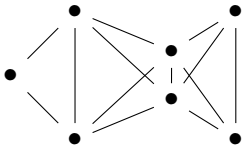

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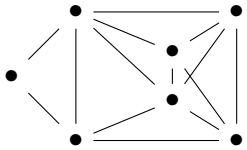

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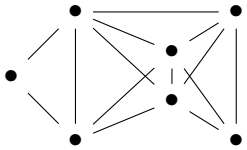

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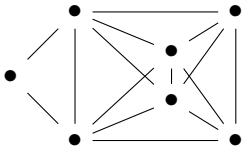

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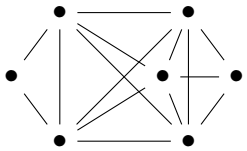

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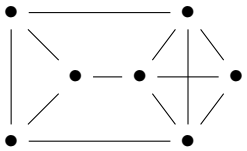

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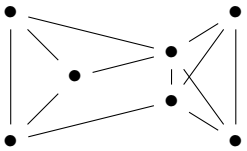

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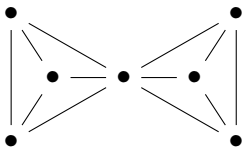

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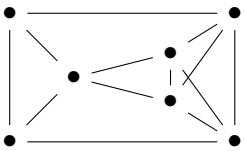

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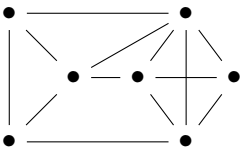

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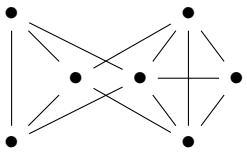

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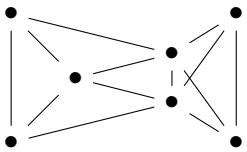

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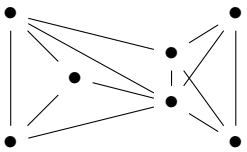

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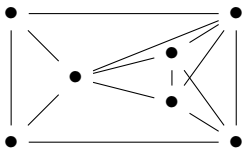

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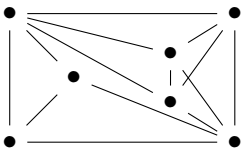

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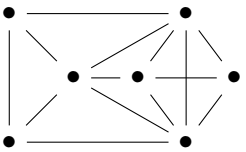

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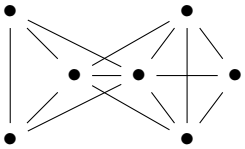

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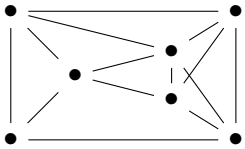

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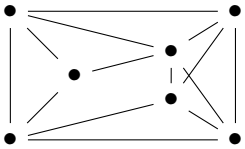

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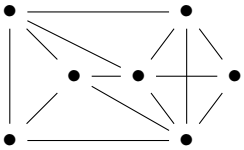

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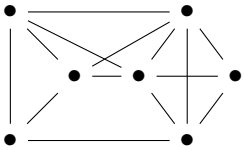

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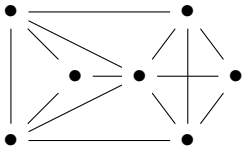

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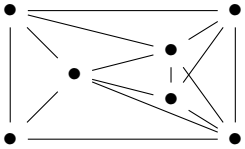

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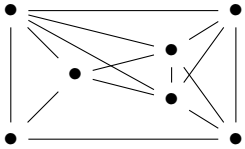

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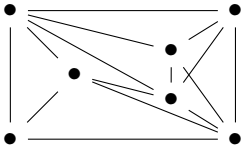

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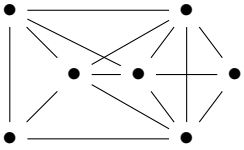

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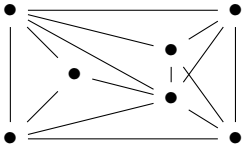

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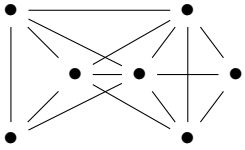

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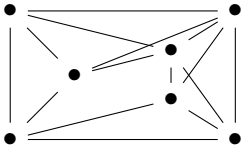

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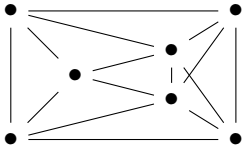

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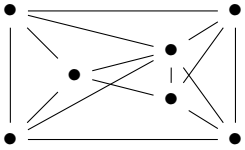

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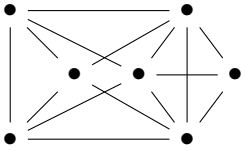

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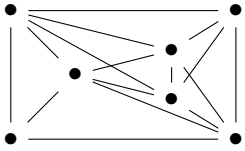

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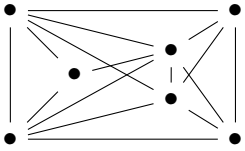

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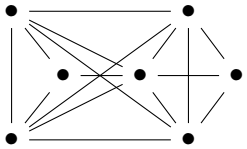

C_{41}


C_{42}

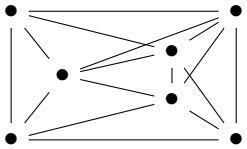

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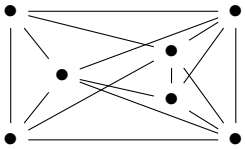

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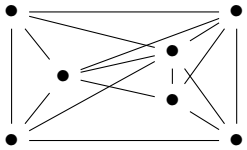
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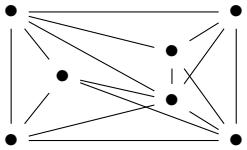
C_{47}



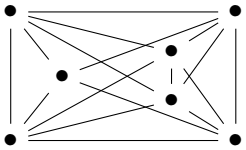
C_{48}



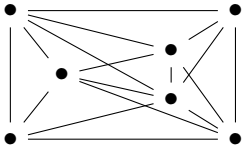
C_{49}



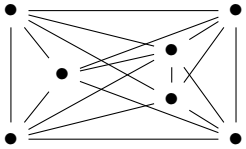
C_{50}



C_{51}



C_{52}



C_{53}

REFERENCES

[1] Z. Akhlaghi – C. Casolo – S. Dolfi – K. Khedri – E. Pacifici: "On the character degree graph of solvable groups", *Proc. Amer. Math. Soc.* 146 (2018), 1505–1513.

[2] M.W. Bissler – J. Laubacher: "Classifying families of character degree graphs of solvable groups", *Int. J. Group Theory* 8 (2019), 37–46.

[3] M.W. Bissler – J. Laubacher – M.L. Lewis: "Classifying character degree graphs with six vertices", *Beitr. Algebra Geom.* 60 (2019), 499–511.

[4] M.W. Bissler – J. Laubacher – M.L. Lewis: "A family of graphs that cannot occur as character degree graphs of solvable groups", *Beitr. Algebra Geom.* 66 (2025), 727–738.

[5] M.W. Bissler – J. Laubacher – C.F. Lyons: "On the absence of a normal nonabelian Sylow subgroup", *Comm. Algebra* 47 (2019), 917–922.

[6] C. Casolo – S. Dolfi – E. Pacifici – L. Sanus: "Groups whose character degree graph has diameter three", *Israel J. Math.* 215 (2016), 523–558.

[7] S. DeGroot – J. Laubacher – M. Medwid: "On prime character degree graphs occurring within a family of graphs (ii)", *Comm. Algebra* 50 (2022), 3307–3319.

[8] C.T. Dugan: "Solvable Groups Whose Character Degree Graphs Have Diameter Three", PhD thesis, *Kent State University*, Kent (2007).

[9] B. Huppert: "Endliche Gruppen I", *Springer*, Berlin (1967).

[10] B. Huppert: "Research in Representation Theory at Mainz (1984–1990)", *Progr. Math.* (1991), 17–36.

[11] I.M. Isaacs: "Character Theory of Finite Groups", Pure Appl. Math., No. 69 *Academic Press*, New York (1976).

[12] J. Laubacher – M. Medwid: "On prime character degree graphs occurring within a family of graphs", *Comm. Algebra* 49 (2021), 1534–1547.

[13] M.L. Lewis: "Solvable groups whose degree graphs have two connected components", *J. Group Theory* 4 (2001), 255–275.

[14] M.L. Lewis: "A solvable group whose character degree graph has diameter 3", *Proc. Amer. Math. Soc.* 130 (2002), 625–630.

[15] M.L. Lewis: "Classifying Character Degree Graphs with 5 Vertices", *de Gruyter*, Berlin (2004).

[16] M.L. Lewis: "An overview of graphs associated with character degrees and conjugacy class sizes in finite groups", *Rocky Mountain J. Math.* 38 (2008), 175–211.

[17] M.L. Lewis – Q. Meng: "Solvable groups whose prime divisor character degree graphs are 1-connected", *Monatsh. Math.* 190 (2019), 541–548.

[18] B.D. McKay – A. Piperno: "Practical graph isomorphism, II", *J. Symbolic Comput.* 60 (2014), 94–112.

[19] P.P. Pálfy: "On the character degree graph of solvable groups. I. Three primes", *Period. Math. Hungar.* 36 (1998), 61–65.

[20] P.P. Pálfy: "On the character degree graph of solvable groups. II. Disconnected graphs", *Studia Sci. Math. Hungar.* 38 (2001), 339–355.

[21] C.B. Sass: "Character degree graphs of solvable groups with diameter three", *J. Group Theory* 19 (2016), 1097–1127.

[22] G. Sivanesan – C. Selvaraj: "Character degree graph of solvable groups with odd degree", (2023); *arXiv*:2305.12324.

[23] G. Sivanesan – C. Selvaraj – T.T Chelvam: "Eulerian character degree graphs of solvable groups", *AKCE Int. J. Graphs. Combin.* 21 (2024), 161–166.

[24] J. Zhang: "On a problem by Huppert", *Beijing Daxue Xuebao Ziran Kexue Ban* 34 (1998), 143–150.

[25] C.P. Morresi Zuccari: "Character graphs with no complete vertices", *J. Algebra* 353 (2012), 22–30.

[26] C.P. Morresi Zuccari: "Regular character degree graphs", *J. Algebra* 411 (2014), 215–224.

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