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$\mathcal{P}$-Characters and Solvability of Finite Groups

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Abstract

The aim of this paper is to show that if all $\mathcal{P}$-characters of a finite group G are normally monomial, then G is solvable. Also, we show that all $\mathcal{P}$-characters of a finite solvable group G are monomial. In the end of paper,we obtain some results about $\mathcal{P}$-characters related to nilpotent groups.

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1 Introduction

Throughout this paper, we only consider finite groups and their complex characters and p is a prime number. Let $\Phi(G)$ be the Frattini subgroup of G,

$$\mathrm{Core}_G(H) = \bigcap_{x \in G} H^x$$

for subgroup H of G, $\mathrm{Irr}(G)$ denote the set of all irreducible characters of G and $\mathrm{cd}(G)$ be the set of all irreducible character degrees of G. For a character χ of G, let $\mathrm{Irr}(\chi)$ be the set of

irreducible constituents of χ. Let N be a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$, the set of

$$I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$$

is the inertia group of θ in G, where θ^g is a conjugate of θ, i.e.,

$$\theta^g(x) = \theta(g^{-1}xg),$$

for all $g \in G$ and all $x \in N$.

Let the group G act on the finite set Ω. The map

$$\begin{aligned}\pi_\Omega : G &\to \mathbb{N}\\ g &\to |\mathrm{Fix}_\Omega(g)| = \left|\{\omega \in \Omega \mid \omega^g = \omega\}\right|\end{aligned}$$

is said the permutation character of the action of G on Ω. A transitive permutation character is the character of a transitive action. Let the subgroup H of G be the stabilizer of an element in a transitive action G on Ω. By [9, Lemma 5.14], $(1_H)^G$ is the permutation character of the action and $\pi_\Omega = (1_H)^G$. Transitive permutation characters provide information about subgroups of G from the character table of G. For example,

$$\frac{(1_H)^G(g).|g^G|}{(1_H)^G(1)} = \left|g^G \cap H\right|$$

where g^G is the G-class of $g \in G$. Therefore, the permutation character $(1_H)^G$ gives information about the conjugacy classes of H, i.e., it is possible to calculate the number of elements of a certain order in H.

An irreducible character of G is called a *$\mathcal{P}$-character* if it is a constituent of $(1_M)^G$ for some maximal subgroup M of G, and $\mathrm{Irr}_{\mathcal{P}}(G)$ denotes the set of all $\mathcal{P}$-characters of G and $\mathrm{cd}_{\mathcal{P}}(G)$ is the set of all $\mathcal{P}$-character degrees of G. Moreover, recall that an irreducible character χ of G is *monomial* if χ induced from a linear character of a subgroup of G, and G is an *M-group*

if all its irreducible characters are monomial. According to a well-known theorem by Taketa, all M-groups are solvable.

Let π be a set of prime numbers, a π-*number* is either 1, or an integer divisible by only those primes in π. A subgroup H of G is called an S_π-*subgroup of* G provided that H is a π-group and $|G : H|$ is divisible by no primes in π. We generally write π' to denote the complement of the set π in the set of all prime numbers, and thus a π'-group is one for which no prime in π divides the group order. A group is called π-*separated* if each of its chief factors is either a π-group or a π'-group. Let us observe that, in [14], the authors proved that in a π-group G with an S_π-subgroup H, every irreducible constituent χ of the permutation character $(1_H)^G$ of group G, whose order is a π-number, is monomial.

Also, a group G is called *normally monomial* (or nM-*group* for short) if every irreducible character of G is normally monomial, i.e., is induced from a linear character of a normal subgroup of G. Moreover, in [7], Guan proved that all metabelian groups are nM-groups. Obviously, nM-groups are M-groups, but the opposite is not true. For example, the symmetric group S_4 is an M-group, but not an nM-group.

A nonlinear character $\chi \in \mathrm{Irr}(G)$ is called a *multiply imprimitive character* (or *m.i character* for short) if there exists a proper subgroup H of G and an irreducible character $\lambda \in \mathrm{Irr}(H)$ such that $\lambda^G = m\chi$ for some non negative integer m. Moreover, a group G is said to be an *MI-group* if every nonlinear irreducible character of G is an m.i character. In [11], the authors proved that all MI-groups are solvable.

A group G is said to be *monolithic* if it has a unique minimal normal subgroup. A character $\chi \in \mathrm{Irr}(G)$ is called *monolithic* if $G/\ker\chi$ is a monolithic. If every irreducible monolithic character of G is monomial, then such group G is defined to be a *weak M-group*. Berkovich and Zhmud in [2] proved that all weak M-groups are solvable.

Some problems appear in [13] grouped as Question 3.9, Ques-

tion 3.10, and Question 3.11. Let us explicitly recall two of them.

Question 1.1 (see [13], Question 3.9, and [10], Problem 20.80) *Suppose that all $\mathcal{P}$-characters of G are monomial. Is G solvable?*

Question 1.2 (see [13], Question 3.10) *Is it possible to determine the $\mathcal{P}$-characters of a finite group from its character table?*

In this paper, in relation to Question 1.1, we demonstrate that if all $\mathcal{P}$-characters of G are normally monomial, then G is solvable. In fact, we simplify the condition of Taketa's theorem which states that an M-group is solvable. Furthermore, we give a partial answer to Question 1.2 for the case of nilpotent finite groups.

2 Preliminary and notations

In this section, we present the basic definitions and known results.

Lemma 2.1 (see [9], Lemma 5.11) *Let θ be a character of $H \subseteq G$. Then*

$$\ker(\theta^G) = \bigcap_{g \in G} (\ker \theta)^g$$

Proposition 2.2 (see [13], Proposition 2.1) *Let χ be a nonprincipal $\mathcal{P}$-character of G with respect to a maximal subgroup M. Then*

$$\ker \chi = \mathrm{Core}_G(M).$$

Definition 2.3 Let H be a subgroup of G. Then we say that a subgroup K of G is a *supplement to H in G*, if $G = HK$.

Theorem 2.4 (see [1], Theorem 1.3) *A finite group G is solvable if and only if every maximal subgroup M of G admits a supplement whose commutator subgroup is contained in M.*

Definition 2.5 For subgroup H of G, the transitive permutation character $(1_H)^G$ is said to be *multiplicity-free* if all of its irreducible constituents are distinct.

Theorem 2.6 (see [5], Theorem 1.19) *Permutation characters of maximal subgroups of solvable groups are multiplicity-free.*

Definition 2.7 Let H and N be two groups. We denote the semidirect product G of N by H with $G = N \ltimes H$. Let L/K be a chief factor of G. Then L/K is called *complemented*, if there exists a subgroup M/K of G/K such that

$$G/K = M/K \ltimes L/K.$$

Lemma 2.8 (see [13], Lemma 2.2) *Let G be a p-solvable group.*

(a) *If L/E is a complemented p-chief factor of G, then there exists a maximal subgroup M of G such that*

$$G = ML \qquad \textit{and} \qquad M \cap L = E.$$

(b) *If M is a maximal subgroup of G with index a power of p, then there exists a complemented p-chief factor L/E of G such that*

$$G = ML \qquad \textit{and} \qquad M \cap L = E.$$

Furthermore, such a chief factor L/E can be chosen so that

$$G = ML \qquad \textit{and} \qquad M \cap L = E = \mathrm{Core}_G(M),$$

in which case L/E is the unique minimal normal subgroup of G/E, and all maximal subgroups of G/E not containing L/E are conjugate.

Proposition 2.9 (see [13], Proposition 2.3) *Let G be a p-solvable group. Suppose that $G = ML$ and $M \cap L = E$, where M is maximal in G and L/E is a p-chief factor of G. Then we have:*

(a) *The permutation character $(1_M)^G$ is a sum of distinct irreducible characters of G.*

(b) *The map* $\Theta : \chi \to \mathrm{Irr}(\chi_L)$ *is a bijection of* $\mathrm{Irr}(1_M)^G$ *onto the set of* G*-orbits of* $\mathrm{Irr}(L/E)$. *Furthermore, if* $\lambda \in \mathrm{Irr}(\chi_L)$, *then* λ *extends to* $I_G(\lambda)$ *and* $\chi(1) = |G : I_G(\lambda)|$.

(c) *For all* $\chi \in \mathrm{Irr}(1_M)^G$, χ *is monomial, monolithic, and*

$$\Phi(G/\ker\chi) = 1.$$

Corollary 2.10 (see [13], Corollary 2.6) *A finite group* G *is nilpotent if and only if* $\mathrm{cd}_{\mathcal{P}}(G) = \{1\}$.

Theorem 2.11 (see [9], Theorem 5.18) *Let* H *be a subset of a finite group* G *and* $\chi = (1_H)^G$. *Then the following hold*:

(a) $\chi(1) \mid |G|$.

(b) $[\chi, \psi] \leqslant \psi(1)$, *for all* $\psi \in \mathrm{Irr}(G)$.

(c) $[\chi, 1_G] = 1$.

(d) $\chi(g)$ *is a nonnegative rational integer, for all* $g \in G$.

(e) $\chi(g) \leqslant \chi(g^m)$, *for all* $g \in G$ *and integer* m.

(f) $\chi(g) = 0$ *if* $|g| \nmid |G|/\chi(1)$.

(g) $\chi(g)\left|g^G\right|/\chi(1)$ *is integral, for all* $g \in G$.

Lemma 2.12 (see [3], Lemma 5) *Let* N *be a minimal normal subgroup of* G, *so that*

$$N = S_1 \times \ldots \times S_t,$$

where $S_i \simeq S$, *a nonabelian simple group. Let* A *be the automorphism group of* S. *If* $\sigma \in \mathrm{Irr}(S)$ *extends to* A, *then*

$$\sigma \times \ldots \times \sigma \in \mathrm{Irr}(N)$$

extends to G.

Corollary 2.13 (see [9], Problem 5.8) *Let* χ *be a monomial character of* G *and suppose* $K \subseteq G$ *with* $\chi_K \in \mathrm{Irr}(K)$. *Then* χ_K *is monomial character of* K.

The following result is adapted from Theorem 1 in [14].

Corollary 2.14 *If every $\mathcal{P}$-character of G is monomial and N is a normal subgroup of G, then every $\mathcal{P}$-character of G/N is monomial.*

Proof — Let $\widehat{\chi} \in \mathrm{Irr}(G/N)$ be a $\mathcal{P}$-character. Then there exists a maximal subgroup M/N of G/N such that $[\widehat{\chi}_{M/N}, 1_{M/N}] \neq 0$. Let $\chi \in \mathrm{Irr}(G)$ be a lift character of $\widehat{\chi}$, then

$$0 \neq [\widehat{\chi}_{M/N}, 1_{M/N}] = \frac{1}{|M/N|} \sum_{Nx \in M/N} \widehat{\chi}_{M/N}(Nx)$$

$$= \frac{1}{|M|}\Big(|N| \sum_{Nx \in M/N} \widehat{\chi}_{M/N}(Nx)\Big) = \frac{1}{|M|} \sum_{x \in M} \chi_M(x) = [\chi_M, 1_M]$$

It follows that χ is a $\mathcal{P}$-character with respect to maximal subgroup M. Hence, by hypothesis, there exists a subgroup H of G and a linear character $\lambda \in \mathrm{Irr}(H)$ such that $\chi = \lambda^G$. Then, by Lemma 2.1,

$$N \subseteq \ker \chi = \ker \lambda^G = \bigcap_{g \in G} (\ker \lambda)^g \subseteq \ker \lambda \subseteq H$$

Hence, $\widehat{\lambda} \in \mathrm{Irr}(H/N)$ and λ is a lift character of $\widehat{\lambda}$. So

$$(\widehat{\lambda})^{G/N}(Nx) = \frac{1}{|H/N|} \sum_{Ng \in G/N} (\widehat{\lambda})^\circ(Ng^{-1}NxNg)$$

$$= \frac{1}{|H|} \sum_{g \in G} \lambda^\circ(g^{-1}xg) = \lambda^G(x) = \chi(x) = \widehat{\chi}(Nx)$$

Therefore, $\widehat{\chi}$ is monomial and the proof is completed. □

3 Proof of the main result

We are now ready to prove the first main result of our paper.

Theorem 3.1 *If all $\mathcal{P}$-characters of G are normally monomial, then G is solvable.*

Proof — Suppose that M is a maximal subgroup of G and that $\chi \in \mathrm{Irr}(G)$ is a nonprincipal $\mathcal{P}$-character with respect to M. Then there exists a normal subgroup H of G and a linear character $\lambda \in \mathrm{Irr}(H)$ such that $\chi = \lambda^G$. This implies that

$$|G : H| = \chi(1) < (1_M)^G(1) = |G : M|.$$

Hence, $|M| < |H|$ and $H \nsubseteq M$. Since M is a maximal subgroup of G , we have $G = MH$. On the other hand, λ is a linear character, then $\ker\lambda$ is proper subgroup of H. Hence,

$$M \subseteq M\ker\lambda \subsetneqq MH = G,$$

from which it follows that $M = M\ker\lambda$ and so $\ker\lambda \subseteq M$. Therefore, $H' \subseteq \ker\lambda \subseteq M$. Then, by Theorem 2.4, we conclude that G is solvable. □

Proposition 3.2 *Let G be a solvable group. Then all its $\mathcal{P}$-characters are monomial.*

Proof — Let G be solvable. Thus, for every maximal subgroup M of G, there exists a prime number p and a positive integer n such that $|G : M| = p^n$, and G is p-solvable. Then, by Lemma 2.8, and Proposition 2.9, all $\mathcal{P}$-characters with respect to M are monomial. □

Lemma 3.3 *Let G be a non-abelian simple group. Then G has a nonlinear $\mathcal{P}$-character which is not monomial.*

Proof — We know the principal character 1_G is the only linear character of G. We prove this Lemma by contrary. Let all nonlinear $\mathcal{P}$-characters of G be monomial.

Suppose that M is maximal subgroup of G with the largest order and χ is a $\mathcal{P}$-character with respect to M. By Theorem 2.11(c), we have that $\left[(1_M)^G,\ 1_G\right] = 1$, hence, χ is a nonlinear character. Then there exists a proper subgroup H of G and a linear

character $\lambda \in \mathrm{Irr}(H)$ such that $\chi = \lambda^G$. Thus, $\chi(1) = |G : H|$ and

$$\chi(1) < (1_M)^G(1) = |G : M|.$$

Therefore, $|M| < |H|$, which is contradiction with the choice of M and the proof is completed. □

Groups	Maximal subgroups	$\mathcal{P}$-character degrees
M_{11}	$A_6.2$	10
M_{12}	$L_2(11)$	45
J_1	$L_2(11)$	56
M_{22}	$L_3(4)$	21
J_2	$U_3(3)$	36
M_{23}	M_{22}	22
${}^2F_4(2)'$	$2.[2^8] : 5 : 4$	78
HS	M_{22}	22
J_3	$L_2(16) : 2$	324
M_{24}	M_{23}	23
McL	$U_4(3)$	22
He	$S_4(4) : 2$	680
Ru	${}^2F_4(2)$	783
Suz	$2^{1+6}.U_4(2)$	143
ON	$L_3(7).2$	10944
Co_3	HS	23
Co_2	McL	23
Fi_{22}	$2.U_6(2)$	429
HN	A_{12}	760
Ly	$G_2(5)$	45694
Th	${}^3D_4(2) : 3$	4123
Fi_{23}	$2.Fi_{22}$	782
Co_1	Co_2	299
J_4	$2^{11} : M_{24}$	1776888
Fi'_{24}	$O^-_{10}(2)$	8671
B	Fi_{23}	4371
M	2.B	196883

Table 1

Remark 3.4 In Table 1, for every sporadic simple group or Tits group S, we have considered the maximal subgroup M of S in the second column. Suppose that $\theta \in \mathrm{Irr}(S)$ has the degree given in the third column. The character θ is selected to satisfy the following properties:

(i) θ is extendible to $\mathrm{Aut}(S)$.

(ii) θ is a $\mathcal{P}$-character with respect to M.

(iii) θ is nonmonomial.

Lemma 3.5 *Let S be a sporadic simple group or Tits group and suppose that G is a almost simple group with socle S. Then, G has a nonmonomial $\mathcal{P}$-character.*

Proof — Let θ be as in Table 1. By Lemma 2.12, θ is extendible to $\chi \in \mathrm{Irr}(G)$, i.e., $\chi_S = \theta$. Then, by Corollary 2.13, χ is not monomial. We show that χ is a $\mathcal{P}$-character.

Let θ be a $\mathcal{P}$-character with respect to maximal subgroup K of S, then, $[\theta_K, 1_K] \neq 0$. Suppose that M is a maximal subgroup of G containing K. Then for every $x \in G$ we have

$$|x^G \cap K| \leqslant |x^G \cap M|.$$

We have

$$\frac{(1_K)^G(x) \cdot |x^G|}{|G : K|} = |x^G \cap K| \leqslant |x^G \cap M| = \frac{(1_M)^G(x) \cdot |x^G|}{|G : M|}$$

Therefore, $(1_K)^G(x) \leqslant |M : K|(1_M)^G(x)$, for every $x \in G$. Hence,

$$\begin{aligned} 0 \neq [\theta_K, 1_K] &= [(\chi_S)_K, 1_K] = [\chi_K, 1_K] = \left[\chi, (1_K)^G\right] \\ &= \frac{1}{|G|}\sum_{x\in G} \chi(x)(1_K)^G(x^{-1}) \leqslant \frac{|M : K|}{|G|}\sum_{x\in G} \chi(x)(1_M)^G(x^{-1}) \\ &= |M : K|\left[\chi, (1_M)^G\right] \end{aligned}$$

Then χ is a $\mathcal{P}$-character with respect to M. □

In the remainder of this paper, we bring two results related to $\mathcal{P}$-characters in the nilpotent groups.

Proposition 3.6 *Let* M *be a maximal subgroup of nilpotent group* G. *Then the number of* $\mathcal{P}$*-characters with respect to* M *is equal to* $|G : M|$.

Proof — Let $\chi_1, \chi_2, \ldots, \chi_k$ be all $\mathcal{P}$-characters with respect to M. By Theorem 2.6, we have $(1_M)^G = \chi_1 + \chi_2 + \ldots + \chi_k$. Then

$$|G : M| = (1_M)^G(1) = \chi_1(1) + \chi_2(1) + \ldots + \chi_k(1).$$

By Corollary 2.10, $\chi_i(1) = 1$, therefore $|G : M| = k$. □

Proposition 3.7 *Let* G *be a nilpotent group. Then the only characters whose kernels are maximal subgroups of* G *are* $\mathcal{P}$*-character.*

Proof — Let $\chi \in \mathrm{Irr}(G)$ and $M = \ker \chi$ be a maximal subgroup of G, then $\left[\chi_M, 1_M\right] \neq 0$ and so χ is a $\mathcal{P}$-character with respect to M.

If χ is a $\mathcal{P}$-character with respect to M, then by Lemma 2.2,

$$\ker \chi = \mathrm{Core}_G(M) = M.$$

Therefore $\ker \chi$ is maximal subgroup of G. □

Conclusions To summarize, in this paper we have presented a new criterion for the solvability of finite groups by simplifying the condition of Taketa's theorem. Taketa showed that all M-groups are solvable. In Theorem 3.1, we established that a group is solvable if all of its $\mathcal{P}$-characters are normally monomial.

In [13], Qian posed the question of whether the $\mathcal{P}$-characters of a finite group can be determined from its character table. In Proposition 3.7, we proved that $\mathcal{P}$-characters of the nilpotent groups can be determined from their character table.

Also, in Proposition 3.6, we calculated the number of $\mathcal{P}$-characters with respect to each maximal subgroup of a nilpotent group.

REFERENCES

[1] B. Baumeister: "A characterization of the finite soluble groups", *Arch. Math.* (*Basel*) 72 (1999), 167–176.

[2] Y. Berkovich – E.M. Zhmud: "On monolithic character", *Houston J. Math.* 22 (1996), 263–278.

[3] M. Bianchi – D. Chillag – M.L. Lewis – E. Pacifici: "Character degree graphs that are complete graphs", *Proc. Amer. Math. Soc.* 135 (2007), 671–676.

[4] T. Breuer – P. Götz: "Finding possible permutation characters", *J. Symbolic Comput.* 26 (1998), 343–354.

[5] T. Chimpinde: "Maximal Subgroups and Character Theory", M.Sc. Thesis, *Central European University*, Vienna (2015).

[6] J.H. Conway – R.T. Curtis – S.P. Norton – R.A. Parker – R.A. Wilson: "ATLAS of Finite Groups", *Oxford University Press*, Eynsham (1985).

[7] H.G.A. How: "Some classes of monomial groups", Ph.D. Thesis, *The Australian National University*, Canberra (1980).

[8] L. He – G. Zhu: "Non solvable normal subgroups and irreducible character degrees", *J. Algebra* 372 (2012), 68–84.

[9] I.M. Isaacs: "Character Theory of Finite Groups", *Academic Press*, New York (1976).

[10] E.I. Khukhro – V.D. Mazurov: "Unsolved Problems in Group Theory. The Kourovka Notebook" (2024); *arXiv*:1401.0300v29.

[11] T. Le – J. Moori – H.P. Tong-Viet: "On a generalization of M-group", *J. Algebra* 374 (2013), 27–41.

[12] G. Qian: "Two results related to the solvability of M-groups", *J. Algebra* 323 (2010), 3134–3141.

[13] G. Qian – Y. Yang: "Permutation characters in finite solvable groups", *Comm. Algebra* 46 (2018), 167–175.

[14] A.V. Romanovskiĭ – A.A. Yadchenko: "Monomial characters and normal subgroups of finite groups", *Ukrain. Mat. Zh.* 43 (1991), 991–996.

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