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Introducing n-Magic Groups and Characterizing 3-Magic Finitely-Generated Abelian Groups

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Abstract

In this paper, we define an n-magic square in a group to be an $n \times n$ array of group elements whose rows, columns, and diagonals have the same product. This definition is similar to the idea of magic squares in the integers. Groups that have an n-magic square are said to be n-*magic*. We begin with some preliminary results and focus much of our attention on 3-magic groups. Through a series of propositions, we ultimately prove a characterization theorem for 3-magic finitely-generated abelian groups. We then discuss some additional results about non-abelian groups as well as n-magic groups where $n > 3$.

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1 Previous work

Magic squares are one of the most well-known topics in recreational mathematics. The notion initially referred to a square array of distinct positive integers in which every row, column, and diagonal sum to the same number. It is well known that magic squares exist for every size larger than a 2×2 array. In fact, many examples of these were found in antiquity.

This paper originated in our work on the problems posed in the Numberphile video "The Parker Square – Numberphile" (see [2]),

and we began considering squares in settings other than the integers. Magic squares over finite fields are discussed in [1]. An investigation of magic squares in the realm of group theory was initiated by [3], which addressed abelian groups of order n^2 and referred to "magic groups" as the automorphism group on a magic square. This greatly differs from our notion of an n-magic group. Our research independently developed many of the same conclusions as this paper, but our paper generalizes these results to determine the existence of 3×3 magic squares for every finite abelian group, not just those of order n^2.

2 Definitions, notation, and organization

We begin this section with the fundamental definition of this paper.

Definition 2.1 We say that group G is n-*magic* if there is an $n \times n$ magic square with entries from G. That is, there exist distinct

$$g_{1,1}, g_{1,2}, \dots, g_{1,n}, g_{2,1}, g_{2,2}, \dots, g_{2,n}, \dots, g_{n,1}, g_{n,2}, \dots, g_{n,n} \in G$$

such that the products

$$\begin{aligned}
\text{rows: } & \prod_{j=1}^{n} g_{i,j},\ 1 \leqslant i \leqslant n \\
\text{columns: } & \prod_{i=1}^{n} g_{i,j},\ 1 \leqslant j \leqslant n \\
\text{main diagonal: } & \prod_{i=1}^{n} g_{i,i} \\
\text{anti-diagonal: } & \prod_{i=1}^{n} g_{n+1-i,i}
\end{aligned}$$

are all equal. We call this common product the *magic product* of the magic square.

It is also worth noting that natural variants of "magic" objects can be defined similarly, such as for semi-groups or for magmas. We leave those for a future investigation.

In the remainder of this paper, we first provide some preliminary results on 3-magic groups. We then prove a series of propositions that act as lemmas for the proof of the characterization of 3-magic finitely-generated abelian groups, which is then presented. Finally, we provide some results related to nonabelian groups and discuss some ideas for future work on this topic.

3 Results

The following two results for a group G are immediate. The first provides a lower bound on the order of a group for it to be n-magic. The second tells us that if a group is n-magic, then any group containing it as a subgroup is also n-magic.

Proposition 3.1 *If a group* G *is* n*-magic, then* $|G| \geqslant n^2$.

Proposition 3.2 *If* $H \leqslant G$ *and* H *is* n*-magic, then* G *is* n*-magic.*

We can quickly see the following.

Theorem 3.3 *No groups are 2-magic.*

Proof — Suppose G is 2-magic, and let

$$\begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix}$$

be a magic square. Then $g_1 g_2 = g_1 g_3$, and through left multiplication by g_1^{-1}, we obtain that $g_2 = g_3$. This contradicts the assumption that the entries in a magic square are distinct. □

Given this, we can begin investigating which groups are 3-magic. By Proposition 3.1, we have that all groups of order 8 or less cannot be 3-magic. We wish to find "fundamental" classes of magic groups for which we can then apply Proposition 3.2 to find a much larger class of groups. To this end, we endeavour to characterize 3-magic finitely-generated abelian groups. The next four propositions are our first steps in this direction.

Proposition 3.4 *All non-torsion groups are* n*-magic, for all* $n > 2$.

Proof — It is well known that $\mathbb{Z}$ is n-magic, for all $n > 2$. The result then follows from every non-torsion group having a subgroup isomorphic to $\mathbb{Z}$. □

This shows us that all infinite, finitely-generated abelian groups are 3-magic.

Proposition 3.5 *Cyclic group C_k is n-magic if and only if $k \geqslant n^2$, for all $n > 2$.*

Proof — Necessity follows immediately from Proposition 3.1.

To prove sufficiency, let $\left[(a_{ij})\right]$ be an n-magic square in $\mathbb{Z}$ with entries from 0 to $n^2 - 1$. Then for $C_k = \langle x \rangle$ we have that $[(x^{a_{ij}})]$ forms an n-magic square in C_k. The fact that $k \geqslant n^2$ guarantees these entries are distinct. □

To see an example of this, let $k \geqslant 25$. A 5-magic square in $\mathbb{Z}$ formed using the Siamese method starting at 0 is given by

$$\begin{bmatrix} 16 & 23 & 0 & 7 & 14 \\ 22 & 4 & 6 & 13 & 15 \\ 3 & 5 & 12 & 19 & 21 \\ 9 & 11 & 18 & 20 & 2 \\ 10 & 17 & 24 & 1 & 8 \end{bmatrix}.$$

Utilizing this, we see that if $C_k = \langle x \rangle$, then

$$\begin{bmatrix} x^{16} & x^{23} & 1 & x^{7} & x^{14} \\ x^{22} & x^{4} & x^{6} & x^{13} & x^{15} \\ x^{3} & x^{5} & x^{12} & x^{19} & x^{21} \\ x^{9} & x^{11} & x^{18} & x^{20} & x^{2} \\ x^{10} & x^{17} & x^{24} & x & x^{8} \end{bmatrix}$$

forms a 5-magic square. The fact that $k \geqslant 25$ guarantees these entries are distinct.

Proposition 3.6 *Groups C_n^2 and C_n^3 are 3-magic if and only if $n \geqslant 3$.*

Proof — Necessity follows immediately from Proposition 3.1.

For sufficiency, given C_n^2, we must cover a few cases. First, note that $C_n \leqslant C_n^2$, for all n so that, by Proposition 3.2, C_n^2 is 3-magic, for

all $n \geqslant 9$. Next, for $C^2_{2k+1} = \langle x, y \rangle$, note that

$$\begin{bmatrix} x^k & y^{k+1} & x^{k+1}y^k \\ xy^k & 1 & x^{2k}y^{k+1} \\ x^k y & y^k & x^{k+1} \end{bmatrix}$$

forms a magic square with a common product of 1. Thus, C^2_3, C^2_5, and C^2_7 are 3-magic. Furthermore, since $C^2_3 \leqslant C^2_6$, by Proposition 3.2, it follows that C^2_6 is also 3-magic. Lastly, for $C^2_4 = \langle x, y \rangle$,

$$\begin{bmatrix} x & y^3 & x^3y \\ x^2y & 1 & x^2y^3 \\ xy^3 & y & x^3 \end{bmatrix}$$

forms a magic square with a common product of 1. Thus, C^2_4 is 3-magic. Since $C^2_4 \leqslant C^2_8$, by Proposition 3.2, C^2_8 is also 3-magic.

Finally, since $C^2_n \leqslant C^3_n$, for all n, Proposition 3.2 provides sufficiency for C^3_n. □

Proposition 3.7 *The group $C_n \times C_{n^2}$ is 3-magic if and only if $n \geqslant 3$.*

Proof — Necessity again follows from Proposition 3.1. Sufficiency follows from Proposition 3.2, after recognizing that

$$C_{n^2} \leqslant C_n \times C_{n^2}.$$

Therefore the claim is proved. □

To proceed further, we need to define a "normalized" magic square.

Definition 3.8 A *normalized n-magic square* in a group G is an n-magic square in G such that one of the entries is the identity element of G. When n is odd, if the center entry of the magic square is the identity, we call it a *centered normalized n-magic square*.

It can be shown that in abelian groups we can convert any magic square to a normalized magic square with the identity in any position that we choose.

Proposition 3.9 *Let G be an abelian group. Then G is n-magic if and only if, for all $i, j \in \{1, 2, \ldots, n\}$, there exists an n-magic square in G whose $g_{i,j}$ entry is the identity. Therefore, an abelian group is n-magic if and only if it admits a normalized magic square.*

Proof — Sufficiency is clear. To prove necessity, choose an n-magic square in G with magic product s. Let $i, j \in \{1, 2, \ldots, n\}$ and multiply each entry by $g_{i,j}^{-1}$. The resulting square has 1 in the i, j position, and each row, column, and diagonal has a product of $g_{i,j}^{-n}s$. We can guarantee that all these entries are unique since left multiplication by a group element is an automorphism. □

The next three results answer the remaining question as to whether we can find an even stronger biconditional that can be used to quickly show that an abelian group is not 3-magic.

Theorem 3.10 *The elements of any 3-magic square in an abelian group can be generated by three elements.*

Proof — Sufficiency is immediate. The proof for necessity is adapted from [4] and translated into the language of groups. If G is 3-magic, let

$$\begin{bmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \end{bmatrix}$$

be a magic square in G with magic product s. We then have

$$g_1g_5g_9 = s,\ g_2g_5g_8 = s,\ g_4g_5g_6 = s,\ g_7g_5g_3 = s,$$
$$g_1g_4g_7 = s,\ g_3g_6g_9 = s,\ g_1g_2g_3 = s,\ \text{and}\ \ g_7g_8g_9 = s.$$

Multiplying the first four equations gives us

$$(g_1g_2g_3)(g_4g_5g_6)(g_7g_8g_9)g_5^3 = s^4,\ s^3g_5^3 = s^4,\ \text{and}\ g_5^3 = s$$

Then, define $a := g_1g_5^{-1}$, $b := g_3g_5^{-1}$, and $c := g_5$. Now note that since $g_1g_2g_3 = g_5^3$ we know

$$g_2 = g_1^{-1}g_3^{-1}g_5^3,$$

since $g_7g_5g_3 = g_5^3$ we have

$$g_7 = g_3^{-1}g_5^2,$$

and since $g_1 g_5 g_9 = g_5^3$ we can state that

$$g_9 = g_1^{-1} g_5^2.$$

Combining these statements allows us to say that since $g_1 g_4 g_7 = g_5^3$ we know

$$g_4 = g_1^{-1} g_5^3 g_7^{-1} = g_1^{-1} g_5^3 (g_3^{-1} g_5^2)^{-1} = g_1^{-1} g_3^{-1} g_5,$$

since $g_2 g_5 g_8 = g_5^3$ we have

$$g_8 = g_2^{-1} g_5^2 = (g_1^{-1} g_3^{-1} g_5^3)^{-1} g_5^2 = g_1 g_3 g_5^{-1},$$

and since $g_3 g_6 g_9 = g_5^3$ we have

$$g_6 = g_3^{-1} g_5^3 g_9^{-1} = g_3^{-1} g_5^3 (g_1^{-1} g_5^2)^{-1} = g_1 g_3^{-1} g_5.$$

Combining these lets us see that

$$\begin{bmatrix} ac & a^{-1}b^{-1}c & bc \\ a^{-1}bc & c & ab^{-1}c \\ b^{-1}c & abc & a^{-1}c \end{bmatrix}$$

$$= \begin{bmatrix} g_1 g_5^{-1} g_5 & (g_1 g_5^{-1})^{-1}(g_3 g_5^{-1})^{-1} g_5 & g_3 g_5^{-1} g_5 \\ (g_1 g_5^{-1})^{-1} g_3 g_5^{-1} g_5 & g_5 & g_1 g_5^{-1} (g_3 g_5^{-1})^{-1} g_5 \\ (g_3 g_5^{-1})^{-1} g_5 & g_1 g_5^{-1} g_3 g_5^{-1} g_5 & (g_1 g_5^{-1})^{-1} g_5 \end{bmatrix}$$

$$= \begin{bmatrix} g_1 & g_1^{-1} g_3^{-1} g_5^3 & g_3 \\ g_1^{-1} g_3 g_5 & g_5 & g_1 g_3^{-1} g_5 \\ g_3^{-1} g_5^2 & g_1 g_3 g_5^{-1} & g_1^{-1} g_5^2 \end{bmatrix} = \begin{bmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \end{bmatrix}.$$

Therefore, the statement is proved. □

This leads to the biconditional result that will be used throughout the remainder of the paper.

Corollary 3.11 *The abelian group* G *is 3-magic if and only if there is a centered normalized 3-magic square in* G *of the form*

$$\begin{bmatrix} a & a^{-1}b^{-1} & b \\ a^{-1}b & 1 & ab^{-1} \\ b^{-1} & ab & a^{-1} \end{bmatrix},$$

where $a, b \in G$.

Proof — This result follows from multiplying every entry in

$$\begin{bmatrix} ac & a^{-1}b^{-1}c & bc \\ a^{-1}bc & c & ab^{-1}c \\ b^{-1}c & abc & a^{-1}c \end{bmatrix}$$

by c^{-1}. □

Hence, for an abelian group G to be 3-magic, there must be two elements $a, b \in G$ such that a, $a^{-1}b^{-1}$, b, $a^{-1}b$, ab^{-1}, b^{-1}, ab, and a^{-1} are distinct. This proves to be especially useful in showing the impossibility of having a 3-magic square in certain classes of groups.

Corollary 3.12 *The elements of any normalized 3-magic square in an abelian group can be generated by two elements of the group.*

Proof — From the previous proposition, every 3-magic square in an abelian group is of the form

$$\begin{bmatrix} ac & a^{-1}b^{-1}c & bc \\ a^{-1}bc & c & ab^{-1}c \\ b^{-1}c & abc & a^{-1}c \end{bmatrix}.$$

If this were normalized, then one of the entries would be 1. This would give us an equation in which we could solve for c in terms of a and b. □

We can now make use of these results to continue determining which abelian groups are 3-magic.

Proposition 3.13 *Group* C_n^k *with* $k \geqslant 4$ *is 3-magic if and only if* $n \geqslant 3$.

Proof — Note first that, since $C_n^3 \leqslant C_n^k$, Proposition 3.2 gives sufficiency for $n \geqslant 3$. Now let $n = 2$. Assume there is a centered normalized 3-magic square in $C_n^3 \leqslant C_n^k$, as given by Corollary 3.11, and use the notation in that corollary for its form. Since every nontrivial element in C_2^k is an involution, $a = a^{-1}$, which violates distinctness. Hence, C_2^k is not 3-magic, for any k. □

Proposition 3.14 *The group* $C_n \times C_{n^k}$, *with* $k \geqslant 3$, *is 3-magic if and only if* $n \geqslant 2$.

PROOF — The fact that $C_n \times C_{n^3}$ is 3-magic for $n \geqslant 3$ follows from Proposition 3.2 after recognizing that $C_{n^3} \leqslant C_n \times C_{n^3}$.

For $C_{2^3} \times C_2 = \langle x, y \rangle$, we have that

$$\begin{bmatrix} xy & x^5 & x^2y \\ x & 1 & x^7 \\ x^6y & x^3 & x^7y \end{bmatrix}$$

forms a magic square with a common product of 1. Hence, for $n = 2$ and $k \geqslant 4$, we can then use Proposition 3.2 in a similar manner as was previously done. □

The following proposition completes the search for abelian 3-magic groups of odd order and leaves us only with those of even order.

Proposition 3.15 *All abelian groups of odd order n with $n \geqslant 9$ are 3-magic.*

PROOF — Let G be a finite abelian group of odd order with

$$|G| = n \geqslant 9.$$

By the Fundamental Theorem of Finite Abelian Groups, we know that $G = C_{n_1} \times C_{n_2} \times \ldots \times C_{n_\ell}$, where ℓ is the rank of G. If $\ell = 1$, then G is 3-magic by Proposition 3.5. Suppose then that $\ell > 1$. Since ℓ is the rank of G, by the Chinese Remainder Theorem, there must be a $p \mid n_i$, for $i \in \{1, 2, \ldots, \ell\}$ where p is an odd prime. Thus $C_p^\ell \leqslant G$ and since $p \geqslant 3$, G is 3-magic by Propositions 3.2, 3.6, and 3.13. □

We now turn our attention to abelian groups of even order.

Proposition 3.16 *The group $C_2^k \times C_4$ is not 3-magic for any k.*

PROOF — Let $C_2^k \times C_4 =< x_1, x_2, \ldots, x_k, y >$, where $|x_i| = 2$ (that is, the order of x_i is 2), for all i and $|y| = 4$. Assume there is a centered normalized 3-magic square in $C_2^k \times C_4$ by Corollary 3.11, and use the notation in that corollary for its form. Since a and b cannot be involutions, $|a| = |b| = 4$. Then $a = X_1 y^\alpha$ and $b = X_2 y^\beta$, where

$$X_1 = x_{i_1} x_{i_2} \ldots x_{i_l}, \quad X_2 = x_{j_1} x_{j_2} \ldots x_{j_m}, \quad \text{and} \quad \alpha, \beta \in \{1, 3\}.$$

Case 1: $\alpha = \beta$. Without loss of generality, we may assume that $\alpha = \beta = 1$. Then

$$a^{-1}b = (X_1 y^3)(X_2 y) = X_1 X_2 = (X_1 y)(X_2 y^3) = ab^{-1},$$

contradicting distinctness.

Case 2: $\alpha \neq \beta$. Without loss of generality, we may assume that $\alpha = 1$ and $\beta = 3$. Then

$$a^{-1}b^{-1} = (X_1y^3)(X_2y) = X_1X_2 = (X_1y)(X_2y^3) = ab,$$

contradicting distinctness. □

In a very similar manner as was previously done, we can determine another family of groups that can be omitted from the search.

Proposition 3.17 *The group* $C_2^k \times C_3$ *is not 3-magic for any* k.

Proof — This is true for $k = 1$ since $|C_6| < 9$. Let $k \geqslant 2$, and let

$$C_2^k \times C_3 \simeq C_2^{k-1} \times C_6 =< x_1, x_2, \ldots, x_{k-1}, y >,$$

where $|x_i| = 2$, for all i and $|y| = 6$. Assume there is a centered normalized 3-magic square in $C_2^{k-1} \times C_6$ by Corollary 3.11, and use the notation used in that corollary for its form. Since a and b cannot be involutions, $|a|, |b| \in \{3, 6\}$. Then $a = X_1y^\alpha$ and $b = X_2y^\beta$, where $X_1 = x_{i_1}x_{i_2}\ldots x_{i_l}$, $X_2 = x_{j_1}x_{j_2}\ldots x_{j_m}$, and $\alpha, \beta \in \{1, 2, 4, 5\}$.

Case 1: $\alpha = \beta$. Then

$$a^{-1}b = (X_1y^{-\alpha})(X_2y^\alpha) = X_1X_2 = (X_1y^\alpha)(X_2y^{-\alpha}) = ab^{-1},$$

contradicting distinctness.

Case 2: $\beta \equiv_6 -\alpha$. Then

$$a^{-1}b^{-1} = (X_1y^{-\alpha})(X_2y^\beta) = X_1X_2 = (X_1y^\alpha)(X_2y^{-\alpha}) = ab,$$

contradicting distinctness.

Case 3: $\{\alpha, \beta\} = \{1, 2\}$. Without loss of generality, we may assume that $\alpha = 1$ and $\beta = 2$. Then

$$a^{-1}b^{-1} = (X_1y^5)(X_2y^4) = X_1y^3X_2 = (X_1y)(X_2y^2) = ab,$$

contradicting distinctness.

Case 4: $\{\alpha, \beta\} = \{1, 4\}$. Without loss of generality, we may assume that $\alpha = 1$ and $\beta = 4$. Then

$$a^{-1}b = (X_1y^5)(X_2y^4) = X_1y^3X_2 = (X_1y)(X_2y^2) = ab^{-1},$$

contradicting distinctness.

Case 5: $\{\alpha, \beta\} = \{2, 5\}$. Without loss of generality, we may assume that $\alpha = 2$ and $\beta = 5$. Then

$$a^{-1}b = (X_1y^4)(X_2y^5) = X_1y^3X_2 = (X_1y^2)(X_2y) = ab^{-1},$$

contradicting distinctness.

Case 6: $\{\alpha, \beta\} = \{4, 5\}$. Without loss of generality, we may assume that $\alpha = 4$ and $\beta = 5$. Then

$$a^{-1}b^{-1} = (X_1y^2)(X_2y) = X_1y^3X_2 = (X_1y^4)(X_2y^5) = ab,$$

contradicting distinctness. □

The next proposition may seem less impactful than the previous ones, but its usefulness will be seen in the next section. It turns out knowing whether $C_4 \times C_8$ is 3-magic or not is an exceptional case in the proving of the characterization theorem; the next proposition handles this case.

Proposition 3.18 *The group* $C_4 \times C_8$ *is 3-magic.*

Proof — Let $C_4 \times C_8 = \langle x, y \rangle$, where $|x| = 4$ and $|y| = 8$. Then

$$\begin{bmatrix} y & x^3y^6 & xy \\ x & 1 & x^3 \\ x^3y^7 & xy^2 & y^7 \end{bmatrix}$$

is a magic square with common product 1. □

4 Characterization of finitely-generated abelian 3-magic groups

In this section, we present the characterization of finitely-generated abelian 3-magic groups, which is built upon many of the previous results, which had a narrower scope.

Theorem 4.1 *Let* G *be a finitely-generated abelian group. It follows that* G *is not 3-magic if and only if*

$$|G| < 9 \quad \text{or} \quad G \simeq C_2^j \times C_k$$

where $j > 0$ *and* $k \in \{1, 3, 4\}$.

Proof — Sufficiency follows quickly from Propositions 3.1, 3.13, 3.16, and 3.17.

For necessity, suppose G is not 3-magic and that $|G| \geqslant 9$. By Proposition 3.4, G must be a torsion group and hence finite (since it's abelian). By Proposition 3.15, $|G|$ must be even. This tells us that G must have a subgroup H_1 isomorphic to a cyclic 2-group. By Proposition 3.5, $|H_1| < 9$. Furthermore, since $|G| \geqslant 9$, there must be another nontrivial cyclic subgroup H_2 with $H_1 \cap H_2 = 1$ and $|H_2| < 9$.

Case 1: $H_1 \simeq C_8$. By Proposition 3.6, $H_2 \not\simeq C_8$ and, by Proposition 3.5, $H_2 \not\simeq C_3, C_5, C_6, C_7$. Moreover, by Proposition 3.18, $H_2 \not\simeq C_4$. Finally, by Proposition 3.14, $H_2 \not\simeq C_2$, and hence, this case cannot hold.

Case 2: $H_1 \simeq C_4$. By Proposition 3.18, $H_2 \not\simeq C_8$ and, by Proposition 3.5, $H_2 \not\simeq C_3, C_5, C_6, C_7$. Furthermore, by Proposition 3.6, we obtain that $H_2 \not\simeq C_4$. Hence, it must be that $G \simeq C_2^j \times C_4$, for some $j > 0$.

Case 3: $H_1 \simeq C_2$. By Proposition 3.14, $H_2 \not\simeq C_8$ and, by Proposition 3.5, we have that $H_2 \not\simeq C_5, C_6, C_7$. Therefore, it must be that either $G \simeq C_2^j \times C_4$, $G \simeq C_2^j \times C_3$, or $G \simeq C_2^j$, for some $j > 0$. □

5 Non-abelian groups

We now briefly turn our attention to nonabelian groups and first present some sufficient conditions for 3-magic groups.

Proposition 5.1 *Let* G *be a group with* $|G| = n$. *If any of the following are true, then* G *is 3-magic.*

(1) *There is a prime,* $p \geqslant 11$, *such that* $p \mid n$.

(2) *There is a prime,* $p \neq 2$, *such that* $p^2 \mid n$.

Proof — (1) By Cauchy's Theorem, $C_p \leqslant G$, which is 3-magic by Proposition 3.5.

(2) As a consequence of the Sylow Theorems, we have that $H \leqslant G$, where $|H| = p^2$. This means that $H \simeq C_{p^2}$ or $H \simeq C_p^2$, both of which are 3-magic by Proposition 3.6. □

Corollary 5.2 *If* G *is a finite group and* $|G| \neq 2^k 3^{\epsilon_1} 5^{\epsilon_2} 7^{\epsilon_3}$, *where* $k \geqslant 0$ *and* $\epsilon_i \in \{0, 1\}$, *then* G *is 3-magic.*

Next we present a nonabelian group that is 3-magic but not covered by the previous proposition.

Example 5.3 The group

$$C_7 \rtimes C_3 = \left\langle a, b \mid a^7 = b^3 = 1, bab^{-1} = a^4 \right\rangle$$

is 3-magic. Indeed, it is sufficient to observe that the following

$$\begin{bmatrix} a & ab & a^3b^2 \\ a^2b^2 & 1 & a^6b \\ a^2b & a^5b^2 & a^6 \end{bmatrix}$$

is a 3-magic square with magic product 1.

Lastly, we present a nonabelian group that is not 3-magic and not covered by the previous proposition.

Example 5.4 The group

$$D_5 = \left\langle r, s \mid r^5 = s^2 = 1, sr = r^{-1}s \right\rangle$$

is not 3-magic. Indeed, notice that the group D_5 contains five reflections of the form $r^i s$, all of which are involutions. If we remove the distinctness requirement, there are exactly three (nearly) 3-magic squares in $\mathbb{Z}_2$:

$$A := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad C := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

A 3-magic square in D_5 must position all reflections in the same positions as the 1s in one of these three formations. As there are only five reflections, the case described by A is not possible.

For the case described by B, we look at the potential 3-magic square characterized by

$$\begin{bmatrix} r^{a_{11}}s & r^{a_{12}} & r^{a_{13}}s \\ r^{a_{21}} & r^{a_{22}} & r^{a_{23}} \\ r^{a_{31}}s & r^{a_{32}} & r^{a_{33}}s \end{bmatrix}.$$

Forcing the products of all the rows, columns, and diagonals to be the same, we get eight equations involving the exponents. Solving the system modulo 5, we arrive at $a_{11} = a_{33}$, contradicting distinctness.

For the case described by C, we look at the potential 3-magic square characterized by

$$\begin{bmatrix} r^{a_{11}} & r^{a_{12}}s & r^{a_{13}} \\ r^{a_{21}}s & r^{a_{22}}s & r^{a_{23}}s \\ r^{a_{31}} & r^{a_{32}}s & r^{a_{33}} \end{bmatrix}.$$

Forcing the products of all the rows, columns, and diagonals to be the same, we similarly get eight equations involving the exponents. Solving the system modulo 5, we again arrive at $a_{11} = a_{33}$, contradicting distinctness.

6 Future research

There are multiple topics of study for future investigations, beyond just continuing the investigation into the nonabelian case. For example, we could explore "magic cubes" or "magic tesseracts" in groups, potentially as an extension of the concept briefly touched on in [3]. Another idea is to investigate whether various structures (e.g. groups, graphs, etc.) have 3-magic automorphism groups. Finally, in our future study, we would like to find groups that are n-magic for $n > 3$.

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