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Prime Factorizations of Finite Groups, II *

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Abstract

This is the second part of our three consecutive papers which develop a theory of prime factorizations of groups. In this part, we show core results. Notably, we completely determine finite *unique factorization groups* (UFGs), i.e. groups with unique prime factorizations (up to equivalence). In fact, we show that a finite group is a UFG if and only if it is cyclic. We also develop methods to construct from prime factorizations of a group, inequivalent prime factorizations of the group.

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The present paper is the second of a three-paper work, the foremost appeared in [17] and the third one that will appear on this journal in [18].

1 Results on UFGs in the special cases

1.1 Nonabelian p-groups are not UFGs

A group with a *unique* prime factorization is called a *unique factorization group* (UFG). In this section, we show that nonabelian p-groups are not UFGs, where by a p-group, we mean a *finite* group of prime

power order. Let us recall that any p-group G has a nontrivial center (see [14, p. 103, Lemma 5.21]); thus G always has a cyclic subgroup $N\,(=\mathbb{Z}_p)$ of order p contained in the center (in particular N is normal in G).

Assumption 1.1 In what follows, let us suppose that G is a nonabelian p-group; so N is proper in G.

Consider the quotient homomorphism $\pi : G \to G' := G/N$. Then G' is a p-group of order $|G|/p$, so any subgroup of G' is a p-groups; in particular any cyclic subgroup is generated by an element of order a power of p.

Claim 1.2 *For a cyclic subgroup $\mathbb{Z}_{p^k}$ of G', its preimage $\pi^{-1}(\mathbb{Z}_{p^k})$ is either of the following*:

(i) $\mathbb{Z}_{p^{k+1}}$.

(ii) $\mathbb{Z}_{p^k} \times N$, *i.e.* $\mathbb{Z}_{p^k}N$ *with* $\mathbb{Z}_{p^k} \cap N = \{1\}$, *where note that any element of* N *commutes with any element of* $\mathbb{Z}_{p^k}$, *as* N *is contained in the center of* G.

Proof — For brevity, set

$$n := p^k \quad \text{and} \quad H := \pi^{-1}(\mathbb{Z}_n).$$

As $1 \in \mathbb{Z}_n$, we have $\pi^{-1}(1) \subset \pi^{-1}(\mathbb{Z}_n)$, that is, $N \subset H$, and

$$H/N = \mathbb{Z}_n. \tag{1.1}$$

In particular, note that $|H| = np$. Now, take $y \in H$ that is mapped to a generator $\overline{y}$ of $\mathbb{Z}_n$. Then, by (1.1), $y^n \in N$ and n is the least of such positive integers. Hence either $y^n = 1$ or y^n is equal to a generator of $N = \mathbb{Z}_p$. We separately consider these cases:

Case 1 $y^n = x$ (a generator of $N = \mathbb{Z}_p$).

Then $y^{pn} = x^p = 1$, and pn is the order of y. This with $|H| = pn$ implies that y generates H. Thus $H = \mathbb{Z}_{pn}$.

Case 2 $y^n = 1$.

This with the fact that n is the least among the positive integers m satisfying $y^m \in N$ implies the following:

(a) n is the order of y; so $K := \langle y \rangle$ is isomorphic to $\mathbb{Z}_n$.

(b) $K \cap N = \{1\}$.

On the other hand, since N is contained in the center of the group G, we have $xz = zx$ for any $z \in G$. This fact together with $K \cap N = \{1\}$ implies that $H = K \times N$. Since $K \simeq \mathbb{Z}_n$, the assertion is confirmed. Note that as $K \cap N = \{1\}$, the homomorphism $\pi : G \to G' := G/N$ isomorphically maps K to $\mathbb{Z}_n$. □

Note that in Claim 1.2, if the p-group G' itself is a cyclic group $\mathbb{Z}_{p^k}$, then $G = \pi^{-1}(\mathbb{Z}_{p^k})$, and G is either $\mathbb{Z}_{p^k} \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^k} \times N$, but both are abelian, which contradicts the assumption that G is nonabelian. Thus G' cannot be cyclic.

Assumption 1.3 In what follows, we only consider the case that the p-group G' is noncyclic.

Since G' is finite, by [17, Theorem 3.16] it has (at least) one prime factorization, say $G' = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_l$, where $\mathfrak{p}_i$ $(i = 1, 2, \dots, l)$ is a cyclic group of order a power of p. Say $\mathfrak{p}_i = \mathbb{Z}_{p^{k_i}}$, then $\pi^{-1}(\mathbb{Z}_{p^{k_i}})$ is either $\mathbb{Z}_{p^{k_i+1}}$ or $\mathbb{Z}_{p^{k_i}} \times N$ (see Claim 1.2), which is of single type or double type as in [17, Definition 4.37]. Hence, we have the result below.

Claim 1.4 *Any prime factorization of* G' *is* π*-primary* (see [17, Definition 4.39]).

We can say more. Indeed, since by assumption $N = \mathbb{Z}_p$, by [17, Claim 4.44], the following result holds.

Claim 1.5 *Any prime factorization of* G' *induces a prime factorization of* G.

We show that any nonabelian p-group G admits at least two (inequivalent) prime factorizations, proceeding by induction on $|G|$. As above, take $N = \mathbb{Z}_p$ in the center of G, and consider the quotient homomorphism $\pi : G \to G' := G/N$. Note that, by Assumption 1.3, the p-group G' is either nonabelian or noncyclic abelian (*caution*: Although G is nonabelian, G' may be abelian, e.g. G is the dihedral group D_4 of degree 4 and $N = \mathbb{Z}_2$ is the center of D_4, for which $D_4/N \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$). We separately consider two cases.

Case 1 The p-group G' is nonabelian.

As $|G'| < |G|$, the induction hypothesis ensures that G' has (at least) two inequivalent prime factorizations, say

$$G' = H_1'H_2' \dots H_n' \quad \text{and} \quad G' = K_1'K_2' \dots K_m'.$$

Then we have their induced prime factorizations (see Claim 1.5), say

$$G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l \quad \text{and} \quad G = \mathfrak{q}_1\mathfrak{q}_2\dots\mathfrak{q}_k.$$

These prime factorizations are inequivalent by [17, Corollary 4.48]. Hence G has (at least) two inequivalent prime factorizations.

Case 2 The p-group G' is noncyclic abelian.

Note that such a group has (at least) two inequivalent prime factorizations (Proposition 2.10). We may thus apply the induction argument as in *Case 1*, to show that G has (at least) two inequivalent prime factorization. This completes the proof of our assertion, and the following is established:

Proposition 1.6 *Any nonabelian finite* p*-group has at least two* (*inequivalent*) *prime factorizations. In particular it is not a unique factorization group.*

1.2 Product factorizations

We will show that any *nonabelian* nilpotent finite groups admits at least two (inequivalent) prime factorizations. This section is devoted to its preparation. Let $G = G_1G_2$ be a factorization of a group G, where G_1 and G_2 are *nontrivial proper* subgroups of G. For factorizations $G_1 = H_1H_2\dots H_n$ and $G_2 = K_1K_2\dots K_m$, we have

$$G_1G_2 = H_1H_2\dots H_nK_1K_2\dots K_m,$$

that is,

$$G = H_1H_2\dots H_nK_1K_2\dots K_m. \tag{1.2}$$

Definition 1.7 We say that (1.2) is the *product factorization* of G constructed from $G_1 = H_1H_2\dots H_n$ and $G_2 = K_1K_2\dots K_m$.

For a factorization $G = G_1G_2$, any element g of G is written in the form $g = xy$ for some $x \in G_1$ and $y \in G_2$. If $G_1 \cap G_2 = \{1\}$, this expression is *unique*; indeed if $g = xy = x'y'$ for $x' \in G_1$ and $y' \in G_2$, then

$$(x')^{-1}x = y'y^{-1} \in G_1 \cap G_2,$$

so $(x')^{-1}x = y'y^{-1} = 1$, that is, $x = x'$ and $y = y'$. If moreover

$$xy = yx \tag{1.3}$$

for any $x \in G_1$ and $y \in G_2$, then for any $x_1y_1, x_2y_2 \in G_1G_2$, their product $x_1y_1x_2y_2$ is rewritten as $x_1x_2y_1y_2$, which implies that the group structure on $G = G_1G_2$ is a direct product, that is, $G = G_1 \times G_2$ with elements xy denoted by (x, y), and (1.3) is given by

$$(x, 1)(1, y) = (1, y)(x, 1)\,(= (x, y)).$$

Yet we avoid using the expression $G = G_1 \times G_2$ until we establish basic properties of product factorizations. The reason is explained in Convention 1.9 below.

Let $G = G_1G_2$ be a direct product group. Hence, $G_1 \cap G_2 = \{1\}$ and $xy = yx$ for any $x \in G_1$ and $y \in G_2$. The latter condition implies the following result.

Lemma 1.8 *For a direct product group $G = G_1G_2$, any subgroup H of G_1 and any subgroup K of G_2 permute, namely, $HK = KH$.*

Convention 1.9 In order to clarify our discussion, for the time being we avoid the expression $G = G_1 \times G_2$. Indeed, while

$$G = G_1G_2 = G_2G_1,$$

we have $G_1 \times G_2 \neq G_2 \times G_1$ but merely $G_1 \times G_2 \simeq G_2 \times G_1$. Similarly, in Lemma 1.10, although $HK = KH$, in general, $H \times K \neq K \times H$ but merely $H \times K \simeq K \times H$. This causes a confusion if we use the expression $G_1 \times G_2$. We thus avoid this expression until we establish basic properties of product factorizations (1.2).

Lemma 1.10 *Let $G = G_1G_2$ be a direct product group. For factorizations*

$$G_1 = H_1H_2 \dots H_n \quad \textit{and} \quad G_2 = K_1K_2 \dots K_m,$$

if the product factorization

$$G = H_1H_2 \dots H_nK_1K_2 \dots K_m$$

is redundant, then $G_1 = H_1H_2 \dots H_n$ or $G_2 = K_1K_2 \dots K_m$ is redundant.

Proof — If $G = H_1H_2 \dots H_nK_1K_2 \dots K_m$ is redundant, then the following cases may occur:

$$\begin{cases} \text{(a) some } H_i \text{ is redundant in } H_1H_2 \dots H_nK_1K_2 \dots K_m, \\ \text{(b) some } K_j \text{ is redundant in } H_1H_2 \dots H_nK_1K_2 \dots K_m. \end{cases} \tag{1.4}$$

Hence, it suffices to show the following:

$$\begin{cases} \text{In (a), the } H_i \text{ is redundant in } H_1H_2\dots H_n. \\ \text{In (b), the } K_j \text{ is redundant in } K_1K_2\dots K_m. \end{cases} \tag{1.5}$$

First in (a), we have

$$G = H_1H_2\dots\breve{H}_i\dots H_nK_1K_2\dots K_m.$$

Setting $G_1' := H_1H_2\dots\breve{H}_i\dots H_n$, then

$$G = G_1'G_2 \quad \text{and} \quad G_1' \subset G_1. \tag{1.6}$$

To show that $G_1 = H_1H_2\dots H_n$ is redundant at H_i, it suffices to show that $G_1' = G_1$ that is, $H_1H_2\dots\breve{H}_i\dots H_n = H_1H_2\dots H_n$. Supposing that $G_1' \subsetneq G_1$, we deduce a contradiction. Since $G = G_1G_2$ is a direct product, if $G_1' \subsetneq G_1$, then $G_1'G_2 \subsetneq G_1G_2$, that is, $G_1'G_2 \subsetneq G$. Here, by (1.6), $G_1'G_2 = G$, so $G \subsetneq G$, which is a contradiction. Similarly we can show the assertion for (b). □

Lemma 1.10 is restated as in the following result.

Lemma 1.11 *Let $G = G_1G_2$ be a direct product group. If factorizations $G_1 = H_1H_2\dots H_n$ and $G_2 = K_1K_2\dots K_m$ are irredundant, then their product factorization $G = H_1H_2\dots H_nK_1K_2\dots K_m$ is irredundant.*

Moreover the following result holds.

Proposition 1.12 *Let $G = G_1G_2$ be a direct product group. Then factorizations $G_1 = H_1H_2\dots H_n$ and $G_2 = K_1K_2\dots K_m$ are irredundant if and only if $G = H_1H_2\dots H_nK_1K_2\dots K_m$ is irredundant.*

Proof — "$\Longrightarrow$" It follows by Lemma 1.11.
"$\Longleftarrow$" Otherwise

$$\begin{cases} \text{(a)} \;\; G_1 = H_1H_2\dots\breve{H}_i\dots H_n \text{ for some } H_i, \text{ or} \\ \text{(b)} \;\; G_2 = K_1K_2\dots\breve{K}_j\dots K_m \text{ for some } K_j. \end{cases} \tag{1.7}$$

Accordingly,

$$\begin{cases} \text{(a)} \;\; G = G_1G_2 = H_1H_2\dots\breve{H}_i\dots H_nK_1K_2\dots K_m, \text{ or} \\ \text{(b)} \;\; G = G_1G_2 = H_1H_2\dots H_nK_1K_2\dots\breve{K}_j\dots K_m. \end{cases} \tag{1.8}$$

In either case, the factorization $G = H_1H_2 \dots H_nK_1K_2 \dots K_m$ is redundant. This contradicts the assumption. □

Corollary 1.13 *Let* $G = G_1G_2$ *be a direct product finite group, and*

$$G_1 = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n \quad \textit{and} \quad G_2 = \mathfrak{q}_1\mathfrak{q}_2 \dots \mathfrak{q}_m$$

be primary factorizations (*i.e. every* $\mathfrak{p}_i$ *and* $\mathfrak{q}_j$ *are cyclic subgroups of prime power orders*). *Then* $G_1 = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ *and* $G_2 = \mathfrak{q}_1\mathfrak{q}_2 \dots \mathfrak{q}_m$ *are prime factorizations if and only if* $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n\mathfrak{q}_1\mathfrak{q}_2 \dots \mathfrak{q}_m$ *is a prime factorization.*

Proof — A primary factorization of a finite group is a prime factorization if and only if it is irredundant [17, Definition 3.15]. Thus the assertion is immediate from Proposition 1.12. □

Rewritings and mutations of product sects

Let $G = G_1G_2$ be a direct product group; G_1 and G_2 are *nontrivial proper* subgroups of G. Let

$$H_1H_2 \dots H_n \quad \text{and} \quad K_1K_2 \dots K_m$$

be sects of G_1 and G_2, respectively. Then,

$$H_1H_2 \dots H_nK_1K_2 \dots K_m$$

is called the *product sect* of $H_1H_2 \dots H_n$ and $K_1K_2 \dots K_m$. Since factors H_i and K_j are subgroups of G_1 and G_2, respectively, we have that

$$H_iK_j = K_jH_i$$

(see Lemma 1.8). Consequently, $H_1H_2 \dots H_nK_1K_2 \dots K_m$ admits various rewritings. For instance, $H_1H_2K_1K_2$ admits the following rewritings:

$$\begin{aligned} H_1H_2K_1K_2 &= H_1K_1H_2K_2 && \text{as } H_2K_1 = K_1H_2 \\ &= K_1H_1H_2K_2 && \text{as } H_1K_1 = K_1H_1 \\ &= K_1H_1K_2H_2 && \text{as } H_2K_2 = K_2H_2 \\ &= K_1K_2H_1H_2 && \text{as } H_1K_2 = K_2H_1. \end{aligned} \tag{1.9}$$

Let us observe that $H_iK_j = K_jH_i$ implies that H_iK_j is a subgroup of G (see [17, Corollary 2.6]) so the above rewritings are mutations. Recall that two sects are equivalent if one is transformed to the other via a mutation (see [17, Definition 3.21]). Thus the following result holds.

Lemma 1.14 *Let $G = G_1G_2$ be a direct product group. Let $H_1H_2\dots H_n$ and $K_1K_2\dots K_m$ be sects of G_1 and G_2 respectively. Then the sects obtained by rewriting*

$$H_1H_2\dots H_nK_1K_2\dots K_m$$

via $H_iK_j = K_jH_i$ are equivalent.

Subgroups of a direct product group $G = G_1G_2$ contained in factors G_1 or G_2 are special: If H and K are *nontrivial* subgroups of G_1 and G_2 respectively, then $H \neq K$ in G. In fact, we have that $H \cap K = \{1\}$ from $G_1 \cap G_2 \neq \{1\}$. In particular, the following hold:

(i) Nontrivial subgroups of G_1 are not subgroups of G_2.

(ii) Nontrivial subgroups of G_2 are not subgroups of G_1.

We formalize this in the following result.

Lemma 1.15 *For a direct product group $G = G_1G_2$, the following hold*:

(i) *If H is a nontrivial subgroup of G_1, then it is not a subgroup of G_2.*

(ii) *If K is a nontrivial subgroup of G_2, then it is not a subgroup of G_1.*

Definition 1.16 Let $L_1L_2\dots L_l$ be a sect of a direct product group

$$G = G_1G_2.$$

If each factor L_i $(i = 1, 2, \dots, l)$ is a subgroup of G_1 or G_2, then

$$L_1L_2\dots L_l$$

is called an *admissible sect*.

An admissible sect of a direct product group $G = G_1G_2$ is rewritten as a product sect of sects of G_1 and G_2.

Lemma 1.17 *Any admissible sect* $L_1L_2\dots L_l$ *of a direct product group*

$$G = G_1G_2$$

is rewritten as a product sect

$$H_1H_2\dots H_nK_1K_2\dots K_m,$$

with $n+m=l$, *of a sect* $H_1H_2\dots H_n$ *of* G_1 *and a sect* $K_1K_2\dots K_m$ *of* G_2.

Proof — We classify the factors $L_1, L_2, \dots, L_l$ into those in G_1 and those in G_2, say $L_{i_1}, L_{i_2}, \dots, L_{i_a}$ with $i_1 < i_2 < \dots < i_a$, are subgroups of G_1 and $L_{j_1}, L_{j_2}, \dots, L_{j_b}$ with $j_1 < j_2 < \dots < j_b$, are subgroups of G_2, where $a+b=l$. Then, by Lemma 1.8,

$$L_{i_u}L_{j_v} = L_{j_v}L_{i_u},$$

with $u = 1, 2, \dots, a$ and $v = 1, 2, \dots, b$. Hence, using this relation, in $L_1L_2\dots L_l$ we may shift

$$L_{i_1}, L_{i_2}, \dots, L_{i_a}$$

to the left and

$$L_{j_1}, L_{j_2}, \dots, L_{j_b}$$

to the right, and we rewrite $L_1L_2\dots L_l$ as

$$L_{i_1}L_{i_2}\dots L_{i_a}L_{j_1}L_{j_2}\dots L_{j_b}.$$

Here, $L_{i_1}L_{i_2}\dots L_{i_a}$ and $L_{j_1}L_{j_2}\dots L_{j_b}$ are sects of G_1 and G_2, respectively, so the assertion is confirmed. □

Remark 1.18 Note that "rewritten as" may be replaced with "equivalent to" by Lemma 1.14.

Definition 1.19 For an admissible sect $L_1L_2\dots L_l$ of a direct product group $G = G_1G_2$, its equivalent sect $H_1H_2\dots H_nK_1K_2\dots K_m$, with $n+m=l$, in Lemma 1.17 is called its *standard form*.

By Lemma 1.17, the following result holds.

Lemma 1.20 *Any admissible sect is equivalent to its standard form.*

As long as we are concerned with the equivalence classes of admissible sects, it thus suffices to consider their standard forms.

Convention 1.21 From now on, a direct product group $G = G_1G_2$ is denoted by $G = G_1 \times G_2$, accordingly the standard form

$$H_1H_2 \dots H_nK_1K_2 \dots K_m$$

of an admissible sect is denoted by

$$H_1H_2 \dots H_n \times K_1K_2 \dots K_m,$$

or simply

$$\mathcal{H} \times \mathcal{K},$$

where $\mathcal{H} := H_1H_2 \dots H_n$ and $\mathcal{K} := K_1K_2 \dots K_m$. The length n of $\mathcal{H}$ is denoted by $\mathrm{length}(\mathcal{H})$.

Definition 1.22 Let $G = G_1 \times G_2$ be a direct product group. For a sect $\mathcal{H}$ of G_1 and a sect $\mathcal{K}$ of G_2, the sect $\mathcal{H} \times \mathcal{K}$ of G is called a *direct product sect*.

Mutations of direct product groups preserve the direct product decompositions of sects.

Lemma 1.23 *Let $\mu : \mathcal{H} \times \mathcal{K} \to \mathcal{H}' \times \mathcal{K}'$ be a mutation between direct product sects $\mathcal{H} \times \mathcal{K}$ and $\mathcal{H}' \times \mathcal{K}'$ of a direct product group $G = G_1 \times G_2$, where $\mathcal{H}$ and $\mathcal{H}'$ are sects of G_1 and $\mathcal{K}$ and $\mathcal{K}'$ are sects of G_2. Then μ maps $\mathcal{H}$ to $\mathcal{H}'$ and $\mathcal{K}$ to $\mathcal{K}'$. Therefore, it gives a mutation from $\mathcal{H}$ to $\mathcal{H}'$ and a mutation from $\mathcal{K}$ to $\mathcal{K}'$. In particular,* $\mathrm{length}(\mathcal{H}) = \mathrm{length}(\mathcal{H}')$ *and* $\mathrm{length}(\mathcal{K}) = \mathrm{length}(\mathcal{K}')$.

Proof — Write

$$\mathcal{H} \times \mathcal{K} = L_1L_2 \dots L_n \times L_{n+1}L_{n+2} \dots L_l,$$

where $L_1, L_2, \dots, L_n$ are nontrivial subgroups of G_1 and, additionally, $L_{n+1}, L_{n+2}, \dots, L_l$ are nontrivial subgroups of G_2. Let σ be a permutation of $1, 2, \dots, l$ that expresses the mutation μ, that is,

$$\begin{aligned}&\mu(L_1L_2 \dots L_n \times L_{n+1}L_{n+2} \dots L_l)\\ &\quad = L_{\sigma(1)}L_{\sigma(2)} \dots L_{\sigma(n)} \times L_{\sigma(n+1)}L_{\sigma(n+2)} \dots L_{\sigma(l)}.\end{aligned}$$

Note that L_i, for $1 \leqslant i \leqslant n$, (resp. $n+1 \leqslant i \leqslant l$) cannot be a subgroup of G_2 (resp. G_1) by Lemma 1.15. Therefore, we obtain that σ permutes $L_1, L_2, \dots, L_n$ (resp. $L_{n+1}, L_{n+2}, \dots, L_l$). Thus, μ maps $\mathcal{H}$ to $\mathcal{H}'$ (resp. $\mathcal{K}$ to $\mathcal{K}'$). □

Proposition 1.24 *Let $\mathcal{H} \times \mathcal{K}$ and $\mathcal{H}' \times \mathcal{K}'$ be direct product sects of a direct product group $G = G_1 \times G_2$. Then $\mathcal{H} \times \mathcal{K} \sim \mathcal{H}' \times \mathcal{K}'$ if and only if $\mathcal{H} \sim \mathcal{H}'$ and $\mathcal{K} \sim \mathcal{K}'$ (in particular, $\mathrm{length}(\mathcal{H}) = \mathrm{length}(\mathcal{H}')$ and $\mathrm{length}(\mathcal{K}) = \mathrm{length}(\mathcal{K}')$).*

Proof — "$\Longrightarrow$" This follows from Lemma 1.23.
"$\Longleftarrow$" Say that $\mathcal{H} \sim \mathcal{H}'$ via a mutation $\tau : \mathcal{H} \to \mathcal{H}'$ and that $\mathcal{K} \sim \mathcal{K}'$ via a mutation $\sigma : \mathcal{K} \to \mathcal{K}'$. Then the product map

$$\tau \times \sigma : \mathcal{H} \times \mathcal{K} \to \mathcal{H}' \times \mathcal{K}'$$

is a mutation. Thus $\mathcal{H} \times \mathcal{K} \sim \mathcal{H}' \times \mathcal{K}'$. □

Generalization to $G = G_1 \times G_2 \times \ldots \times G_n$
Let

$$G = G_1 G_2 \ldots G_n$$

be a factorization of a group G, where $G_1, G_2, \ldots, G_n$ are nontrivial proper subgroups of G.

Definition 1.25 We say that $G = G_1 G_2 \ldots G_n$ is a *direct product group* if the following hold:

(D1) $G_i \cap G_1 G_2 \ldots \check{G}_i \ldots G_n = \{1\}$ for any $i = 1, 2, \ldots, n$.

(D2) $xy = yx$ for any $x \in G_i$, $y \in G_j$ $(i \neq j)$. In particular, we have that $G_i G_j = G_j G_i$.

Let us note that, by (D1), no element $y \in G_i \setminus \{1\}$ can be written as an element of $G_1 G_2 \ldots \check{G}_i \ldots G_n$. Otherwise,

$$y = x_1 x_2 \ldots \check{x}_i \ldots x_n \in G_i \cap G_1 G_2 \ldots \check{G}_i \ldots G_n,$$

but then $y = 1$ by (D1), which contradicts $y \neq 1$.

Remark 1.26 Definition 1.25 is analogous to the definition of a direct product of vector spaces, while a simpler definition begins with a direct product set $G_1 \times G_2 \times \ldots \times G_n$, on which a posteriori a group structure is given. We however at present adopt the former definition, as it fits sects.

Lemma 1.27 *Let $G = G_1 G_2 \ldots G_n$ be a direct product group. Then, for any distinct i and j, $G_i \cap G_j = \{1\}$.*

Proof — Note that, by [17, Lemma 2.1]

$$G_j \subset G_1 G_2 \dots \breve{G}_i \dots G_n.$$

This with (D1) implies $G_i \cap G_j = \{1\}$. □

Definition 1.28 Let $G = G_1 G_2 \dots G_n$ be a direct product group. For sects $\mathcal{H}_i$ of G_i $(i = 1, 2, \dots, n)$, a sect $\mathcal{H}_1 \mathcal{H}_2 \dots \mathcal{H}_n$ of G is called a *direct product sect*.

Notation 1.29 In what follows, a direct product group

$$G = G_1 G_2 \dots G_n$$

is denoted by

$$G = G_1 \times G_2 \times \dots \times G_n$$

and a direct product sect

$$\mathcal{H}_1 \mathcal{H}_2 \dots \mathcal{H}_n$$

is denoted by

$$\mathcal{H}_1 \times \mathcal{H}_2 \times \dots \times \mathcal{H}_n.$$

In the following result, we generalize results on $G = G_1 \times G_2$ to $G = G_1 \times G_2 \times \dots \times G_n$.

Lemma 1.30 *Let $G = G_1 \times G_2 \times \dots \times G_n$ $(n \geqslant 2)$ be a direct product group. Let $\mu : \mathcal{H}_1 \times \mathcal{H}_2 \times \dots \times \mathcal{H}_n \to \mathcal{H}'_1 \times \mathcal{H}'_2 \times \dots \times \mathcal{H}'_n$ be a mutation between direct product sects, where $\mathcal{H}_i$ and $\mathcal{H}'_i$ $(i = 1, 2, \dots, n)$ are sects of G_i. Then μ maps $\mathcal{H}_i$ to $\mathcal{H}'_i$, and hence it gives a mutation from $\mathcal{H}_i$ to $\mathcal{H}'_i$. In particular,* $\mathrm{length}(\mathcal{H}_i) = \mathrm{length}(\mathcal{H}'_i)$.

Proof — We show this by contradiction on n. When $n = 2$, the assertion is already proved in Lemma 1.23. Under the hypothesis that the assertion is valid up to n, we show it for $n+1$. Set $\widetilde{G}_n := G_n \times G_{n+1}$ and write $G = G_1 \times G_2 \times \dots \times \widetilde{G}_n$. Similarly, set $\widetilde{\mathcal{H}}_n := \mathcal{H}_n \times \mathcal{H}_{n+1}$ and $\widetilde{\mathcal{H}}'_n := \mathcal{H}'_n \times \mathcal{H}'_{n+1}$, then the mutation in question is

$$\mu : \mathcal{H}_1 \times \mathcal{H}_2 \times \dots \times \widetilde{\mathcal{H}}_n \to \mathcal{H}'_1 \times \mathcal{H}'_2 \times \dots \times \widetilde{\mathcal{H}}'_n.$$

By induction hypothesis, μ maps $\mathcal{H}_i$ to $\mathcal{H}'_i$ $(i = 1, 2, \dots, n)$ and $\widetilde{\mathcal{H}}_n$ to $\widetilde{\mathcal{H}}'_n$, that is, $\mathcal{H}_n \times \mathcal{H}_{n+1}$ to $\mathcal{H}'_n \times \mathcal{H}'_{n+1}$. By Lemma 1.23, the last

fact implies that μ maps $\mathcal{H}_n$ to $\mathcal{H}'_n$ and $\mathcal{H}_{n+1}$ to $\mathcal{H}'_{n+1}$. This completes the induction step. □

Theorem 1.31 *Let $\mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_n$ and $\mathcal{H}'_1 \times \mathcal{H}'_2 \times \ldots \times \mathcal{H}'_n$ be a sect of a direct product group $G = G_1 \times G_2 \times \ldots \times G_n$, Then*

$$\begin{aligned} &\mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_n \sim \mathcal{H}'_1 \times \mathcal{H}'_2 \times \ldots \times \mathcal{H}'_n \\ &\iff \mathcal{H}_i \sim \mathcal{H}'_i \ (i = 1, 2, \ldots, n), \end{aligned}$$

and, in this case, $\mathrm{length}(\mathcal{H}_i) = \mathrm{length}(\mathcal{H}'_i)$ $(i = 1, 2, \ldots, n)$.

Proof — "$\Longrightarrow$" This follows from Lemma 1.30.
"$\Longleftarrow$" Say that $\mathcal{H}_i \sim \mathcal{H}'_i$ via a mutation $\mu_i : \mathcal{H}_i \to \mathcal{H}'_i$ $(i = 1, 2, \ldots, n)$. Then the product map

$$\mu_1 \times \mu_2 \times \ldots \times \mu_n : \mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_n \to \mathcal{H}'_1 \times \mathcal{H}'_2 \times \ldots \times \mathcal{H}'_n$$

is a mutation. Hence $\mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_n \sim \mathcal{H}'_1 \times \mathcal{H}'_2 \times \ldots \times \mathcal{H}'_n$, which completes the proof. □

The next criterion following from Theorem 1.31 is useful.

Criterion 1.32 *For direct product sects*

$$\mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_n \quad \textit{and} \quad \mathcal{H}'_1 \times \mathcal{H}'_2 \times \ldots \times \mathcal{H}'_n$$

of a direct product group $G = G_1 \times G_2 \times \ldots \times G_n$, if for some i, $\mathcal{H}_i$ is not equivalent to $\mathcal{H}'_i$, then we obtain that $\mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_n$ is not equivalent to $\mathcal{H}'_1 \times \mathcal{H}'_2 \times \ldots \times \mathcal{H}'_n$.

1.3 Nonabelian nilpotent finite groups are not UFGs

We shall show that nonabelian nilpotent finite groups are not unique factorization groups. Note first that any nilpotent finite group G is expressed as a direct product of its Sylow subgroups

$$G = S_{p_1} \times S_{p_2} \times \ldots \times S_{p_k}$$

(see [14, p. 216, Theorem 10.9 (viii)]). Suppose that G is nonabelian; then at least one of S_{p_i} is nonabelian. Without loss of generality, we may assume that S_{p_1} is nonabelian — a nonabelian finite p_1-group. Thus, S_{p_1} admits at least two (inequivalent) prime factorizations (see Proposition 1.6), say

$$S_{p_1} = \mathfrak{p}_1 \mathfrak{p}_2 \ldots \mathfrak{p}_l \qquad \text{and} \qquad S_{p_1} = \mathfrak{p}'_1 \mathfrak{p}'_2 \ldots \mathfrak{p}'_{l'}. \tag{1.10}$$

For brevity, setting $\mathcal{P} := \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l$ and $\mathcal{P}' := \mathfrak{p}'_1\mathfrak{p}'_2\dots\mathfrak{p}'_{l'}$, we write them as $S_{p_1} = \mathcal{P}_1$ and $S_{p_1} = \mathcal{P}'_1$. Next, for each S_{p_i} ($i = 2, 3, \dots, k$), take a prime factorization

$$S_{p_i} = \mathfrak{q}_{i,1}\mathfrak{q}_{i,2}\dots\mathfrak{q}_{i,m_i}.$$

For brevity, set $\mathcal{Q}_i := \mathfrak{q}_{i,1}\mathfrak{q}_{i,2}\dots\mathfrak{q}_{i,m_i}$. Then,

$$G = \mathcal{P}_1 \times \mathcal{Q}_2 \times \dots \times \mathcal{Q}_k \quad \text{and} \quad G = \mathcal{P}'_1 \times \mathcal{Q}_2 \times \dots \times \mathcal{Q}_k$$

are two prime factorizations of G. As $\mathcal{P}_1 \not\sim \mathcal{P}'_1$, they are inequivalent (see Criterion 1.32). We thus obtain the following result.

Proposition 1.33 *Any nonabelian nilpotent finite group admits at least two (inequivalent) prime factorizations. In particular it is not a unique factorization group.*

2 Results on UFGs in the finite abelian case

Throughout this section, groups are assumed to be finite.

2.1 Cyclic groups are UFGs

Let $\mathbb{Z}_n$ be a cyclic group of order n. Let $n = p_1^{m_1}p_2^{m_2}\dots p_l^{m_l}$ be the prime factorization. Then $\mathbb{Z}_n$ has a prime factorization

$$\mathbb{Z}_n = \mathbb{Z}_{p_1^{m_1}}\mathbb{Z}_{p_2^{m_2}}\dots\mathbb{Z}_{p_l^{m_l}}.$$

In fact, by the fundamental theorem of finite abelian groups, this is a direct product $\mathbb{Z}_n = \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \dots \times \mathbb{Z}_{p_l^{m_l}}$. We claim that $\mathbb{Z}_n$ has no other prime factorization. First in the case that $l = 1$, $\mathbb{Z}_n$ is of prime power order, so $\mathbb{Z}_n$ is atomic (see [17, Theorem 3.13]) so this assertion trivially holds. We consider the case $l \geqslant 2$. Note first that any subgroup of $\mathbb{Z}_n$ is cyclic. Moreover, we note the following result.

Claim 2.1 *The following statements are satisfied*:

(1) *For each* $i = 1, 2, \dots, l$, $\mathbb{Z}_{p_i^{m_i}}$ *is a unique Sylow* p_i*-subgroup of* $\mathbb{Z}_n$.

(2) *If* $\mathfrak{p}$ *is a cyclic subgroup of prime power order of* $\mathbb{Z}_n$, *say* $|\mathfrak{p}|$ *is a power of* p_i, *then* $\mathfrak{p} \subset \mathbb{Z}_{p_i^{m_i}}$.

(3) *The maximal cyclic subgroups of prime power order of* $\mathbb{Z}_n$ *are*

$$\mathbb{Z}_{p_1^{m_1}}, \mathbb{Z}_{p_2^{m_2}}, \dots, \mathbb{Z}_{p_l^{m_l}}.$$

Proof — (1) As the order of $\mathbb{Z}_{p_i^{m_i}}$ is $p_i^{m_i}$, it is a Sylow p_i-subgroup of $\mathbb{Z}_n$. As $\mathbb{Z}_n$ is abelian, a Sylow p_i-subgroup is normal, so it is unique by Sylow's third theorem.

(2) As $\mathbb{Z}_{p_i^{m_i}}$ is a unique Sylow p_i-subgroup by (1), $\mathfrak{p} \subset \mathbb{Z}_{p_i^{m_i}}$ by using Sylow's second theorem.

(3) As the maximal prime power divisors of n are

$$p_1^{m_1}, p_2^{m_2}, \dots, p_l^{m_l},$$

the subgroups $\mathbb{Z}_{p_1^{m_1}}, \mathbb{Z}_{p_2^{m_2}}, \dots, \mathbb{Z}_{p_l^{m_l}}$ are maximal cyclic subgroups of prime power orders of $\mathbb{Z}_n$. They indeed exhaust all maximal cyclic subgroups of prime power orders of $\mathbb{Z}_n$. Indeed, if M is a maximal cyclic group of prime power order, say a power of p_i, then its order is at most $p_i^{m_i}$, and by (2), it follows that $M \subset \mathbb{Z}_{p_i^{m_i}}$. The maximality of M then implies $M = \mathbb{Z}_{p_i^{m_i}}$. □

Proposition 2.2 *Any finite cyclic group is a unique factorization group: for* $\mathbb{Z}_n$*, let* $n = p_1^{m_1} p_2^{m_2} \dots p_l^{m_l}$ *be the prime factorization of* n*, then the unique prime factorization of* $\mathbb{Z}_n$ *is given by* $\mathbb{Z}_n = \mathbb{Z}_{p_1^{m_1}} \mathbb{Z}_{p_2^{m_2}} \dots \mathbb{Z}_{p_l^{m_l}}$.

Proof — It suffices to show that if $\mathbb{Z}_n = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_u$ is a prime factorization, then it coincides with $\mathbb{Z}_n = \mathbb{Z}_{p_1^{m_1}} \mathbb{Z}_{p_2^{m_2}} \dots \mathbb{Z}_{p_l^{m_l}}$, that is, $u = l$ and $\mathfrak{p}_i = \mathbb{Z}_{p_i^{m_i}}$ $(i = 1, 2, \dots, l)$ possibly after reordering. Note first the following statements.

(A) By Claim 2.1 (2), $\mathfrak{p}_i$ is contained in some factor of

$$\mathbb{Z}_{p_1^{m_1}} \mathbb{Z}_{p_2^{m_2}} \dots \mathbb{Z}_{p_l^{m_l}},$$

say $\mathfrak{p}_i \subset \mathbb{Z}_{p_k^{m_k}}$ for some k.

(B) In (A), $\mathfrak{p}_i$ and $\mathfrak{p}_j$ $(i \neq j)$ cannot be contained in the same $\mathbb{Z}_{p_k^{m_k}}$. Otherwise

$$\mathfrak{p}_i \subset \mathfrak{p}_j \quad \text{or} \quad \mathfrak{p}_i \supset \mathfrak{p}_j \tag{2.1}$$

because, by [17, Corollary 3.3], the subgroup poset of a cyclic group of prime power order is "straight", i.e.,

$$\mathbb{Z}_{p^m} \supsetneq \mathbb{Z}_{p^{m-1}} \supsetneq \dots \supsetneq \mathbb{Z}_{p^2} \supsetneq \mathbb{Z}_p \supsetneq \{1\}.$$

As $\mathbb{Z}_n$ is abelian, we may shift $\mathfrak{p}_j$ in the sect $\mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_u$ and rewrite $\mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_u$ as $\mathfrak{p}_1\dots\mathfrak{p}_i\mathfrak{p}_j\dots\mathfrak{p}_u$. Here, (2.1) implies that

$$\mathfrak{p}_i\mathfrak{p}_j = \mathfrak{p}_j \quad \text{or} \quad \mathfrak{p}_i\mathfrak{p}_j = \mathfrak{p}_i.$$

Thus $\mathfrak{p}_i$ or $\mathfrak{p}_j$ is redundant, which contradicts the irredundancy of $\mathbb{Z}_n = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_u$.

From (A) and (B), the number of factors of $\mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_u$ is equal to the number of factors of $\mathbb{Z}_{p_1^{m_1}}\mathbb{Z}_{p_2^{m_2}}\dots\mathbb{Z}_{p_l^{m_l}}$, that is, $u = l$. Hence, we have that $\mathbb{Z}_n = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l$. Possibly after reordering the factors of $\mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l$, we may assume that in (A),

$$\mathfrak{p}_i \subset \mathbb{Z}_{p_i^{m_i}} \ (i = 1, 2, \dots, l).$$

We claim that $\mathfrak{p}_i = \mathbb{Z}_{p_i^{m_i}}$ $(i = 1, 2, \dots, l)$. Otherwise,

$$|\mathbb{Z}_n| = |\mathfrak{p}_1||\mathfrak{p}_2|\dots|\mathfrak{p}_l| < |\mathbb{Z}_{p_1^{m_1}}||\mathbb{Z}_{p_2^{m_2}}|\dots|\mathbb{Z}_{p_l^{m_l}}| = |\mathbb{Z}_n|,$$

which is absurd. Hence, the prime factorization $\mathbb{Z}_n = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l$ coincides with the prime factorization $\mathbb{Z}_n = \mathbb{Z}_{p_1^{m_1}}\mathbb{Z}_{p_2^{m_2}}\dots\mathbb{Z}_{p_l^{m_l}}$. □

2.2 Noncyclic abelian p-groups are not UFGs

For distinct primes p and q, the direct product group $\mathbb{Z}_p \times \mathbb{Z}_q$ is isomorphic to a cyclic group $\mathbb{Z}_{pq}$, so it is a unique factorization group (see Proposition 2.2). In contrast, $G = \mathbb{Z}_p \times \mathbb{Z}_p$ is not a unique factorization group. To see this, set $\mathfrak{p}_1 := \mathbb{Z}_p \times \{1\}$ and $\mathfrak{p}_2 := \{1\} \times \mathbb{Z}_p$. Then

$$G = \mathfrak{p}_1 \times \mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_2.$$

Take a generator x of $\mathfrak{p}_1$ and a generator y of $\mathfrak{p}_2$, and consider a "diagonal subgroup" $\mathfrak{p}_3$ of G generated by $xy \in G$. Let us observe that $(xy)^p = 1$, and $\mathfrak{p}_3 \simeq \mathbb{Z}_p$. We claim that $G = \mathfrak{p}_1\mathfrak{p}_3$. It suffices to show that $\mathfrak{p}_1 \subset \mathfrak{p}_1\mathfrak{p}_3$ and $\mathfrak{p}_2 \subset \mathfrak{p}_1\mathfrak{p}_3$. In fact, once this is shown, then $\mathfrak{p}_1\mathfrak{p}_2 \subset \mathfrak{p}_1\mathfrak{p}_3$, namely $G \subset \mathfrak{p}_1\mathfrak{p}_3$, so $G = \mathfrak{p}_1\mathfrak{p}_3$. First, clearly we have that $\mathfrak{p}_1 \subset \mathfrak{p}_1\mathfrak{p}_3$. On the other hand, noticing that $x^{-1} \in \mathfrak{p}_1$

and $xy \in \mathfrak{p}_3$, we have

$$x^{-1} \cdot xy \in \mathfrak{p}_1\mathfrak{p}_3,$$

i.e. $y \in \mathfrak{p}_1\mathfrak{p}_3$. More generally, for any $i = 1, 2, \ldots, p$, we have

$$x^{-i} \cdot (xy)^i \in \mathfrak{p}_1\mathfrak{p}_3,$$

i.e. $y^i \in \mathfrak{p}_1\mathfrak{p}_3$. Thus $\mathfrak{p}_2 \subset \mathfrak{p}_1\mathfrak{p}_3$. Hence, $G = \mathfrak{p}_1\mathfrak{p}_3$. Here, $\mathfrak{p}_1 \cap \mathfrak{p}_3 = \{1\}$, so $G = \mathfrak{p}_1 \times \mathfrak{p}_3$. We can similarly show that $G = \mathfrak{p}_2\mathfrak{p}_3$ and $\mathfrak{p}_2 \cap \mathfrak{p}_3 = \{1\}$, so $G = \mathfrak{p}_2 \times \mathfrak{p}_3$. Therefore, G has three prime factorizations:

$$G = \mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_3 = \mathfrak{p}_2\mathfrak{p}_3 \quad (\text{or } G = \mathfrak{p}_1 \times \mathfrak{p}_2 = \mathfrak{p}_1 \times \mathfrak{p}_3 = \mathfrak{p}_2 \times \mathfrak{p}_3). \tag{2.2}$$

As $\mathfrak{p}_1, \mathfrak{p}_2$, and $\mathfrak{p}_3$ are distinct, these prime factorizations are indeed inequivalent. Consequently, we obtain the following result.

Lemma 2.3 $G = \mathbb{Z}_p \times \mathbb{Z}_p$ *is not a unique factorization group.*

Besides the three prime factorizations (2.2), the group $G = \mathbb{Z}_p \times \mathbb{Z}_p$ has more prime factorizations. In fact, the total number of prime factorizations of G is

$$\frac{p(p+1)}{2}$$

(see [18, Example 2.4] for details).

More generally we consider an abelian p-group $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where m and n are positive integers. Let x be a generator of $\mathbb{Z}_{p^m}$ and y be a generator of $\mathbb{Z}_{p^n}$. To clarify subsequent discussion, we rewrite $\mathbb{Z}_{p^m}$ by $\mathfrak{p}_1$ $(= \langle x \rangle)$ and $\mathbb{Z}_{p^n}$ by $\mathfrak{p}_2$ $(= \langle y \rangle)$. Then

$$G = \mathfrak{p}_1 \times \mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_2.$$

Let $\mathfrak{p}_3 := \langle xy \rangle$. As for the case $m = n = 1$, we claim that

$$G = \mathfrak{p}_1\mathfrak{p}_3 = \mathfrak{p}_2\mathfrak{p}_3.$$

We first show $G = \mathfrak{p}_1\mathfrak{p}_3$. Clearly,

$$\mathfrak{p}_1 \subset \mathfrak{p}_1\mathfrak{p}_3. \tag{2.3}$$

On the other hand,

$$\mathfrak{p}_2 \subset \mathfrak{p}_1\mathfrak{p}_3, \tag{2.4}$$

which follows from $x^{-i} \cdot (xy)^i \in \mathfrak{p}_1\mathfrak{p}_3$ for any integer i, that is,

$$y^i \in \mathfrak{p}_1\mathfrak{p}_3.$$

From (2.3) and (2.4), $G = \mathfrak{p}_1\mathfrak{p}_2 \subset \mathfrak{p}_1\mathfrak{p}_3$, so $G = \mathfrak{p}_1\mathfrak{p}_3$. We can similarly show that $G = \mathfrak{p}_1\mathfrak{p}_3$. We thus have three prime factorizations

$$G = \mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_3 = \mathfrak{p}_2\mathfrak{p}_3.$$

Here, while $\mathfrak{p}_1\mathfrak{p}_2$ is a direct product, $\mathfrak{p}_1\mathfrak{p}_3$ and $\mathfrak{p}_2\mathfrak{p}_3$ are not necessarily direct products.

Claim 2.4 *The following statements are satisfied*:

(1) *If* $m > n$, *then* $G = \mathfrak{p}_1\mathfrak{p}_3$ *is not a direct product.*

(2) *If* $m < n$, *then* $G = \mathfrak{p}_2\mathfrak{p}_3$ *is not a direct product.*

Proof — (1) If $m > n$, then the order of xy is p^m, so $\mathfrak{p}_3 \simeq \mathbb{Z}_{p^m}$. Thus, if $G = \mathfrak{p}_1 \times \mathfrak{p}_3$, then $|G| = |\mathfrak{p}_1||\mathfrak{p}_3|$, that is, $p^m p^n = p^m p^m$, which cannot hold.

(2) If $m < n$, then the order of xy is p^n, so $\mathfrak{p}_3 \simeq \mathbb{Z}_{p^n}$. Therefore, if $G = \mathfrak{p}_2 \times \mathfrak{p}_3$, then $|G| = |\mathfrak{p}_2||\mathfrak{p}_3|$, that is, $p^m p^n = p^n p^n$, which cannot hold. □

Actually, if $m \geqslant 2$ or $n \geqslant 2$, then $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ has more prime factorizations:

(A) For each $k = 0, 1, \ldots, m-1$, we set

$$\mathfrak{q}^{(k)} := \langle x^{p^k} y \rangle.$$

Note that $\mathfrak{q}^{(0)} = \mathfrak{p}_2$. We claim that $G = \mathfrak{p}_1\mathfrak{q}^{(k)}$. It suffices to show that $\mathfrak{p}_1 \subset \mathfrak{p}_1\mathfrak{q}^{(k)}$ and $\mathfrak{p}_2 \subset \mathfrak{p}_1\mathfrak{q}^{(k)}$. The first inclusion is trivial. The second inclusion follows from the fact that for any integer i, $x^{-ip^k} \cdot (x^{p^k} y)^i \in \mathfrak{p}_1\mathfrak{q}^{(k)}$, that is, $y^i \in \mathfrak{p}_1\mathfrak{q}^{(k)}$. Hence,

$$G = \mathfrak{p}_1\mathfrak{q}^{(k)}.$$

Here, the right hand side is primary and irredundant, so this gives a prime factorization of G. We thus obtain prime factorizations $G = \mathfrak{p}_1\mathfrak{q}^{(k)}$ $(k = 1, 2, \ldots, m-1)$.

(B) For each $l = 0, 1, \dots, n-1$, we set

$$\mathfrak{r}^{(l)} := \langle xy^{p^l} \rangle.$$

Note that $\mathfrak{r}^{(0)} = \mathfrak{p}_3$. As for (A), we can show that G has prime factorizations

$$G = \mathfrak{p}_2 \mathfrak{r}^{(l)} \ (l = 1, 2, \dots, n-1).$$

Therefore, the following prime factorizations of G are obtained (in total $1 + m + n$):

$$G = \begin{cases} \mathfrak{p}_1 \mathfrak{p}_2, & \\ \mathfrak{p}_1 \mathfrak{q}^{(k)} & (k = 0, 1, \dots, m-1), \\ \mathfrak{p}_2 \mathfrak{r}^{(l)} & (l = 0, 1, \dots, n-1). \end{cases} \tag{2.5}$$

Remark 2.5 As will be shown in Lemma 2.7,

$$G = \mathfrak{p}_1 \mathfrak{q}^{(k)} \quad \text{and} \quad G = \mathfrak{p}_2 \mathfrak{r}^{(l)}$$

are not necessarily direct products.

The prime factorizations of G in (2.5) are mutually not equivalent, as $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{q}^{(k)}, \mathfrak{r}^{(l)}$ are distinct. We thus obtain the following result.

Lemma 2.6 *An abelian* p*-group* $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ (*where* m *and* n *are positive integers*) *has at least* $m + n + 1$ *prime factorizations given by* (2.5). *In particular,* G *is not a unique factorization group.*

The prime factorizations $G = \mathfrak{p}_1 \mathfrak{q}^{(k)}$ and $G = \mathfrak{p}_2 \mathfrak{r}^{(l)}$ in (2.5) are not necessarily direct products, as we show below.

Lemma 2.7 *The following statements are satisfied*:

(1) *If* $m - k > n$, *then* $G = \mathfrak{p}_1 \mathfrak{q}^{(k)}$ *is not a direct product.*

(2) *If* $m < n - l$, *then* $G = \mathfrak{p}_2 \mathfrak{r}^{(l)}$ *is not a direct product.*

Proof — (1) If $m - k > n$ then the order of $x^{p^k} y$ is p^{m-k}, so

$$\mathfrak{q}^{(k)} \simeq \mathbb{Z}_{p^{m-k}}.$$

Thus, if $G = \mathfrak{p}_1 \times \mathfrak{q}^{(k)}$, then $|G| = |\mathfrak{p}_1||\mathfrak{q}^{(k)}|$, that is,

$$p^m p^n = p^m p^{m-k},$$

which cannot hold.

(2) If $m < n - l$, then the order of xy^{p^l} is p^{n-l}, so $\mathfrak{r}^{(l)} \simeq \mathbb{Z}_{p^{n-l}}$. Thus if $G = \mathfrak{p}_2 \times \mathfrak{r}^{(l)}$, then $|G| = |\mathfrak{p}_2||\mathfrak{r}^{(l)}|$, that is, $p^m p^n = p^n p^{n-l}$, which cannot hold. □

Note that the removal

$$H_1 H_2 \dots \breve{H}_i \dots H_n$$

of H_i from a sect $H_1 H_2 \dots H_n$ amounts to the replacement

$$H_1 H_2 \dots \{1\} \dots H_n$$

of H_i with $\{1\}$. Based on this, we adopt the following convention.

Convention 2.8 For a direct product $H_1 \times H_2 \times \dots \times H_n$ of nontrivial groups $H_1, H_2, \dots, H_n$, the removal of H_i is defined as a direct product $H_1 \times H_2 \times \dots \times \{1\} \times \dots \times H_n$, which is a subgroup of the group $H_1 \times H_2 \times \dots \times H_n$. By abuse of notation it is denoted by

$$H_1 \times H_2 \times \dots \times \breve{H}_i \times \dots \times H_n.$$

We now present a technical lemma that will be used later.

Lemma 2.9 *The following statements are satisfied*:

(1) *A direct product of nontrivial groups is irredundant.*

(2) *A direct product of irredundant sects of groups is irredundant.*

Proof — (1) Let $H_1 \times H_2 \times \dots \times H_n$ be a direct product of nontrivial finite groups $H_1, H_2, \dots, H_n$. Then,

$$H_1 \times H_2 \times \dots \times \breve{H}_i \times \dots \times H_n \subsetneq H_1 \times H_2 \times \dots \times H_n$$

for any i, as $\{1\} \subsetneq H_i$.

(2) For direct products of two irredundant sects, we first show this. For irredundant sects $H_1 H_2 \dots H_n$ and $K_1 K_2 \dots K_m$ of a group, we

show that their direct product

$$H_1H_2\dots H_n \times K_1K_2\dots K_m$$

is also irredundant. Note first that the removal of a factor from it is either of the following:

(i) $H_1H_2\dots \breve{H}_i \dots H_n \times K_1K_2\dots K_m$ or

(ii) $H_1H_2\dots H_n \times K_1K_2\dots \breve{K}_j \dots K_m$.

For the case (i), as $H_1H_2\dots \breve{H}_i \dots H_n \subsetneq H_1H_2\dots H_n$, we have

$$H_1H_2\dots \breve{H}_i \dots H_nK_1K_2\dots K_m \subsetneq H_1H_2\dots H_nK_1K_2\dots K_m, \quad (2.6)$$

consequently,

$$H_1H_2\dots \breve{H}_i \dots H_n \times K_1K_2\dots K_m \subsetneq H_1H_2\dots H_n \times K_1K_2\dots K_m.$$

Hence, $H_1H_2\dots H_n \times K_1K_2\dots K_m$ is irredundant. For the case (ii), the assertion is similarly shown. This argument works for direct products of an arbitrary number of irredundant sects. □

We now show the following result.

Proposition 2.10 *Noncyclic abelian* p*-groups are not unique factorization groups.*

Proof — Let G be a noncyclic abelian p-group. By the fundamental theorem of finite abelian groups, G is written as a direct product

$$\mathbb{Z}_{p^{m_1}} \times \mathbb{Z}_{p^{m_2}} \times \dots \times \mathbb{Z}_{p^{m_k}},$$

where $m_1, m_2, \dots, m_l$ are positive integers. Note that $l \geqslant 2$ as G is assumed to be noncyclic. Set $H := \mathbb{Z}_{p^{m_1}} \times \mathbb{Z}_{p^{m_2}}$ and write

$$G = H \times \mathbb{Z}_{p^{m_3}} \times \dots \times \mathbb{Z}_{p^{m_l}}.$$

Here H has at least $m_1 + m_2 + 1$ prime factorizations (see Lemma 2.6); they are factorizations into two cyclic subgroups of prime power orders. Write them as $H = \mathfrak{s}_i\mathfrak{t}_i$ $(i = 1, 2, \dots, m_1 + m_2 + 1)$. Accordingly,

$$G = \mathfrak{s}_i\mathfrak{t}_i \times \mathbb{Z}_{p^{m_3}} \times \dots \times \mathbb{Z}_{p^{m_l}}$$

with $i = 1, 2, \ldots, m_1 + m_2 + 1$. They are not only primary but also irredundant (see Lemma 2.9 (2)). We thus obtain $m_1 + m_2 + 1$ prime factorizations of G. Here $\mathfrak{s}_i\mathfrak{t}_i$ and $\mathfrak{s}_j\mathfrak{t}_j$ $(i \neq j)$ are not equivalent, so the prime factorizations

$$G = \mathfrak{s}_i\mathfrak{t}_i \times \mathbb{Z}_{p^{m_3}} \times \ldots \times \mathbb{Z}_{p^{m_l}}$$

and

$$G = \mathfrak{s}_j\mathfrak{t}_j \times \mathbb{Z}_{p^{m_3}} \times \ldots \times \mathbb{Z}_{p^{m_l}}$$

are not equivalent by Theorem 1.31. Therefore G is not a unique factorization group. □

Remark 2.11 Let us observe that n the proof of Proposition 2.10, instead of $H := \mathbb{Z}_{p^{m_1}} \times \mathbb{Z}_{p^{m_2}}$, taking

$$H_i := \mathbb{Z}_{p^{m_i}} \times \mathbb{Z}_{p^{m_{i+1}}},$$

for each $i = 1, 2, \ldots, l-1$, or more generally a mixture of "nonadjacent" H_i's, we may produce many other prime factorizations of

$$G = \mathbb{Z}_{p^{m_1}} \times \mathbb{Z}_{p^{m_2}} \times \ldots \times \mathbb{Z}_{p^{m_l}}.$$

2.3 Noncyclic abelian groups are not UFGs

We next consider the case that G is a noncyclic finite abelian group.

Proposition 2.12 *Any noncyclic abelian finite group G has more than one prime factorization. Hence it is not a unique factorization group.*

Proof — Let

$$|G| = p_1^{m_1} p_2^{m_2} \ldots p_u^{m_u}$$

be the prime factorization of the order of G, where possibly $l = 1$ (not that this is the case when G is a noncyclic abelian p-group). For each $i = 1, 2, \ldots, u$, let S_{p_i} be the Sylow p_i-subgroup of G, where as G is abelian, S_{p_i} is normal in G, so a Sylow p_i-subgroup is unique by Sylow's third theorem. Recall that any abelian finite group is nilpotent, so G is written as a direct product of Sylow subgroups

$$G = S_{p_1} \times S_{p_2} \times \ldots \times S_{p_u} \tag{2.7}$$

(for more details, see [14, p.216, Theorem 10.9 (viii)]). Here at least one S_{p_i} is noncyclic; otherwise every S_{p_i} is cyclic, say $S_{p_i} = \mathbb{Z}_{p_i^{m_i}}$,

then

$$G = \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \ldots \times \mathbb{Z}_{p_u^{m_u}} \simeq \mathbb{Z}_{p_1^{m_1} p_2^{m_2} \ldots p_u^{m_u}}$$

by the Chinese remainder theorem, as the primes $p_1, p_2, \ldots, p_u$ are distinct. This contradicts the assumption that G is noncyclic. Now, possibly after reordering $S_{p_1}, S_{p_2}, \ldots, S_{p_u}$, we may assume that S_{p_i} is noncyclic for $i = 1, 2, \ldots, v$ and cyclic for $i = v+1, v+2, \ldots, u$, where $1 \leqslant v \leqslant u$ and in the case that $v = u$, cyclic S_{p_i} does not exist. For $i = 1, 2, \ldots, v$, noting that S_{p_i} has more than one prime factorization (see Proposition 2.10), we take two (inequivalent) prime factorizations

$$S_{p_i} = \mathfrak{p}_{i,1}\mathfrak{p}_{i,2}\ldots\mathfrak{p}_{i,l_i} \qquad \text{and} \qquad S_{p_i} = \mathfrak{q}_{i,1}\mathfrak{q}_{i,2}\ldots\mathfrak{q}_{i,k_i}.$$

For brevity, we set $\mathcal{P}_i := \mathfrak{p}_{i,1}\mathfrak{p}_{i,2}\ldots\mathfrak{p}_{i,l_i}$ and $\mathcal{Q}_i := \mathfrak{q}_{i,1}\mathfrak{q}_{i,2}\ldots\mathfrak{q}_{i,m_i}$. Then,

$$S_{p_i} = \mathcal{P}_i \qquad \text{and} \qquad S_{p_i} = \mathcal{Q}_i.$$

Next, for cyclic S_{p_i} if any, take a prime factorization

$$S_{p_i} = \mathfrak{r}_{i,1}\mathfrak{r}_{i,2}\ldots\mathfrak{r}_{i,k_i}$$

with $i = v+1, v+2, \ldots, u$. Set

$$\mathcal{R}_i := \mathfrak{r}_{i,1}\mathfrak{r}_{i,2}\ldots\mathfrak{r}_{i,k_i},$$

then $S_{p_i} = \mathcal{R}_i$. Now, G has two prime factorizations

$$G = \begin{cases} \mathcal{P}_1 \times \mathcal{P}_2 \times \ldots \times \mathcal{P}_v \times \mathcal{R}_{v+1} \times \ldots \times \mathcal{R}_u, \\ \mathcal{Q}_1 \times \mathcal{Q}_2 \times \ldots \times \mathcal{Q}_v \times \mathcal{R}_{v+1} \times \ldots \times \mathcal{R}_u, \end{cases} \tag{2.8}$$

which are not equivalent by Theorem 1.31. □

Propositions 2.2 and 2.12 together yield the following result.

Theorem 2.13 *A finite abelian group is a unique factorization group if and only if it is cyclic. Here, for a cyclic group* G *of order* n *with prime factorization*

$$n = p_1^{n_1} p_2^{n_2} \ldots p_l^{n_l},$$

its unique prime factorization is

$$G = \mathbb{Z}_{p_1^{n_1}} \mathbb{Z}_{p_2^{n_2}} \ldots \mathbb{Z}_{p_l^{n_l}}.$$

3 Finite nonabelian groups with permutative prime factorizations

Let us initially recall that, according to [17, Theorem 3.16], any finite group has a prime factorization

Definition 3.1 A prime factorization $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l$ of a group G is *permutative* if

$$\mathfrak{p}_i\mathfrak{p}_j = \mathfrak{p}_j\mathfrak{p}_i$$

for any $i, j \in \{1, 2, \dots, l\}$.

Example 3.2 Any prime factorization of an abelian group (or more generally of a quasi-Dedekind group) is permutative, as any subgroup is permutable.

We shall show that any nonabelian finite group with a permutative prime factorization admits another (inequivalent) prime factorization; so it is not a unique factorization group. We first show some technical results needed for the proof.

Lemma 3.3 *Let H_1 and H_2 be finite subgroups of a group G. For the order of a sect H_1H_2, the following hold*:

(1) $|H_1H_2| = \dfrac{|H_1||H_2|}{|H_1 \cap H_2|}$.

(2) *If H_1 and H_2 are p-groups, then $|H_1H_2|$ is a power of p; in particular, under the assumption that $K = H_1H_2$ is a factorization of a subgroup K of G, it follows that K is a p-group.*

Proof — (1) Consider a surjective map $p : H_1 \times H_2 \to H_1H_2$ given by $(h_1, h_2) \mapsto h_1h_2$. For $h_1h_2 \in H_1H_2$, its preimage consists of

$$(k_1, k_2) \in H_1 \times H_2$$

such that $k_1k_2 = h_1h_2$, that is, $k_1^{-1}h_1 = k_2h_2^{-1}$, which is an element of $H_1 \cap H_2$. Setting $t := k_1^{-1}h_1 = k_2h_2^{-1}$, then

$$(k_1, k_2) = (h_1t^{-1}, th_2).$$

Thus

$$p^{-1}(h_1h_2) = \{(h_1t^{-1}, th_2) : t \in H_1 \cap H_2\}. \tag{3.1}$$

This consists of $|H_1 \cap H_2|$ elements. Hence, for any point of H_1H_2, the number of elements of its preimage under p is $|H_1 \cap H_2|$. Thus

$$|H_1 \times H_2| = |H_1H_2||H_1 \cap H_2|,$$

and so

$$|H_1H_2| = \frac{|H_1||H_2|}{|H_1 \cap H_2|}.$$

(2) From (1), it suffices to show that $\frac{|H_1||H_2|}{|H_1 \cap H_2|}$ is a power of p. Since H_1 and H_2 are p-group, their intersection $H_1 \cap H_2$, being a subgroup of a p-group, is also a p-group. Hence all of $|H_1|$, $|H_2|$, and $|H_1 \cap H_2|$ are powers of p. Thus $\frac{|H_1||H_2|}{|H_1 \cap H_2|}$ is a power of p. □

For a sect $H_1H_2 \dots H_n$ of length $n \geqslant 3$, a formula like

$$|H_1H_2| = \frac{|H_1||H_2|}{|H_1 \cap H_2|}$$

in Lemma 3.3 (1) does not hold, and a generalization of Lemma 3.3 (2) fails. Indeed, for a factorization $K = H_1H_2 \dots H_n$ ($n \geqslant 3$) of a subgroup K of G, even if $H_1, H_2, \dots, H_n$ are p-groups, K is not necessarily a p-group.

Example 3.4 Let H_1, H_2 and H_3 be subgroups of the symmetric group $\mathfrak{S}_3$ of degree 3 generated by transpositions $(1\,2)$, $(2\,3)$ and $(1\,3)$, respectively. They are cyclic subgroups of order 2. We can show

$$\mathfrak{S}_3 = H_1H_2H_3$$

by computation. Here $H_1H_2H_3$ is not only primary but also irredundant — the removal of any factor is a sect of order 4 (see Remark 3.5) which is strictly smaller than $H_1H_2H_3$ that is of order 6. Hence, we obtain that

$$\mathfrak{S}_3 = H_1H_2H_3$$

is a prime factorization of $\mathfrak{S}_3$. Here, notice that, although H_1, H_2 and H_3 are 2-groups, $\mathfrak{S}_3$ is not a 2-group since $|\mathfrak{S}_3| = 6$.

Remark 3.5 For $\mathfrak{S}_3 = H_1H_2H_3$ in Example 3.4, we have

$$H_i \cap H_j = \{1\}$$

for any distinct $i, j \in \{1, 2, 3\}$. Thus $|H_iH_j| = |H_i||H_j|$ from the formula in Lemma 3.3 (1). Here $|H_i| = |H_j| = 2$, therefore $|H_iH_j| = 4$.

According to [17, Theorem 3.16], recall that any finite group admits a prime factorization. We shall show that any finite nonabelian group G admits at least two (inequivalent) prime factorizations; so it is not a unique factorization group. We first take a prime factorization

$$G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_l$$

that is irredundant and $\mathfrak{p}_i$ is a cyclic of prime power order, where note that $l \geqslant 2$ (indeed if $l = 1$, then $G = \mathfrak{p}_1$ is abelian). We shall show that G has another prime factorization. We show this by induction on $l\,(\geqslant 2)$. In what follows, the normalizer of a subgroup H of G is denoted by $N_G(H)$.

Lemma 3.6 *Let* $G = HK$ *be a factorization of a group* G, *where* H *and* K *are subgroups of* G. *Then the following hold*:

(1) *If* $H \subset N_G(K)$, *then* K *is normal in* G.

(2) *If* $K \subset N_G(H)$, *then* H *is normal in* G.

Proof — (1) It suffices to show that $xKx^{-1} = K$ for any $x \in G$. Note that, as $G = HK$, we may express $x = hk$ for some $h \in H, k \in K$. Then

$$xKx^{-1} = hkKk^{-1}h^{-1} = hKh^{-1} = K \tag{3.2}$$

where in the second equality we used that $kKk^{-1} = K$ for any $k \in K$, and in the last one, we used that $h \in N_G(K)$ since $H \subset N_G(K)$.

(2) As HK is a group, by [17, Corollary 2.6], we have that

$$HK = KH.$$

Thus we may apply the argument in (2) interchanging H and K. □

Special case (length $l = 2$)

We now show the following result.

Lemma 3.7 *If a nonabelian finite group* G *has a prime factorization of length* 2, *then* G *admits another* (*inequivalent*) *prime factorization.*

Proof — Let us consider a prime factorization $G = \mathfrak{p}_1\mathfrak{p}_2$ of length 2. It follows that, as $\mathfrak{p}_1\mathfrak{p}_2$ is a group, it is permutative, namely $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1$ (see [17, Corollary 2.6]). Say that $|\mathfrak{p}_1|$ is a power of a prime p and $|\mathfrak{p}_2|$ is a power of a prime q. We separately consider the cases $p \neq q$ and $p = q$.

Case $p \neq q$.

Then the orders $|\mathfrak{p}_1|$ and $|\mathfrak{p}_2|$ are relatively prime, so $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{1\}$. We separate into two subcases:

(A) $\mathfrak{p}_1 \subset N_G(\mathfrak{p}_2)$ and $\mathfrak{p}_2 \subset N_G(\mathfrak{p}_1)$.

(B) $\mathfrak{p}_1 \not\subset N_G(\mathfrak{p}_2)$ or $\mathfrak{p}_2 \not\subset N_G(\mathfrak{p}_1)$.

(A) In this subcase, we show that $G = \mathfrak{p}_1 \times \mathfrak{p}_2$, which contradicts the assumption that G is nonabelian; hence this case actually does not occur. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{1\}$, to show that $G = \mathfrak{p}_1 \times \mathfrak{p}_2$ it suffices to demonstrate $xy = yx$ for any $x \in \mathfrak{p}_1$ and $y \in \mathfrak{p}_2$. Note first that

$$xyx^{-1}y^{-1} \in \mathfrak{p}_1 \cap \mathfrak{p}_2,$$

indeed

$$\begin{cases} x(yx^{-1}y^{-1}) \in \mathfrak{p}_1, \text{ since } yx^{-1}y^{-1} \in \mathfrak{p}_1 \text{ as } y \in \mathfrak{p}_2 \subset N_G(\mathfrak{p}_1), \\ (xyx^{-1})y^{-1} \in \mathfrak{p}_2, \text{ since } xyx^{-1} \in \mathfrak{p}_2 \text{ as } x \in \mathfrak{p}_1 \subset N_G(\mathfrak{p}_2). \end{cases} \tag{3.3}$$

Thus $xyx^{-1}y^{-1} \in \mathfrak{p}_1 \cap \mathfrak{p}_2$. Here $d\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{1\}$, so $xyx^{-1}y^{-1} = 1$, that is, $xy = yx$.

(B) First consider that $\mathfrak{p}_1 \not\subset N_G(\mathfrak{p}_2)$. Take $a \in \mathfrak{p}_1 \setminus N_G(\mathfrak{p}_2)$, then we have that $a\mathfrak{p}_2a^{-1} \neq \mathfrak{p}_2$. For brevity, set $\mathfrak{p}_2' := a\mathfrak{p}_2a^{-1}$; then $\mathfrak{p}_2' \neq \mathfrak{p}_2$. Now, as $a \in \mathfrak{p}_1$ and $\mathfrak{p}_1$ is a group, we have $\mathfrak{p}_1 a = \mathfrak{p}_1$, so we may rewrite $G = \mathfrak{p}_1\mathfrak{p}_2$ as

$$G = \mathfrak{p}_1 a\mathfrak{p}_2.$$

Multiplying this by a^{-1}, from the right yields

$$Ga^{-1} = \mathfrak{p}_1 a\mathfrak{p}_2a^{-1},$$

that is, $G = \mathfrak{p}_1\mathfrak{p}_2'$. Here $\mathfrak{p}_1$ and $\mathfrak{p}_2'$ are cyclic groups of prime power orders. Moreover, $G = \mathfrak{p}_1\mathfrak{p}_2'$ is irredundant, which is seen as follows.

First, as $G = \mathfrak{p}_1\mathfrak{p}_2$ is irredundant, we have $\mathfrak{p}_1\widetilde{\mathfrak{p}_2} \subsetneq G$, so

$$\mathfrak{p}_1\widetilde{\mathfrak{p}_2'} \subsetneq G.$$

Secondly, the factor $\mathfrak{p}_1$ is irredundant; otherwise $G = \widetilde{\mathfrak{p}_1}\mathfrak{p}_2'$, that is,

$$G = \mathfrak{p}_2' \,(= a\mathfrak{p}a^{-1}),$$

so $G = \mathfrak{p}_2$. This however implies that

$$G = \widetilde{\mathfrak{p}_1}\mathfrak{p}_2,$$

that is, the factor $\mathfrak{p}_1$ is redundant, which contradicts the irredundancy of $G = \mathfrak{p}_1\mathfrak{p}_2$. This confirms that $G = \mathfrak{p}_1\mathfrak{p}_2'$ is irredundant. Consequently, this is a prime factorization. We claim this is *not* equivalent to $G = \mathfrak{p}_1\mathfrak{p}_2$. Otherwise,

(i) $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_1\mathfrak{p}_2'$ or

(ii) $\mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{p}_2'$.

That is, as sectwords, in the case (i),

$$\mathfrak{p}_1 = \mathfrak{p}_1 \quad \text{and} \quad \mathfrak{p}_2 = \mathfrak{p}_2',$$

which contradicts the assumption. In the case (ii),

$$\mathfrak{p}_2 = \mathfrak{p}_1 \quad \text{and} \quad \mathfrak{p}_1 = \mathfrak{p}_2',$$

so $\mathfrak{p}_2 = \mathfrak{p}_2'$, which contradicts the assumption. Therefore, $G = \mathfrak{p}_1\mathfrak{p}_2$ and $G = \mathfrak{p}_1\mathfrak{p}_2'$ are not equivalent. As $G = \mathfrak{p}_2\mathfrak{p}_1$ (see [17, Corollary 2.6]) in the case that $\mathfrak{p}_2 \not\subset N_G(\mathfrak{p}_1)$ we may similarly show that there exists another (inequivalent) prime factorization.

Case $p = q$.

Then $G = \mathfrak{p}_1\mathfrak{p}_2$ is a factorization of G into two p-groups, thus G is a p-group (see Lemma 3.3 (2)); more specifically, we have that G is a nonabelian p-group, so $G = \mathfrak{p}_1\mathfrak{p}_2$ has another (inequivalent) prime factorization (see Proposition 1.6). This confirms the assertion. □

General case

We now show the following statement.

Assertion 3.8 *If a nonabelian finite group* G *has a* permutative *prime factorization* $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l$ $(l \geqslant 2)$, *then* G *has another* (*inequivalent*) *prime factorization.*

We already showed this for $l = 2$ (see Lemma 3.7). Assuming that Assertion 3.8 is true for $l-1$ $(\geqslant 2)$, we show that it is true for l. For each $k = 1, 2, \dots, l$, we set

$$G'_k := \mathfrak{p}_1\mathfrak{p}_2\dots\breve{\mathfrak{p}}_k\dots\mathfrak{p}_l.$$

Since $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \breve{\mathfrak{p}}_k, \dots, \mathfrak{p}_l$ are permutable with each other, we obtain that G'_k is a subgroup of G (see [17, Corollary 2.3]). We take a generator x_i of $\mathfrak{p}_i$ $(i = 1, 2, \dots, l)$. If G'_k is abelian, then $x_1, x_2, \dots, \breve{x}_k, \dots, x_l$ commute each other. In particular, if all G'_k $(k = 1, 2, \dots, n)$ are abelian, then all generators $x_1, x_2, \dots, x_l$ commute each other, so the group $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l$ is abelian. This contradicts the assumption that G is nonabelian. Hence, the following holds:

Claim 3.9 *At least one of* $G'_1, G'_2, \dots, G'_l$ *is nonabelian.*

Say that G'_k is nonabelian. We may assume that k is l, because in

$$G'_k = \mathfrak{p}_1\mathfrak{p}_2\dots\breve{\mathfrak{p}}_k\dots\mathfrak{p}_l,$$

we may hence freely rearrange the order of $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \breve{\mathfrak{p}}_k, \dots, \mathfrak{p}_l$ as they are permutable with each other (by assumption), we may rewrite the group $G'_k = \mathfrak{p}_1\mathfrak{p}_2\dots\breve{\mathfrak{p}}_k\dots\mathfrak{p}_l$ as

$$G'_k = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l\breve{\mathfrak{p}}_k.$$

Renumbering the indices, in what follows, G'_l is nonabelian. For brevity, we remove l and write G'_l as G'.

Claim 3.10 $G' = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_{l-1}$ *is irredundant.*

Proof — Otherwise $G' = \mathfrak{p}_1\mathfrak{p}_2\dots\breve{\mathfrak{p}}_i\dots\mathfrak{p}_{l-1}$ for some $\mathfrak{p}_i$. Multiplying this by $\mathfrak{p}_l$ from the right yields $G'\mathfrak{p}_l = \mathfrak{p}_1\mathfrak{p}_2\dots\breve{\mathfrak{p}}_i\dots\mathfrak{p}_{l-1}\mathfrak{p}_l$, that is,

$$G = \mathfrak{p}_1\mathfrak{p}_2\dots\breve{\mathfrak{p}}_i\dots\mathfrak{p}_{l-1}\mathfrak{p}_l.$$

Nevertheless, this contradicts the assumption that $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_l$ is irredundant. □

As $G' = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_{l-1}$ is primary and irredundant, the following result holds.

Claim 3.11 $G' = p_1 p_2 \dots p_{l-1}$ *is a prime factorization of* G'.

Since G' is a nonabelian finite group and the length of the prime factorization $G' = p_1 p_2 \dots p_{l-1}$ is $l-1$, the induction hypothesis (on the length of a prime factorization of a nonabelian finite group) ensures that besides this prime factorization, G' has another (inequivalent) prime factorization, say

$$G' = q_1 q_2 \dots q_m.$$

Multiplying this by p_l from the right yields

$$G' p_l = q_1 q_2 \dots q_m p_l,$$

that is,

$$G = q_1 q_2 \dots q_m p_l.$$

Here the factor p_l is irredundant since

$$G' = q_1 q_2 \dots q_m \breve{p}_l \subsetneq G = q_1 q_2 \dots q_m p_l.$$

However there may be redundant factors among $q_1, q_2, \dots, q_m$. Take an irredundantization of $G = q_1 q_2 \dots q_m p_l$, which is necessarily of the following form:

$$G = q_{i_1} q_{i_2} \dots q_{i_{m'}} p_l \quad (1 \leqslant i_1 < i_2 < \dots < i_{m'} \leqslant m). \tag{3.4}$$

Let us note that, if $G = q_1 q_2 \dots q_m p_l$ is irredundant, then this itself is as in (3.4). Since (3.4) is irredundant and primary, (3.4) is a prime factorization of G. If this is not equivalent to $G = p_1 p_2 \dots p_l$, then G has two (inequivalent) prime factorizations, and we are done. Otherwise, i.e. if

(a) $G = q_{i_1} q_{i_2} \dots q_{i_{m'}} p_l$ and

(b) $G = p_1 p_2 \dots p_l$

are equivalent, then we may deduce a contradiction as follows. Indeed, first the equivalence of (a) and (b) implies that their lengths are equal, i.e. $m' = l-1$, and

$$q_{i_1}, q_{i_2}, \dots, q_{i_{l-1}}, p_l \text{ is a permutation of } p_1, p_2, \dots, p_l. \tag{3.5}$$

Here, $p_1, p_2, \dots, p_l$ are permutable with each other by assumption.

This with the irredundancy of $p_1 p_2 \dots p_l$ implies that $p_1, p_2, \dots, p_l$ are distinct (see [17, Corollary 2.14]) consequently, the permutation in (3.5) maps p_l to itself, and so

$$q_{i_1}, q_{i_2}, \dots, q_{i_{l-1}} \text{ is a permutation of } p_1, p_2, \dots, p_{l-1}. \tag{3.6}$$

Here, since $p_1, p_2, \dots, p_{l-1}$ are permutable with each other, we have that $q_{i_1}, q_{i_2}, \dots, q_{i_{l-1}}$ are also permutable with each other. In particular, any of $q_{i_1}, q_{i_2}, \dots, q_{i_{l-1}}$ is permutable with any q_k ($k = 1, 2, \dots, m$). Indeed, $q_{i_j} q_k = q_k q_{i_j}$ ($j = 1, 2, \dots, l-1$). Thus, in $G' = q_1 q_2 \dots q_m$, we may shift $q_{i_1}, q_{i_2}, \dots, q_{i_{l-1}}$ to the left, and renumber their indices as

$$i_1 = 1, i_2 = 2, \dots, i_{l-1} = l-1.$$

Accordingly,

$$G = q_{i_1} q_{i_2} \dots q_{i_{l-1}} p_l$$

is rewritten as $G = q_1 q_2 \dots q_{l-1} p_l$. Then (3.6) is restated as

$$q_1, q_2, \dots, q_{l-1}$$

is a permutation of $p_1, p_2, \dots, p_{l-1}$, that is, there is a permutation σ of $1, 2, \dots, l-1$ such that

$$q_1 = p_{\sigma(1)}, q_2 = p_{\sigma(2)}, \dots, q_{l-1} = p_{\sigma(l-1)}.$$

Then the prime factorization $G' = q_1 q_2 \dots q_m$ is rewritten as

$$G' = p_{\sigma(1)} p_{\sigma(2)} \dots p_{\sigma(l-1)} q_l q_{l+1} \dots q_m. \tag{3.7}$$

Note that, as $p_1, p_2, \dots, p_{l-1}$ are normal, by [17, Corollary 2.5],

$$p_{\sigma(1)} p_{\sigma(2)} \dots p_{\sigma(l-1)} = p_1 p_2 \dots p_{l-1}$$

Thus, (3.7) is rewritten as

$$G' = p_1 p_2 \dots p_{l-1} q_l q_{l+1} \dots q_m. \tag{3.8}$$

Comparing this prime factorization with another prime factorization

$$G' = p_1 p_2 \dots p_{l-1},$$

we see that

$$q_l q_{l+1} \cdots q_m$$

is removable from (3.8), i.e. redundant, which contradicts the irredundancy of (3.8). This completes the proof of our assertion: *A nonabelian finite group with a permutative prime factorization admits another* (*inequivalent*) *prime factorization.* Next consider the abelian case and note that any prime factorization of an abelian group is permutative. We already showed that an abelian finite group admits more than one prime factorization if and only if it is noncyclic (see Theorem 6.43).

By combining these facts, we obtain the following result.

Theorem 3.12 *If a finite noncyclic group has a permutative prime factorization, then it has another* (*inequivalent*) *prime factorization; thus it is not a unique factorization group.*

For later convenience, we introduce the following notion.

Definition 3.13 A group is *permutative* if it admits a permutative prime factorization.

Theorem 3.12 is then restated in the following form.

Theorem 3.14 *A noncyclic permutative finite group has more than one prime factorization; thus it is not a unique factorization group.*

4 Replacement of subsects

A sect $H_1 H_2 \dots H_n$ is said to be *permutative* if all factors are permutable with each other, i.e.

$$H_i H_j = H_j H_i$$

for any $i, j \in \{1, 2, \dots, n\}$. This is "inner permutability"; cf. "outer permutability" — a subgroup H of a group G is *permutable* if

$$HK = KH$$

for any subgroup K of G.

Lemma 4.1 *The following statements are satisfied*:

(1) *A permutative sect* $H_1H_2\dots H_n$ *of a group* G *is a subgroup of* G; *so it is regular. By permutativity, this subgroup is independent of the order of* $H_1, H_2, \dots, H_n$, *i.e.*

$$H_1H_2\dots H_n = H_{\sigma(1)}H_{\sigma(2)}\dots H_{\sigma(n)}$$

for any permutation $\sigma \in \mathfrak{S}_n$.

(2) *If a permutative sect* $H_1H_2\dots H_n$ *is irredundant, it follows that the subgroups* $H_1, H_2, \dots, H_n$ *are distinct.*

Proof — It suffices to observe that statements (1) and (2) are restatements of [17, Corollary 2.3] and [17, Lemma 2.12] respectively. □

Let us now state the following result.

Lemma 4.2 *Any subsect of an irredundant sect is irredundant.*

Proof — We show that if a sect $H_1H_2\dots H_n$ is irredundant, then any subsect $H_iH_{i+1}\dots H_j$ $(1 \leqslant i < j \leqslant n)$ is irredundant. Otherwise,

$$H_iH_{i+1}\dots \breve{H}_k \dots H_j = H_iH_{i+1}\dots H_j \quad \text{for some } k\ (i \leqslant k \leqslant j).$$

Substituting this into $H_1H_2\dots H_n$ yields

$$H_1H_2\dots \underbrace{H_iH_{i+1}\dots \breve{H}_k \dots H_j}\dots H_n = H_1H_2\dots \underbrace{H_iH_{i+1}\dots H_j}\dots H_n,$$

that is,

$$H_1H_2\dots \breve{H}_k \dots H_n = H_1H_2\dots H_n,$$

which contradicts the irredundancy of $H_1H_2\dots H_n$. □

Corollary 4.3 *Let* $H_1H_2\dots H_n$ *be an irredundant sect. Then the factors of any permutative subsect are distinct.*

Proof — By Lemma 4.2, any subsect is irredundant. If it is permutative, then its factors are distinct (see Lemma 4.1 (2)). □

For an irredundant sect $H_1H_2\dots H_n$, its ith removal

$$H_1H_2\dots \breve{H}_i \dots H_n$$

is generally not irredundant. However, for $i = 1, n$, it is always irredundant. In fact the following result holds.

Proposition 4.4 *Let* $H_1H_2\dots H_n$ *be an irredundant sect. Then, the following statements are satisfied*:

(1) *The subsects obtained from* $H_1H_2\dots H_n$ *by successively removing the rightmost factor, that is,*

$$H_1H_2\dots H_{n-1},\ H_1H_2\dots H_{n-2},\ H_1H_2\dots H_{n-3}$$

and so on, are irredundant.

(2) *The subsects obtained from* $H_1H_2\dots H_n$ *by successively removing the leftmost factor, that is,*

$$H_2H_3\dots H_n,\ H_3H_4\dots H_n,\ H_4H_5\dots H_n$$

and so on, are irredundant.

Proof — It suffices to notice that statements (1) and (2) follow from Lemma 4.2, as the subsects in assertion are subsects of the irredundant sect $H_1H_2\dots H_n$. □

Remark 4.5 For an irredundant sect $\mathcal{H}$, let $\overset{\vee\vee}{\mathcal{H}}$ denote a sect obtained by removing (possibly more than one) factors, such as

$$H_1H_2\dots\overset{\vee}{H}_i\dots\overset{\vee}{H}_j\dots\overset{\vee}{H}_k\dots H_n.$$

Its components outside the removed factors,

$$H_1H_2\dots H_{i-1},\ H_{i+1}\dots H_{j-1},\ H_j\dots H_{k-1},\ H_{k+1}\dots H_n$$

are subsects of the irredundant sect $\mathcal{H}$, therefore they are irredundant by Lemma 4.2. Namely, $\overset{\vee\vee}{\mathcal{H}}$ is piecewise irredundant even if it were redundant.

Irredundantization after replacement

Notation Let $\mathcal{H} := H_1H_2\dots H_n$ be a sect and let

$$\mathcal{A} := H_iH_{i+1}\dots H_j \quad (1 \leqslant i < j \leqslant n)$$

be a subsect. Then $\mathcal{H}$ is denoted as

$$H_1 H_2 \dots H_{i-1} \mathcal{A} H_{j+1} \dots H_n,$$

where if $i = 1$ then $H_1 H_2 \dots H_{i-1}$ is empty and if $j = n$ then we get $H_{j+1} \dots H_n$ is empty.

We later apply the subsequent results to factorizations obtained by replacing subsects in the factorizations with other sects — from a factorization

$$G = H_1 H_2 \dots H_{i-1} \mathcal{A} H_{j+1} \dots H_n$$

with a subsect $\mathcal{A}$, we intend to construct another *inequivalent* factorization

$$G = H_1 H_2 \dots H_{i-1} \mathcal{B} H_{j+1} \dots H_n$$

replacing $\mathcal{A}$ with another sect $\mathcal{B}$. Note that if this is the case, the following equation must hold:

$$H_1 H_2 \dots H_{i-1} \mathcal{A} H_{j+1} \dots H_n = H_1 H_2 \dots H_{i-1} \mathcal{B} H_{j+1} \dots H_n. \quad (4.1)$$

To ensure this to hold, in what follows we impose the condition

$$\mathcal{A} = \mathcal{B} \quad (4.2)$$

equal as subsets of G, or a relation in the sect monoid $S(G)$.

We discuss sects before going into factorizations.

Lemma 4.6 *Let*

$$\mathcal{H} := H_1 H_2 \dots H_{i-1} \mathcal{A} H_j \dots H_n$$

be a sect with a subsect $\mathcal{A}$. Let $\mathcal{B}$ be a sect such that $\mathcal{A} = \mathcal{B}$, and consider a sect

$$\mathcal{H}' := H_1 H_2 \dots H_{i-1} \mathcal{B} H_j \dots H_n.$$

Let $\mathcal{H}'_{\text{irred}}$ be an irredundantization of $\mathcal{H}'$. Then

$$\mathcal{H} = \mathcal{H}' = \mathcal{H}'_{\text{irred}}.$$

Proof — From $\mathcal{A} = \mathcal{B}$ we have $\mathcal{H} = \mathcal{H}'$, while from the definition of $\mathcal{H}'_{\text{irred}}$ we have $\mathcal{H}' = \mathcal{H}'_{\text{irred}}$. □

Suppose that

$$\mathcal{H} := H_1 H_2 \dots H_{i-1} \mathcal{A} H_j \dots H_n$$

is an irredundant sect with a subsect $\mathcal{A}$. Then $\mathcal{A}$ is irredundant by Lemma 4.2. Replacing $\mathcal{A}$ in $\mathcal{H}$ with another sect $\mathcal{B}$ such that $\mathcal{A} = \mathcal{B}$ yields an equation

$$H_1 H_2 \dots H_{i-1} \mathcal{A} H_{j+1} \dots H_n = H_1 H_2 \dots H_{i-1} \mathcal{B} H_{j+1} \dots H_n. \quad (4.3)$$

While the left hand side is irredundant by assumption, the right hand side is not necessarily irredundant even if $\mathcal{B}$ is irredundant. For example, consider an irredundant sect

$$\mathcal{H} := H_1 \underbrace{H_2 H_3}_{\mathcal{A}} H_4$$

and an irredundant sect of the form $\mathcal{B} = H_1 K_2 K_3$ such that $\mathcal{A} = \mathcal{B}$, then

$$\mathcal{H}' = H_1 \underbrace{H_1 K_2 K_3}_{\mathcal{B}} H_4$$

is redundant. We claim that an irredundantization of

$$\mathcal{H}' := H_1 H_2 \dots H_{i-1} \mathcal{B} H_{j+1} \dots H_n$$

is obtained by removing some factors in $\mathcal{B}$ — no factors outside $\mathcal{B}$ need not to be removed. An irredundantization of $\mathcal{H}'$ is a priori obtained by removing factors of $\mathcal{B}$ and factors, say

$$H_{i_1}, H_{i_2}, \dots, H_{i_m},$$

outside $\mathcal{B}$. Namely, it is of the form

$$H_1 H_2 \dots \breve{H}_{i_1} \dots \breve{H}_{i_2} \dots \breve{\breve{\mathcal{B}}} \dots \breve{H}_{i_m} \dots H_n, \quad (4.4)$$

where $\breve{\breve{\mathcal{B}}}$ denotes a sect obtained by removing factors from $\mathcal{B}$. As this is an irredundantization of $H_1 H_2 \dots \mathcal{B} \dots H_n$, we have

$$H_1 H_2 \dots \breve{H}_{i_1} \dots \breve{H}_{i_2} \dots \breve{\breve{\mathcal{B}}} \dots \breve{H}_{i_m} \dots H_n = H_1 H_2 \dots \mathcal{B} \dots H_n. \quad (4.5)$$

On the other hand, the following inclusions hold:

$$\begin{aligned}
&H_1H_2\dots\breve{H}_{i_1}\dots\breve{H}_{i_2}\dots\breve{\breve{\mathcal{B}}}\dots\breve{H}_{i_m}\dots H_n\\
&\quad\subset H_1H_2\dots\breve{H}_{i_1}\dots\breve{H}_{i_2}\dots\mathcal{B}\dots\breve{H}_{i_m}\dots H_n \qquad \text{as } \breve{\breve{\mathcal{B}}}\subset\mathcal{B}\\
&\quad\subset H_1H_2\dots H_{i_1}\dots H_{i_2}\dots\mathcal{B}\dots H_{i_m}\dots H_n\\
&\quad= H_1H_2\dots\mathcal{B}\dots H_n.
\end{aligned}$$

This with (4.5) implies that

$$\begin{aligned}
&H_1H_2\dots\breve{H}_{i_1}\dots\breve{H}_{i_2}\dots\breve{\breve{\mathcal{B}}}\dots\breve{H}_{i_m}\dots H_n\\
&\quad= H_1H_2\dots\breve{H}_{i_1}\dots\breve{H}_{i_2}\dots\mathcal{B}\dots\breve{H}_{i_m}\dots H_n\\
&\quad= H_1H_2\dots\mathcal{B}\dots H_n.
\end{aligned}\tag{4.6}$$

In particular,

$$H_1H_2\dots\breve{H}_{i_1}\dots\breve{H}_{i_2}\dots\mathcal{B}\dots\breve{H}_{i_m}\dots H_n = H_1H_2\dots\mathcal{B}\dots H_n.$$

We however claim that this cannot hold. Indeed, as $\mathcal{B}=\mathcal{A}$, we may replace $\mathcal{B}$ with $\mathcal{A}$. Then

$$H_1H_2\dots\breve{H}_{i_1}\dots\breve{H}_{i_2}\dots\mathcal{A}\dots\breve{H}_{i_m}\dots H_n = H_1H_2\dots\mathcal{A}\dots H_n \tag{4.7}$$

that contradicts the irredundancy of $H_1H_2\dots\mathcal{A}\dots H_n$. Thus, this assertion holds: *In irredundantizing*

$$H_1H_2\dots\mathcal{B}\dots H_n,$$

factors are removed only from $\mathcal{B}$. We however point out that not all of the factors of $\mathcal{B}$ are removed. Otherwise,

$$H_1H_2\dots H_{i-1}H_{j+1}\dots H_n = H_1H_2\dots H_{i-1}\mathcal{B}H_{j+1}\dots H_n. \tag{4.8}$$

Replacing $\mathcal{B}$ with $\mathcal{A}$ yields

$$H_1H_2\dots H_{i-1}H_{j+1}\dots H_n = H_1H_2\dots H_{i-1}\mathcal{A}H_{j+1}\dots H_n, \tag{4.9}$$

that is,

$$H_1H_2\dots H_{i-1}H_{j+1}\dots H_n = H_1H_2\dots H_n,$$

which contradicts the irredundancy of $H_1H_2\dots H_n$. Hence, we formalize these results as follows.

Lemma 4.7 *Let $\mathcal{H} := H_1H_2\dots H_n$ be an irredundant sect of a group G and let $\mathcal{A}$ be its subsect; write*

$$\mathcal{H} = H_1H_2\dots H_{i-1}\mathcal{A}H_{j+1}\dots H_n. \tag{4.10}$$

Let $\mathcal{B}$ be a sect of G such that $\mathcal{A} = \mathcal{B}$, and replace $\mathcal{A}$ in $\mathcal{H}$ with $\mathcal{B}$:

$$\mathcal{H}' := H_1H_2\dots H_{i-1}\mathcal{B}H_{j+1}\dots H_n.$$

Then any irredundantization $\mathcal{H}'_{\text{irred}}$ of $\mathcal{H}'$ is obtained by removing (not all) factors of $\mathcal{B}$, that is, of the form

$$\mathcal{H}'_{\text{irred}} = H_1H_2\dots H_{i-1}\overset{\approx}{\mathcal{B}}H_{j+1}\dots H_n, \tag{4.11}$$

where $\overset{\approx}{\mathcal{B}}$ denotes a sect obtained by removing factors from $\mathcal{B}$.

Remark 4.8 In Lemma 4.7, note the following:

(1) As $\mathcal{H}'_{\text{irred}}$ is irredundant, its subsect $\overset{\approx}{\mathcal{B}}$ is irredundant by Lemma 4.2.

(2) In general, $\overset{\approx}{\mathcal{B}}$ is different from an irredundantization $\mathcal{B}_{\text{irred}}$ of $\mathcal{B}$. Even if $\mathcal{B}$ itself is irredundant, i.e. $\mathcal{B}_{\text{irred}} = \mathcal{B}$, in general $\overset{\approx}{\mathcal{B}} \neq \mathcal{B}$. For example, let $\mathcal{H} := H_1H_2H_3H_4$ be an irredundant sect and consider its subsect $\mathcal{A} := H_2H_3$. For an irredundant sect of the form $\mathcal{B} = H_1K_2K_3$ such that $\mathcal{A} = \mathcal{B}$, replacing $\mathcal{A}$ in $\mathcal{H}$ with $\mathcal{B}$ yields a redundant sect

$$\mathcal{H}' = H_1\underbrace{H_1K_2K_3}_{\mathcal{B}}H_4,$$

where H_1 in $\mathcal{B}$ is removable from $\mathcal{H}'$, so $\overset{\approx}{\mathcal{B}} \neq \mathcal{B}$.

(3) The condition $\mathcal{A} = \mathcal{B}$ is essential. Otherwise, the whole of $\mathcal{B}$ may be removable from $\mathcal{H}'$. For instance, consider the case

that $\mathcal{A}$ consists of a single factor H_i and also $\mathcal{B}$ consists of a single factor, so $\mathcal{B}$ is a subgroup of G, such that

$$H_{i-1} \supset \mathcal{B} \quad \text{or} \quad \mathcal{B} \subset H_{i+1}.$$

Then $\mathcal{B}$ is removable from

$$\mathcal{H}' := H_1 H_2 \dots H_{i-1} \mathcal{B} H_{i+1} \dots H_n,$$

because

$$H_{i-1}\mathcal{B} = H_{i-1} \quad \text{or} \quad \mathcal{B}H_{i+1} = H_{i+1}$$

depending on whether

$$H_{i-1} \supset \mathcal{B} \quad \text{or} \quad \mathcal{B} \subset H_{i+1}.$$

4.1 Construction of inequivalent sects by replacement

In the subsequent discussion, keep in mind the difference between the following notations.

Notation 4.9 For sects $H_1 H_2 \dots H_n$ and $K_1 K_2 \dots K_m$ of a group G,

(i) $H_1 H_2 \dots H_n = K_1 K_2 \dots K_m$ means that they are equal as subsets of G. In other words, this equation is a relation between words in the sect monoid of G.

(ii) $H_1 H_2 \dots H_n \equiv K_1 K_2 \dots K_m$ means that they are identical, namely $n = m$ and $H_i = K_i$, for $i = 1, 2, \dots, n$.

Note that if $n = m = 1$, then (i) and (ii) coincide — both mean that H_1 and K_1 are the same subgroup of G.

(iii) $H_1 H_2 \dots H_n \sim K_1 K_2 \dots K_m$ means that they are equivalent (via a mutation). In this case, one is a permutation of the other, so they have the same length, i.e. $n = m$.

Remark 4.10 Even if $H_1 H_2 \dots H_n = K_1 K_2 \dots K_n$, possibly

$$H_1 H_2 \dots H_n \not\sim K_1 K_2 \dots K_n.$$

For instance, for the symmetric group $\mathfrak{S}_3$ of degree 3, let H_1, H_2 and H_3 be its cyclic subgroups of order two generated by transpositions $(1\,2)$, $(2\,3)$ and $(3\,1)$, respectively. Then

$$H_1 H_2 H_3 = H_1 H_2 H_1 \ (= \mathfrak{S}_3),$$

but

$$H_1H_2H_3 \not\sim H_1H_2H_1,$$

as $H_1H_2H_1$ is not a permutation of $H_1H_2H_3$.

Notation 4.11 If $\mathcal{A}$ is a subsect of a sect $\mathcal{H}$, then we express

$$\mathcal{H} := H_1H_2 \dots H_u\mathcal{A}H_v \dots H_n,$$

where subsects $H_1H_2 \dots H_u$ or $H_v \dots H_n$ may be empty, i.e. possibly

$$\mathcal{H} := \mathcal{A}H_v \dots H_n \quad \text{or} \quad \mathcal{H} := H_1H_2 \dots H_u\mathcal{A}.$$

Let

$$\mathcal{H} := H_1H_2 \dots H_u\mathcal{A}H_v \dots H_n$$

be an irredundant sect with $\mathcal{A}$ its subsect. Let us notice that $\mathcal{A}$ is irredundant by Lemma 4.2. Let $\mathcal{B}$ be an irredundant sect such that $\mathcal{A} = \mathcal{B}$ but $\mathcal{A} \not\sim \mathcal{B}$ (in particular $\mathcal{A} \not\equiv \mathcal{B}$). In $\mathcal{H}$, replacing $\mathcal{A}$ with $\mathcal{B}$ yields a sect

$$\mathcal{H}' := H_1H_2 \dots H_u\mathcal{B}H_v \dots H_n.$$

From $\mathcal{A} = \mathcal{B}$, we have $\mathcal{H} = \mathcal{H}'$, but $\mathcal{H}'$ is not necessarily irredundant. Let $\mathcal{H}'_{\text{irred}}$ be an irredundantization of $\mathcal{H}$, which is of the form

$$\mathcal{H}'_{\text{irred}} = H_1H_2 \dots H_u\dddot{\mathcal{B}}H_v \dots H_n,$$

where $\dddot{\mathcal{B}}$ is a sect obtained by removing factors from $\mathcal{B}$ (see Lemma 4.7). Note that the irredundancy of $\mathcal{B}$ is needed for $\mathcal{H}'_{\text{irred}}$ to be of this form. We will show that $\mathcal{H}$ and $\mathcal{H}'_{\text{irred}}$ are not equivalent. We begin with preparation. For the time being, unless otherwise mentioned,

$$\mathcal{H} := H_1H_2 \dots H_u\mathcal{A}H_vH_{v+1} \dots H_n$$

need not to be irredundant. The outside $\mathcal{H} \setminus \mathcal{A}$ of $\mathcal{A}$ in $\mathcal{H}$ is the underbraced part below:

$$\mathcal{H} = \underbrace{H_1H_2 \dots H_u}\,\mathcal{A}\,\underbrace{H_vH_{v+1} \dots H_n}. \tag{4.12}$$

Note that, if $\mathcal{H} = \mathcal{A}$, then $\mathcal{H} \setminus \mathcal{A}$ is empty. Moreover, let us observe that for $H_i \in \mathcal{H} \setminus \mathcal{A}$, we mean that H_i is a factor of $\mathcal{H} \setminus \mathcal{A}$. Now let σ be a permutation of the factors of $\mathcal{H}$ that transforms $\mathcal{H}$ to a sect of

the form

$$\mathcal{H}' := H_1 H_2 \dots H_u \mathcal{A}' H_v H_{v+1} \dots H_n$$

instead the outside $\mathcal{H}' \setminus \mathcal{A}'$ coincides with the outside $\mathcal{H} \setminus \mathcal{A}$) such that the subsect $\mathcal{A}'$ has the same length as $\mathcal{A}$ (*caution*: In general, σ does not map $\mathcal{A}$ to $\mathcal{A}'$. In fact, σ does not necessarily map factors of $\mathcal{A}$ to factors of $\mathcal{A}'$).

Example 4.12 Consider a sect $\mathcal{H} := H_1 H_2 H_3 \underbrace{H_4 H_5}_{\mathcal{A}} H_6$. Then a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 4 & 3 & 2 & 6 \end{pmatrix}$$

transforms $\mathcal{H}$ to

$$\mathcal{H}' := H_1 H_5 H_4 \underbrace{H_3 H_2}_{\mathcal{A}'} H_6.$$

For $\mathcal{H}' \setminus \mathcal{A}'$ and $\mathcal{H} \setminus \mathcal{A}$ to coincide, we impose $H_5 = H_2$ and $H_4 = H_3$. Then

$$\mathcal{H}' := H_1 H_2 H_3 \underbrace{H_5 H_4}_{\mathcal{A}'} H_6.$$

Note that $\mathcal{A} := H_4 H_5$ turns into $\mathcal{A}' := H_5 H_4$. This however does not imply that σ maps $\mathcal{A}$ to $\mathcal{A}'$. In fact, the transposition $\tau = (4\,5)$ giving $H_4 H_5 \mapsto H_5 H_4$ is not a restriction of σ.

In spite of the fact that σ generally does not map the factors of $\mathcal{A}$ to those of $\mathcal{A}'$, we will show that the factors of $\mathcal{A}'$ are always a permutation of the factors of $\mathcal{A}$. Then we simply say that $\mathcal{A}'$ is a permutation of $\mathcal{A}$. We point out that such a permutation is generally not unique, for instance in Example 4.12, if $H_4 = H_5$, then besides the transposition $H_4 H_5 \mapsto H_5 H_4$, the identity permutation $H_4 H_5 \mapsto H_4 H_5$ is also a permutation from $\mathcal{A}$ to $\mathcal{A}'$. In the case that the factors of $\mathcal{A}$ are distinct, this phenomenon does not occur — a permutation from $\mathcal{A}$ to $\mathcal{A}'$ is necessarily unique. In what follows, no matter whether the factors of $\mathcal{A}$ are distinct or not, from σ we will explicitly (and canonically) construct a permutation from $\mathcal{A}$ to $\mathcal{A}'$. Before going into the general case, we point out that in the particular case that all factors of $\mathcal{H}$ are distinct, its construction is immediate. Indeed, in this case σ necessarily maps the outside of $\mathcal{A}$ to the outside of $\mathcal{A}'$, accordingly σ maps $\mathcal{A}$ to $\mathcal{A}'$, so the factors of $\mathcal{A}'$ are necessarily a permutation of the factors of $\mathcal{A}$. This permutation is unique (as the factors of $\mathcal{A}$ are distinct)

and is the restriction of σ. To treat the general case, the following is needed.

Claim 4.13 *Under σ, if a factor $H_i \in \mathcal{H}$ is mapped to the position of a factor $H_j \in \mathcal{H}' \setminus \mathcal{A}'$ $(= \mathcal{H} \setminus \mathcal{A})$, then $H_i = H_j$.*

Proof — Originally, $\mathcal{H}'$ is of the form

$$\mathcal{H}' = \ldots \mathcal{A}' \ldots H_j \ldots$$

It has another expression given as the image of $\mathcal{H}$ under σ, indeed, by assumption, $\mathcal{H}$ becomes of the form

$$\mathcal{H}' = \ldots \mathcal{A}' \ldots H_i \ldots$$

Note that, in both of the preceding forms, H_j and H_i lie at the same position of $\mathcal{H}'$, thus $H_j = H_i$. □

Claim 4.13 is symbolically paraphrased as: *Under the assumption that $H_i \in \mathcal{H} \overset{\sigma}{\mapsto} H_j \in \mathcal{H}' \setminus \mathcal{A}'$, we obtain that $H_i = H_j$.*

We then generalized this to an orbital sequence, as shown below.

Claim 4.14 *For $H_i \in \mathcal{H}$, let*

$$H_i \overset{\sigma}{\mapsto} H_{j_1} \overset{\sigma}{\mapsto} H_{j_2} \overset{\sigma}{\mapsto} \ldots \overset{\sigma}{\mapsto} H_{j_q}$$

be an orbital sequence caused by iterated applications of σ. If $H_{j_k} \in \mathcal{H} \setminus \mathcal{A}$ with $k = 1, 2, \ldots, q$, then

$$H_i = H_{j_1} = H_{j_2} = \ldots = H_{j_q}.$$

Proof — This is confirmed by successive applications of Claim 4.13. First of all, its application to $H_i \overset{\sigma}{\mapsto} H_{j_1}$ gives $H_i = H_{j_1}$. Next its application to $H_{j_1} \overset{\sigma}{\mapsto} H_{j_2}$ gives $H_{j_1} = H_{j_2}$, so

$$H_i = H_{j_1} = H_{j_2}.$$

Thirdly, its application to $H_{j_2} \overset{\sigma}{\mapsto} H_{j_3}$ gives $H_{j_2} = H_{j_3}$, so

$$H_i = H_{j_1} = H_{j_2} = H_{j_3}.$$

Repeating this yields $H_i = H_{j_1} = H_{j_2} = \ldots = H_{j_q}$. □

The following assertion is essential for our subsequent discussion.

Claim 4.15 *Under the permutation σ, if a factor A of $\mathcal{A}$ is replaced with a factor H of $\mathcal{H} \setminus \mathcal{A}$, then H is equal to some factor of $\mathcal{A}$ (which is generally different from A).*

Proof — Let m be the minimal positive integer such that $\sigma^m(H) = H$. Then the orbit $\mathcal{O}_H$ of H under σ is

$$H, \sigma(H), \sigma^2(H), \dots, \sigma^{m-1}(H).$$

Note also that $\sigma(H) = A$ by assumption. Among them, let

$$A_{i_0} = A, A_{i_1}, A_{i_2}, \dots, A_{i_p}$$

be the factors of $\mathcal{A}$ appearing in this order. These are called the *$\mathcal{A}$-factors* in the orbit $\mathcal{O}_H$. After the last $\mathcal{A}$-factor A_{i_p}, only factors of $\mathcal{H} \setminus \mathcal{A}$ appear, say $H_{j_1}, H_{j_2}, \dots, H_{j_q} = H$:

$$\begin{aligned} H \overset{\sigma}{\mapsto} A_{i_0} = A \overset{\sigma}{\mapsto} \dots \overset{\sigma}{\mapsto} A_{i_1} \overset{\sigma}{\mapsto} \\ \dots \overset{\sigma}{\mapsto} A_{i_p} \overset{\sigma}{\mapsto} H_{j_1} \overset{\sigma}{\mapsto} H_{j_2} \overset{\sigma}{\mapsto} \dots \overset{\sigma}{\mapsto} H_{j_q} = H. \end{aligned} \tag{4.13}$$

Since $H_{j_k} \in \mathcal{H} \setminus \mathcal{A}$ ($k = 1, 2, \dots, q$), it follows from Claim 4.14 that

$$A_{i_p} = H_{j_1} = H_{j_2} = \dots = H_{j_q} = H.$$

In particular, $A_{i_p} = H$. This confirms that H coincides with a factor A_{i_p} of $\mathcal{A}$. Note that, in general, $A_{i_p} \neq A$. □

Suppose that under the permutation‹σ, a factor A of $\mathcal{A}$ is replaced with a factor H of $\mathcal{H}$. Then $\mathcal{H}$ is a factor of $\mathcal{A}'$ and any factor of $\mathcal{A}'$ arises in this way. This is symbolically expressed as

$$\mathcal{A} = \dots A \dots \overset{\sigma}{\longmapsto} \mathcal{A}' = \dots H \dots . \tag{4.14}$$

Let us note that any factor of $\mathcal{A}'$ is equal to some factor of $\mathcal{A}$, indeed in (4.14):

- If $H \in \mathcal{A}$, this is trivial.
- If $H \in \mathcal{H} \setminus \mathcal{A}$, the factor H of $\mathcal{A}'$ is equal to a factor $A_{i_p} \in \mathcal{A}$ appeared in the proof of Claim 4.15.

The following statement is thus obtained.

Claim 4.16 *Any factor of $\mathcal{A}'$ is equal to a factor of $\mathcal{A}$.*

We shall modify the mapping (4.14) in the case that $H \in \mathcal{H} \setminus \mathcal{A}$. Note first that since $H = A_{i_p}$ (as in the proof of Claim 4.15), replacing H in $\mathcal{A}'$ with A_{i_p} does not change $\mathcal{A}'$, so (4.14) is modified as

$$\mathcal{A} = \dots A \dots \longmapsto \mathcal{A}' = \dots A_{i_p} \dots \tag{4.15}$$

which is however not a mapping by σ. Based on the above observation, we define a map τ from $\mathcal{A}$ to $\mathcal{A}'$ by

$$\tau : A \mapsto \begin{cases} A_{i_p} & \text{if } \sigma \text{ replaces } A \text{ with } H \in \mathcal{H} \setminus \mathcal{A}, \\ H & \text{if } \sigma \text{ replaces } A \text{ with } H \in \mathcal{A}. \end{cases} \tag{4.16}$$

To uniformly treat the two cases in (4.16), we consider the orbit $\mathcal{O}_A$ of A under σ, which coincides with $\mathcal{O}_H$ as $A \in \mathcal{O}_H$, so we keep the notation $A_{i_1}, A_{i_2}, \dots, A_{i_p}$ in (4.13). Then the orbital sequence starting from A is of the form:

$$A_{i_0} = A \overset{\sigma}{\mapsto} \dots \overset{\sigma}{\mapsto} A_{i_1} \overset{\sigma}{\mapsto} \dots \overset{\sigma}{\mapsto} \dots \overset{\sigma}{\mapsto} A_{i_p} \overset{\sigma}{\mapsto} \dots \overset{\sigma}{\mapsto} A_{i_0} = A. \tag{4.17}$$

The $\mathcal{A}$-factor A_{i_p} appearing in this sequence is used in the first case of (4.16). For brevity we set $B := A_{i_p}$. Formally, note that B is defined as the $\mathcal{A}$-factor such that $\sigma^a(A) = B$ with $a\,(< m)$ the maximal positive integer satisfying this. Alternatively, B is defined as the $\mathcal{A}$-factor such that $\sigma^b(B) = A$ with b minimal positive integer satisfying this. Observe that $b = m - a$.

Definition 4.17 Noting that B is uniquely defined by A, it is denoted by B_A, and called the *ancestor* of A. The positive integer b in $\sigma^b(B) = A$ is called the *ancestor number* of A.

Example 4.18 In the first case of (4.16), the ancestor B_A is A_{i_p}, while in the second case, B_A is H itself.

Claim 4.19 *The following statements are satisfied*:

(1) *For $A, A' \in \mathcal{A}$ $(A \neq A')$, we have $B_A \neq B_{A'}$.*

(2) *The map $\tau : A \in \mathcal{A} \mapsto B_A \in \mathcal{A}'$ is bijective, so a permutation — thus $\mathcal{A}'$ is a permutation of $\mathcal{A}$.*

Proof — (1) Let b and b' be the ancestor numbers of B_A and $B_{A'}$. Hence,

$$\sigma^b(B_A) = A \quad \text{and} \quad \sigma^{b'}(B_{A'}) = A'.$$

Now, supposing that $B_A = B_{A'}$, we deduce a contradiction. Set

$$B := B_A = B_{A'},$$

then

$$\sigma^b(B) = A \qquad \text{and} \qquad \sigma^{b'}(B) = A'.$$

Here $b \neq b'$ because $A \neq A'$. Without loss of generality, we may assume that $b < b'$. Then

$$\sigma^{b'-b}(A) = A'$$

from $\sigma^b(B) = A$ and $\sigma^{b'}(B) = A'$. Here $b' - b < b'$, therefore B with $\sigma^{b'}(B) = A'$ is not the ancestor of A'. This is a contradiction.

(2) By (1), the map

$$\tau : A \in \mathcal{A} \mapsto B_A \in \mathcal{A}'$$

is injective. Here the lengths of $\mathcal{A}$ and $\mathcal{A}'$ are equal by assumption, i.e. the numbers of factors of $\mathcal{A}$ and $\mathcal{A}'$ are equal. Thus it follows that the injection $\tau : \mathcal{A} \to \mathcal{A}'$ is necessarily bijective. □

We now define $\tau : \mathcal{A} \to \mathcal{A}'$ in (4.16) without separating into two cases as follows:

$$\tau : A \mapsto B_A \text{ (the ancestor of } A). \tag{4.18}$$

By Claim 4.19 (2), this is a permutation and the following assertion is confirmed.

Claim 4.20 *The subset $\mathcal{A}'$ is a permutation of $\mathcal{A}$ via τ (caution: As seen from the construction, τ is generally not a restriction of σ).*

Remark 4.21 Among the orbit

$$A, \sigma(A), \sigma^2(A), \ldots, \sigma^{m-1}(A)$$

of $A \in \mathcal{A}$, the $\mathcal{A}$-factors

$$A_{i_0} = A, A_{i_1}, A_{i_2}, \ldots, A_{i_p},$$

appearing in this order as in (4.17), are called the $\mathcal{A}$-*orbit* $\mathcal{P}$ of A. Say $A_{i_k} = \sigma^{n_k}(A)$ $(k = 0, 1, \ldots, p)$, where

$$n_0 = 0 < n_1 < n_2 < \ldots < n_p < m.$$

Note that A_{i_k} $(k = 0, 1, \ldots, p-1)$ is the ancestor of $A_{i_{k+1}}$, and A_{i_p} is the ancestor of A_{i_0}. Indeed, setting

$$b_k := n_{k+1} - n_k \quad \text{and} \quad b_p := m - n_p \quad (k = 0, 1, \ldots, p-1)$$

then

$$A_{i_{k+1}} = \sigma^{b_k}(A_{i_k}) \quad \text{and} \quad A_{i_0} = \sigma^{b_p}(A_{i_p}) \quad (k = 0, 1, \ldots, p-1)$$

such that each b_k $(k = 0, 1, \ldots, p)$ is the minimum satisfying this. Thus from the definition of τ,

$$\tau : \begin{cases} A_{i_k} \mapsto A_{i_{k+1}} \ (k = 0, 1, \ldots, p-1), \\ A_{i_p} \mapsto A_{i_0}. \end{cases} \tag{4.19}$$

Namely, τ cyclically permutes $A_{i_0} = A, A_{i_1}, A_{i_2}, \ldots, A_{i_p}$, which is expressed by a cyclic permutation

$$\tau_{\mathcal{P}} := \begin{pmatrix} A_{i_0} & A_{i_1} & A_{i_2} & \cdots & A_{i_p} \\ A_{i_1} & A_{i_2} & A_{i_3} & \cdots & A_{i_0} \end{pmatrix}. \tag{4.20}$$

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_u$ be the $\mathcal{A}$-orbits. Then $\mathcal{A}$ is their disjoint union. Thus, letting $\tau_{\mathcal{P}_j}$ $(j = 1, 2, \ldots, u)$ be the cyclic permutation associated with $\mathcal{P}_j$ (see (4.20)), then $\tau = \tau_{\mathcal{P}_1}\tau_{\mathcal{P}_2}\ldots\tau_{\mathcal{P}_u}$.

For the subsequent application, we formalize Claim 4.20 as follows.

Lemma 4.22 *Let $\mathcal{H} := H_1H_2\ldots H_u\mathcal{A}H_vH_{v+1}\ldots H_n$ be a sect with a subsect $\mathcal{A}$. Let σ be a permutation of the factors of $\mathcal{H}$ that transforms $\mathcal{H}$ to a sect of the form*

$$\mathcal{H}' := H_1H_2\ldots H_u\mathcal{A}'H_vH_{v+1}\ldots H_n,$$

where $\mathcal{A}'$ is a subsect of the same length as $\mathcal{A}$. Then $\mathcal{A}'$ is a permutation of $\mathcal{A}$.

Remark 4.23 Let us observe that if the factors of $\mathcal{H}$ are not distinct — even if the factors of $\mathcal{A}$ are distinct — this permutation is not necessarily a restriction of σ (see Example 4.12).

Corollary 4.24 *Let $\mathcal{H} := H_1H_2\dots H_u\mathcal{A}H_vH_{v+1}\dots H_n$ be a sect with a subsect $\mathcal{A}$ and let $\mathcal{H}' := H_1H_2\dots H_u\mathcal{A}'H_vH_{v+1}\dots H_n$ be a sect with a subsect $\mathcal{A}'$ of the same length as $\mathcal{A}$. If there exists a mutation between $\mathcal{H}$ and $\mathcal{H}'$, then $\mathcal{A}'$ is a permutation of $\mathcal{A}$.*

Proof — Let σ be the permutation of the factors of $\mathcal{H}$ induced by the mutation in the assertion. Then apply Lemma 4.22. □

Corollary 4.24 implies that if $\mathcal{A}'$ is not a permutation of $\mathcal{A}$, then there exists no mutation between $\mathcal{H}$ and $\mathcal{H}'$. Namely the following result holds.

Proposition 4.25 *Let $\mathcal{H} := H_1H_2\dots H_u\mathcal{A}H_vH_{v+1}\dots H_n$ be a sect with a subsect $\mathcal{A}$ and let $\mathcal{H}' := H_1H_2\dots H_u\mathcal{A}'H_vH_{v+1}\dots H_n$ be a sect with a subsect $\mathcal{A}'$ of the same length as $\mathcal{A}$. If $\mathcal{A}'$ is not a permutation of $\mathcal{A}$, then there exists no mutation between $\mathcal{H}$ and $\mathcal{H}'$; consequently, $\mathcal{H}$ and $\mathcal{H}'$ are not equivalent.*

So far we have considered a sect $\mathcal{H}'$ that replaces a subsect $\mathcal{A}$ in a sect $\mathcal{H}$ with a sect $\mathcal{A}'$ of the same length as $\mathcal{A}$. More generally we consider a sect that replaces $\mathcal{A}$ in $\mathcal{H}$ with a sect of arbitrary length. To describe it, we need the following result.

Lemma 4.26 *Let $\mathcal{A} := A_1A_2\dots A_k$ be a permutative sect. Suppose that*

$$\mathcal{B} := B_1B_2\dots B_m$$

is an irredundant sect such that $\mathcal{A} = \mathcal{B}$ and $\mathcal{A} \not\sim \mathcal{B}$ (inequivalent; in particular $\mathcal{A} \not\equiv \mathcal{B}$). Then any sect $\overset{\approx}{\mathcal{B}}$ obtained by removing (not all) factors from $\mathcal{B}$ is not a permutation of $\mathcal{A}$; consequently, $\mathcal{A} \not\sim \overset{\approx}{\mathcal{B}}$.

Proof — Let us proceed by showing the statement by contradiction. If $\overset{\approx}{\mathcal{B}}$ is a permutation of $\mathcal{A}$, then $A_1, A_2, \dots, A_k$ appear in $\mathcal{B}$, say in the order $A_{i_1}, A_{i_2}, \dots, A_{i_k}$. Then $\mathcal{B}$ is of the form

$$\mathcal{B} = \dots A_{i_1} \dots A_{i_2} \dots A_{i_k} \dots .$$

On the other hand, as $\mathcal{A}$ is permutative, we have $\mathcal{A} = A_{i_1}A_{i_2}\dots A_{i_k}$. From the assumption $\mathcal{A} = \mathcal{B}$, we have

$$A_{i_1}A_{i_2}\dots A_{i_k} = \dots A_{i_1} \dots A_{i_2} \dots A_{i_k} \dots . \tag{4.21}$$

This implies that removing factors from $\mathcal{B}$ other than $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ yields $\mathcal{A}$. Therefore $\mathcal{B}$ is redundant, which contradicts the assumption. □

We now give a key result on equivalences between irredundant sects.

Proposition 4.27 *Let $\mathcal{H} := H_1 H_2 \ldots H_u \mathcal{A} H_v H_{v+1} \ldots H_n$ be an irredundant sect with a permutative subsect $\mathcal{A}$ (which is irredundant by* Lemma 4.2*). Let $\mathcal{B}$ be a sect of the arbitrary length such that $\mathcal{A} = \mathcal{B}$ and $\mathcal{A} \not\sim \mathcal{B}$, and consider a sect*

$$\mathcal{H}' := H_1 H_2 \ldots H_u \mathcal{B} H_v H_{v+1} \ldots H_n.$$

Let $\mathcal{H}'_{\mathrm{irred}}$ be an irredundantization of $\mathcal{H}'$. Then the following are satisfied:

(1) $\mathcal{H} = \mathcal{H}'_{\mathrm{irred}}$.

(2) $\mathcal{H} \not\sim \mathcal{H}'_{\mathrm{irred}}$.

Proof — (1) follows from $\mathcal{H} = \mathcal{H}'$ and $\mathcal{H}' = \mathcal{H}'_{\mathrm{irred}}$.

(2) First note that $\mathcal{H}'_{\mathrm{irred}}$ is of the form

$$H_1 H_2 \ldots H_u \overset{\approx}{\mathcal{B}} H_v H_{v+1} \ldots H_n,$$

where $\overset{\approx}{\mathcal{B}}$ is a sect obtained by removing some (not all) factors from $\mathcal{B}$ (see Lemma 4.7). Note also that, since $\mathcal{H}'_{\mathrm{irred}}$ is irredundant, its subsect $\overset{\approx}{\mathcal{B}}$ is irredundant (see Lemma 4.2). We separate into two cases depending on whether the lengths of $\mathcal{A}$ and $\overset{\approx}{\mathcal{B}}$ are equal or not.

Case 1 $\mathrm{length}(\mathcal{A}) \neq \mathrm{length}(\overset{\approx}{\mathcal{B}})$.

Then $\mathrm{length}(\mathcal{H}) \neq \mathrm{length}(\mathcal{H}'_{\mathrm{irred}})$, so there exists no mutation between $\mathcal{H}$ and $\mathcal{H}'_{\mathrm{irred}}$ (recall that a mutation preserves lengths of sects). Thus $\mathcal{H} \not\sim \mathcal{H}'_{\mathrm{irred}}$.

Case 2 $\mathrm{length}(\mathcal{A}) = \mathrm{length}(\overset{\approx}{\mathcal{B}})$.

Since $\mathcal{A}$ is permutative, in this case we have that $\mathcal{A} \not\sim \overset{\approx}{\mathcal{B}}$ by Lemma 4.26. Then, $\mathcal{H} \not\sim \mathcal{H}'_{\mathrm{irred}}$ by Proposition 4.25.

Consequently, the assertion is established. □

We now arrive at the following classification result.

Theorem 4.28 *Let $\mathcal{H}$ be an irredundant sect with a permutative subsect $\mathcal{A}$, and let $\mathcal{H}'$ be a sect obtained by replacing $\mathcal{A}$ in $\mathcal{H}$ with a sect $\mathcal{B}$. Next, let $\mathcal{H}'_{\mathrm{irred}}$ be an irredundantization of $\mathcal{H}'$, which is obtained by replacing $\mathcal{A}$ in $\mathcal{H}$ with a sect $\overset{\vee}{\mathcal{B}}$ removing some (not all) factors from $\mathcal{B}$ (see* Lemma 4.7*). Then whether the irredundant sects $\mathcal{H}'_{\mathrm{irred}}$ and $\mathcal{H}$ are equivalent or not is classified as follows*:

(a) *If* $\mathrm{length}(\mathcal{H}'_{\mathrm{irred}}) \neq \mathrm{length}(\mathcal{H})$ (*or equiv.* $\mathrm{length}\overset{\vee}{\mathcal{B}} \neq \mathrm{length}(\mathcal{A})$), *then $\mathcal{H}$ and $\mathcal{H}'_{\mathrm{irred}}$ are not equivalent.*

(b) *If* $\mathrm{length}(\mathcal{H}'_{\mathrm{irred}}) = \mathrm{length}(\mathcal{H})$, *then there are two cases*:

(b.1) *If $\mathcal{H}'_{\mathrm{irred}}$ is obtained from $\mathcal{H}$ by permuting only the factors of $\mathcal{A}$, then $\mathcal{H}'_{\mathrm{irred}}$ and $\mathcal{H}$ are equivalent.*

(b.2) *Otherwise $\mathcal{H}'_{\mathrm{irred}}$ and $\mathcal{H}$ are not equivalent.*

Proof — (a) and (b.2) were already shown in Proposition 4.27. Let us show (b.1). As $\mathcal{A}$ is permutative, the permutation in (a.1) is a mutation of $\mathcal{A}$. Extend this to the whole of $\mathcal{H}$ trivially, i.e. as the identity map on the outside $\mathcal{H} \setminus \mathcal{A}$. The resulting mutation on $\mathcal{H}$ gives an equivalence between $\mathcal{H}$ and $\mathcal{H}'_{\mathrm{irred}}$. □

4.2 Construction of inequivalent factorizations by replacement

In Section 4.1, given a sect we studied (in)equivalence between it and a sect obtained by replacing its subsect with another sect. In this section, given a prime factorization of a group we study (in)equivalence between it and a prime factorization of the group obtained by replacing a subsect in the factorization with another sect. We begin with preparation.

Definition 4.29 A sect $H_1H_2 \dots H_n$ is called an *atomic sect* (resp. a *primary sect*) if every factor H_i ($i = 1, 2, \dots, H_n$) is an atomic group (resp. a primary group, i.e. a cyclic group of prime power order). If $H_1H_2 \dots H_n$ is atomic and irredundant, then $H_1H_2 \dots H_n$ is called a *prime sect*.

By definition, any subsect of an atomic (resp. primary) sect is also atomic (resp. primary). On the other hand, any subsect of an irredundant sect is also irredundant (Lemma 4.2). In partiular the following holds:

Lemma 4.30 *Any subsect of a prime sect is a prime sect.*

Note next the following result.

Lemma 4.31 *Let* $\mathcal{H} := H_1H_2 \dots H_u\mathcal{A}H_vH_{v+1} \dots H_n$ *be an atomic* (resp. *primary*) *sect with a subsect* $\mathcal{A}$*. Then for any atomic* (resp. *primary*) *sect* $\mathcal{B}$*, a sect* $\mathcal{H}' := H_1H_2 \dots H_u\mathcal{B}H_vH_{v+1} \dots H_n$ *is atomic* (resp. *primary*)*.*

Proof — By assumption, the factors of

$$H_1H_2 \dots H_u,\ \mathcal{B}, \quad \text{and} \quad H_vH_{v+1} \dots H_n$$

are atomic (resp. primary), thus $\mathcal{H}'$ is atomic (resp. primary). □

Definition 4.32 A sect is *prime* if it is irredundant and atomic. For example, an irredundant primary sect is prime.

Lemma 4.33 *Let* $\mathcal{H} := H_1H_2 \dots H_n$ *be an atomic* (resp. *primary*) *sect. Then any irredundantization* $\mathcal{H}_{\text{irred}}$ *of* $\mathcal{H}$ *is irredundant and atomic* (resp. *primary*)*. Namely,* $\mathcal{H}_{\text{irred}}$ *is prime.*

Proof — By definition, $\mathcal{H}_{\text{irred}}$ is irredundant. It is also atomic (resp. primary) because it is obtained by removing factors from the atomic (resp. primary) sect $\mathcal{H}$. □

Corollary 4.34 *Let* $\mathcal{H} := H_1H_2 \dots H_u\mathcal{A}H_vH_{v+1} \dots H_n$ *be an atomic* (resp. *primary*) *sect with a subsect* $\mathcal{A}$*. Then for any atomic* (resp. *primary*) *sect* $\mathcal{B}$*, where possibly* $\mathcal{A} \neq \mathcal{B}$*, the following hold*:

(1) *A sect* $\mathcal{H}' := H_1H_2 \dots H_u\mathcal{B}H_vH_{v+1} \dots H_n$ *is atomic* (resp. *primary*)*.*

(2) *Any irredundantization* $\mathcal{H}'_{\text{irred}}$ *of* $\mathcal{H}'$ *is prime.*

Proof — By Lemma 4.31, $\mathcal{H}'$ is atomic (resp. primary), so (1) is confirmed. Then by Lemma 4.33, its irredundantization $\mathcal{H}'_{\text{irred}}$ is prime, so (2) is confirmed. □

If the factors of a sect are a permutation of the factors of $\mathcal{A}$, then the sect is said to be a permutation of $\mathcal{A}$. Such a sect has the same length as $\mathcal{A}$.

Proposition 4.35 *Let* $\mathcal{H} := H_1H_2 \dots H_u\mathcal{A}H_vH_{v+1} \dots H_n$ *be an atomic* (resp. *primary*) *sect with an irredundant subsect* $\mathcal{A}$*. Let* $\mathcal{B}$ *be an atomic* (resp. *primary*) *irredundant sect of the same length as* $\mathcal{A}$ *such that* $\mathcal{B}$ *is* not *a permutation of* $\mathcal{A}$*. Then the following are satisfied*:

(1) *A sect* $\mathcal{H}' := H_1H_2 \dots H_u\mathcal{B}H_vH_{v+1} \dots H_n$ *is not equivalent to* $\mathcal{H}$. (*Note*: $\mathcal{H}'$ *is atomic* (*resp. primary*) *by* Lemma 4.31.)

(2) *If* $\mathcal{A} = \mathcal{B}$ *and* $\mathcal{H}$ *is irredundant, then* $\mathcal{H}'_{\text{irred}}$ *is not equivalent to* $\mathcal{H}$. (*Note*: *From* $\mathcal{A} = \mathcal{B}$, *we have* $\mathcal{H} = \mathcal{H}' = \mathcal{H}'_{\text{irred}}$ *by* Lemma 4.6.)

Proof — (1) Since $\text{length}(\mathcal{A}) = \text{length}(\mathcal{B})$ and $\mathcal{B}$ is not a permutation of $\mathcal{A}$, the sect $\mathcal{H}'$ is not equivalent to $\mathcal{H}$ by Proposition 4.25.

(2) Since $\text{length}(\mathcal{A}) = \text{length}(\mathcal{B})$ and $\mathcal{B}$ is not a permutation of $\mathcal{A}$, there exists no mutation between $\mathcal{A}$ and $\mathcal{B}$, so $\mathcal{A}$ is not equivalent to $\mathcal{B}$, i.e. $\mathcal{A} \not\sim \mathcal{B}$. This together with the assumption that $\mathcal{A} = \mathcal{B}$ and $\mathcal{H}$ is irredundant implies that $\mathcal{H}$ is not equivalent to $\mathcal{H}'_{\text{irred}}$ by Proposition 4.27 (2). □

Before proceeding, note the following result.

Lemma 4.36 *If a sect* $H_1H_2 \dots H_n$ *is prime, then any subsect*

$$H_iH_{i+1} \dots H_j \qquad (1 \leqslant i < j \leqslant n)$$

is also prime.

Proof — Since $H_iH_{i+1} \dots H_j$ is a subsect of irredundant $H_1H_2 \dots H_n$, it is irredundant (see Lemma 4.2). Since $H_1H_2 \dots H_n$ is prime, every factor is atomic, in particular every factor of $H_iH_{i+1} \dots H_j$ is atomic. This confirms that $H_iH_{i+1} \dots H_j$ is prime. □

Note that in the following, the length of $\mathcal{B}$ is not necessarily equal to that of $\mathcal{A}$, while $\mathcal{A}$ is assumed to be permutative.

Lemma 4.37 *Let* $\mathcal{H} := H_1H_2 \dots \mathcal{A} \dots H_n$ *be a prime sect with a permutative subsect* $\mathcal{A}$. *Suppose that there exists a prime sect* $\mathcal{B}$ *such that* $\mathcal{A} = \mathcal{B}$ *and* $\mathcal{A} \not\sim \mathcal{B}$. *Set*

$$\mathcal{H}' := H_1H_2 \dots \mathcal{B} \dots H_n$$

and let $\mathcal{H}'_{\text{irred}}$ *be an irredundantization of* $\mathcal{H}'$. *Then the following are satisfied*:

(1) $\mathcal{H}'_{\text{irred}}$ *is prime.*

(2) $\mathcal{H} \not\sim \mathcal{H}'_{\text{irred}}$.

Proof — (1) follows from Corollary 4.34 (2), where the permutativity of $\mathcal{A}$ is unnecessary. (2) follows from Proposition 4.27 (2). □

Let us now state the multi-version of Lemma 4.37.

Corollary 4.38 *Let* $\mathcal{H} := H_1H_2 \dots \mathcal{A} \dots H_n$ *be a prime sect with a permutative subsect* $\mathcal{A}$*. Suppose that there exist prime sects* $\mathcal{B}_i$ $(i = 1, 2, \dots, k)$ *such that* $\mathcal{A} = \mathcal{B}_1 = \mathcal{B}_2 = \dots = \mathcal{B}_k$ *but they are mutually inequivalent. Set* $\mathcal{H}'_i := H_1H_2 \dots \mathcal{B}_i \dots H_n$ *and let* $\mathcal{H}'_{i,\mathrm{irred}}$ *be an irredundantization of* $\mathcal{H}'_i$*. Then the following are satisfied*:

(1) $\mathcal{H}'_{i,\mathrm{irred}}$ *is prime.*

(2) $\mathcal{H}, \mathcal{H}'_{1,\mathrm{irred}}, \mathcal{H}'_{2,\mathrm{irred}}, \dots, \mathcal{H}'_{k,\mathrm{irred}}$ *are mutually inequivalent.*

The prime factorization version of Corollary 4.38 is stated as follows.

Corollary 4.39 *Let* $G = H_1H_2 \dots \mathcal{A} \dots H_n$ *be a prime factorization of a group* G *with a permutative subsect* $\mathcal{A}$*. Suppose that there exist prime sects* $\mathcal{B}_i$ $(i = 1, 2, \dots, k)$ *such that* $\mathcal{A} = \mathcal{B}_1 = \mathcal{B}_2 = \dots = \mathcal{B}_k$ *but these are mutually inequivalent. Set*

$$\mathcal{H} := H_1H_2 \dots \mathcal{A} \dots H_n \quad \textit{and} \quad \mathcal{H}'_i := H_1H_2 \dots \mathcal{B}_i \dots H_n,$$

and let $\mathcal{H}'_{i,\mathrm{irred}}$ *be an irredundantization of* $\mathcal{H}'_i$*. Then*

$$G = \mathcal{H},\ G = \mathcal{H}'_{1,\mathrm{irred}},\ G = \mathcal{H}'_{2,\mathrm{irred}}, \dots, G = \mathcal{H}'_{k,\mathrm{irred}}$$

are mutually inequivalent prime factorizations.

For the subsequent discussion, the following is needed.

Lemma 4.40 *Let* $\mathcal{H} := H_1H_2 \dots H_u\mathcal{A}H_vH_{v+1} \dots H_n$ *be a sect with a subsect* $\mathcal{A}$*. Suppose that a permutation of the factors of* $\mathcal{H}$ *transforms* $\mathcal{H}$ *to a sect of the form* $\mathcal{H}' := H_1H_2 \dots H_u\mathcal{A}'H_vH_{v+1} \dots H_n$*, where* $\mathcal{A}'$ *is a subsect of the same length as* $\mathcal{A}$*. If* $\mathcal{A}$ *is permutative and irredundant and* $\mathcal{A}'$ *is irredundant, then the following hold*:

(1) *The factors of* $\mathcal{A}$ *are distinct.*

(2) $\mathcal{A}'$ *is permutative and the factors of* $\mathcal{A}'$ *are distinct.*

Proof — (1) Since $\mathcal{A}$ is permutative and irredundant, the factors of $\mathcal{A}$ are distinct by Lemma 4.1 (2).

(2) Any factor of $\mathcal{A}'$ is equal to some factor of $\mathcal{A}$ (see Claim 4.16). Here the factors of $\mathcal{A}$ are permutable with each other, and consequently the factors of $\mathcal{A}'$ are permutable with each other. Hence $\mathcal{A}'$ is permutative. Besides, $\mathcal{A}'$ is irredundant by assumption. Hence the factors of $\mathcal{A}'$ are distinct by Lemma 4.1 (2). □

Note next the following result.

Lemma 4.41 *Let $\mathcal{A} := A_1 A_2 \dots A_k$ be a permutative prime sect — so $\mathcal{A}$ may be regarded as a group* (see Lemma 4.1 (1)) *for which $A_1 A_2 \dots A_k$ is a permutative prime factorization. Suppose that there exists another prime sect $\mathcal{B}$ that gives an inequivalent prime factorization of the group $\mathcal{A}$,* i.e.

$$\mathcal{A} = \mathcal{B} \quad \textit{and} \quad \mathcal{A} \not\sim \mathcal{B}.$$

Then any sect $\overset{\approx}{\mathcal{B}}$ obtained by removing (*not all*) *factors from $\mathcal{B}$ is not a permutation of $A_1 A_2 \dots A_k$.*

Proof — First since $\mathcal{A}$ and $\mathcal{B}$ are factorizations of the same group, say K, we have $\mathcal{A} = \mathcal{B}\,(= K)$. Then the assertion is confirmed by contradiction. Suppose that $\overset{\approx}{\mathcal{B}}$ is a permutation of $A_1 A_2 \dots A_k$. Then we have that $A_1, A_2, \dots, A_k$ appear in $\mathcal{B}$, say in the order $A_{i_1}, A_{i_2}, \dots, A_{i_k}$. Hence $\mathcal{B}$ is of the form

$$\mathcal{B} = \dots A_{i_1} \dots A_{i_2} \dots A_{i_k} \dots \ .$$

On the other hand, as $\mathcal{A}$ is permutative, we have

$$\mathcal{A} = A_{i_1} A_{i_2} \dots A_{i_k}. \tag{4.22}$$

Since $\mathcal{A} = \mathcal{B}$, we have

$$A_{i_1} A_{i_2} \dots A_{i_k} = \dots A_{i_1} \dots A_{i_2} \dots A_{i_k} \dots \ . \tag{4.23}$$

This implies that $\mathcal{A}$ is obtained by removing factors from $\mathcal{B}$ other than $A_{i_1}, A_{i_2}, \dots, A_{i_k}$. Hence $\mathcal{B}$ is redundant, which contradicts the irredundancy of the prime sect $\mathcal{B}$. □

We may regard a permutative sect as a group (see Lemma 4.1 (1)); this group is called the *underlying group* of the sect. If the underlying group is (non)cyclic, then the permutative sect is said to be (non)cyclic. In the subsequent construction, we use noncyclic permutative *finite* sects, for which the underlying groups are noncyclic finite groups.

Lemma 4.42 *If an irredundant sect $\mathcal{H} := H_1 H_2 \dots H_n$ contains a noncyclic permutative* finite *subsect $\mathcal{A}$ of length greater than one, then the following hold*:

(1) *There exists a prime sect* $\mathcal{B}$ *such that* $\mathcal{A} = \mathcal{B}$ *and* $\mathcal{A} \not\sim \mathcal{B}$ (*not equivalent, in particular,* $\mathcal{A} \not\equiv \mathcal{B}$ (*not identical*)).

(2) *Let* $\mathcal{H}'$ *be a sect obtained by replacing* $\mathcal{A}$ *in* $\mathcal{H}$ *with* $\mathcal{B}$ *in (1), and let* $\mathcal{H}'_{\text{irred}}$ *be an irredundantization of* $\mathcal{H}'$ *which is obtained by replacing* $\mathcal{B}$ *in* $\mathcal{H}'$ *with a sect* $\overset{\curlyvee}{\mathcal{B}}$ *obtained from* $\mathcal{B}$ *by removing some* (*not all*) *factors* (*see* Lemma 4.7). *Then the following are satisfied*:

(2.a) $\overset{\curlyvee}{\mathcal{B}}$ *is not a permutation of* $\mathcal{A}$.

(2.b) *The irredundant sect* $\mathcal{H}'_{\text{irred}}$ *is not equivalent to* $\mathcal{H}$.

Proof — (1) By Theorem 3.12.

(2.a) By Lemma 4.26.

(2.b) Write $\mathcal{H} = H_1 H_2 \dots H_u \mathcal{A} H_v \dots H_n$. Then

$$\mathcal{H}' = H_1 H_2 \dots H_u \mathcal{B} H_v \dots H_n$$

and its irredundantization is of the form

$$\mathcal{H}'_{\text{irred}} = H_1 H_2 \dots H_u \overset{\curlyvee}{\mathcal{B}} H_v \dots H_n$$

(see Lemma 4.7). Here by (2.a), $\overset{\curlyvee}{\mathcal{B}}$ is not a permutation of $\mathcal{A}$. Hence the assertion follows from Proposition 4.27 (2). □

For later application, we extract (2.b) of Lemma 4.42 as follows.

Proposition 4.43 *If an irredundant sect* $\mathcal{H} := H_1 H_2 \dots H_n$ *contains a noncyclic permutative* finite *subsect* $\mathcal{A}$ *of length greater than one, then there exists an irredundant sect* $\mathcal{K}$ *such that* $\mathcal{H} = \mathcal{K}$ *and* $\mathcal{H} \not\sim \mathcal{K}$ (*explicitly* $\mathcal{K}$ *is given by* $\mathcal{H}'_{\text{irred}}$ *in* (2.b) *of* Lemma 4.42).

In Proposition 4.43, in addition to $\mathcal{H}$ being irredundant, if moreover $\mathcal{H}$ is atomic, then $\mathcal{H}'$ is also atomic, accordingly its irredundantization $\mathcal{H}'_{\text{irred}}$ is atomic. Therefore, in Proposition 4.43, if $\mathcal{H}$ is a prime sect, then $\mathcal{H}'_{\text{irred}}$ is also a prime sect. Thus Proposition 4.43 is sharpened as follows.

Proposition 4.44 *If a prime sect* $\mathcal{H} := H_1 H_2 \dots H_n$ *contains a noncyclic permutative* finite *subsect of length greater than one, then there exists a prime sect* $\mathcal{K}$ *such that* $\mathcal{H} = \mathcal{K}$ *and* $\mathcal{H} \not\sim \mathcal{K}$. (*Explicitly* $\mathcal{K}$ *is given by* $\mathcal{H}'_{\text{irred}}$ *in* (2.b) *of* Lemma 4.42).

Applying Proposition 4.44 to a prime *factorization* yields the following result.

Theorem 4.45 *Let* $G = H_1H_2 \dots \mathcal{A} \dots H_n$ *be a prime factorization of a group* G *such that* $\mathcal{A}$ *is a noncyclic permutative* finite *subsect of length greater than one. Then there exists a prime sect* $\mathcal{B}$ *such that* $\mathcal{A} = \mathcal{B}$ *and* $\mathcal{A} \not\sim \mathcal{B}$*, and* G *admits an inequivalent prime factorization* (*which is an irredundantization of a factorization* $G = H_1H_2 \dots \mathcal{B} \dots H_n$).

The multi-version of Theorem 4.45 is the following (see Corollary 4.39).

Corollary 4.46 *Let* $G = H_1H_2 \dots \mathcal{A} \dots H_n$ *be a prime factorization of a group* G *such that* $\mathcal{A}$ *is a noncyclic permutative* finite *subsect of length greater than one. Let* $\mathcal{B}_i$ $(i = 1, 2, \dots, k)$ *be mutually inequivalent prime sects such that* $\mathcal{A} = \mathcal{B}_i$ *and* $\mathcal{A} \not\sim \mathcal{B}_i$*. Set*

$$\mathcal{H} := H_1H_2 \dots \mathcal{A} \dots H_n \quad \textit{and} \quad \mathcal{H}'_i := H_1H_2 \dots \mathcal{B}_i \dots H_n,$$

and let $\mathcal{H}'_{i,\mathrm{irred}}$ *be an irredundantization of* $\mathcal{H}'_i$*. Then*

$$G = \mathcal{H},\ G = \mathcal{H}'_{1,\mathrm{irred}},\ G = \mathcal{H}'_{2,\mathrm{irred}},\ \dots,\ G = \mathcal{H}'_{k,\mathrm{irred}}$$

are mutually inequivalent prime factorizations of G.

Remark 4.47 In Corollary 4.46, when $\mathcal{A}$ is regarded as a group, note the following:

(1) The equations $\mathcal{A} = \mathcal{B}_i$ $(i = 1, 2, \dots, k)$ give mutually inequivalent prime factorizations of $\mathcal{A}$.

(2) If $\mathcal{A}$ is *finite noncyclic abelian*, then $\mathcal{A}$ generally admits many inequivalent prime factorizations (for example, see Lemma 2.6), accordingly from Corollary 4.46, G admits many inequivalent prime factorizations.

5 Construction of inequivalent factorizations by end conjugation

In Section 4.2, we constructed from (resp. prime) factorizations of groups, their inequivalent (resp. prime) factorizations by replacing

subsects in the factorizations with other sects. In this section, we construct other inequivalent (resp. prime) factorizations by means of *partial conjugations* of subsects. We first prepare terminology. For a factorization

$$G = H_1 H_2 \dots H_n$$

of a group G, its *conjugate* by $g \in G$ is

$$gGg^{-1} = gH_1 H_2 \dots H_n g^{-1},$$

that is,

$$G = (gH_1 g^{-1})(gH_2 g^{-1}) \dots (gH_n g^{-1}).$$

Two factorizations

$$G = H_1 H_2 \dots H_n \quad \text{and} \quad G = K_1 K_2 \dots K_n$$

of the same length are *conjugate* if there exists an element g of G such that

$$G = (gH_1 g^{-1})(gH_2 g^{-1}) \dots (gH_n g^{-1})$$

coincides with

$$G = K_1 K_2 \dots K_n,$$

i.e. $gH_i g^{-1} = K_i$ for $i = 1, 2, \dots, n$ (*simultaneous conjugation*). We point out that it is in general hard to check whether conjugate factorizations are inequivalent. To overcome this difficulty, we use *partial* conjugations of factorizations to construct inequivalent factorizations. Here a *partial conjugation* of a sect $H_1 H_2 \dots H_n$ is given by a conjugation of some subsect, i.e. of the form:

$$H_1 H_2 \dots H_k (gH_{k+1} g^{-1})(gH_{k+1} g^{-1}) \dots (gH_l g^{-1}) H_{l+1} \dots H_n. \quad (5.1)$$

In this section, we only use special partial conjugations called *end conjugations* (more general partial conjugations will be used in Section 6). Yet we point out that given a factorization $G = H_1 H_2 \dots H_n$, to construct inequivalent factorizations of G by partial conjugations is actually subtle. In fact, while $G = (gH_1 g^{-1})(gH_2 g^{-1}) \dots (gH_n g^{-1})$ holds for any $g \in G$, in general

$$G \neq H_1 H_2 \dots H_k (gH_{k+1} g^{-1})(gH_{k+1} g^{-1}) \dots (gH_l g^{-1}) H_{l+1} \dots H_n.$$

Moreover even if

$$G = H_1H_2\dots H_k(gH_{k+1}g^{-1})(gH_{k+1}g^{-1})\dots(gH_lg^{-1})H_{l+1}\dots H_n,$$

it may be merely equivalent to $G = H_1H_2\dots H_n$. We thus have to find partial conjugations that produce inequivalent factorizations of G. We begin with preparation.

Definition 5.1 A sect $H_1H_2\dots H_n$ $(n \geqslant 2)$ is *rigid* if no adjacent factors are permutable, i.e.

$$H_iH_{i+1} \neq H_{i+1}H_i$$

for any $i = 1, 2, \dots, n-1$, or equivalently $H_1H_2\dots H_n$ contains no permutative subsect at all.

For a sect $\mathcal{H} := H_1H_2\dots H_n$, its *left fringe* is a subsect H_1H_2 and its *right fringe* is a subsect $H_{n-1}H_n$:

$$\mathcal{H} := \underbrace{H_1H_2}_{\text{left fringe}} \dots \underbrace{H_{n-1}H_n}_{\text{right fringe}}. \tag{5.2}$$

Definition 5.2 A sect $H_1H_2\dots H_n$ $(n \geqslant 2)$ is:

(1) *tight* if its both fringes are nonpermutative, i.e. $H_1H_2 \neq H_2H_1$ (*left tight*) and $H_{n-1}H_n \neq H_nH_{n-1}$ (*right tight*).

(2) *semitight* if at least one of both fringes is nonpermutative, i.e.

$$H_1H_2 \neq H_2H_1 \quad \text{or} \quad H_{n-1}H_n \neq H_nH_{n-1}.$$

Remark 5.3 The above notions are arranged in the strong order as follows:

$$\{\text{rigid sects}\} \subset \{\text{tight sects}\} \subset \{\text{semitight sects}\}. \tag{5.3}$$

The following is often used in our subsequent construction.

Lemma 5.4 *Let* H *and* K *be subgroups of a group* G. *If* $HK \neq KH$, *then the following are satisfied*:

(1) $H \not\subset N_G(K)$.

(2) $K \not\subset N_G(H)$, *where* $N_G(K)$ *and* $N_G(H)$ *denote the normalizers of* K *and* H *in* G.

Proof — (1) Let us prove the claim by contradiction. Indeed, if we assume that $H \subset N_G(K)$, then we obtain that $hK = Kh$ for any $h \in H$. Consequently, $HK = KH$, which contradicts the assumption.

(2) The statement is confirmed similarly to (1). □

Definition 5.5 A factorization $G = H_1H_2 \dots H_n$ of a group G is called *rigid, tight, or semitight* respectively if $H_1H_2 \dots H_n$ is so.

5.1 Construction by end conjugation

We shall show that if a factorization of a group is semitight, then the group admits an inequivalent factorization — in particular, any tight or rigid factorization admits an inequivalent factorization. Let

$$G = H_1H_2 \dots H_n \quad (n \geqslant 2)$$

be a semitight factorization of a group G, i.e.

$$H_1H_2 \neq H_2H_1 \quad \text{or} \quad H_{n-1}H_n \neq H_nH_{n-1}.$$

We consider these two cases separately:

Case 1 $H_{n-1}H_n \neq H_nH_{n-1}$.

Then by Lemma 5.4 (1), $H_{n-1} \not\subset N_G(H_n)$, so we may take

$$g \in H_{n-1} \setminus N_G(H_n).$$

Note the following:

$$gH_ng^{-1} \neq H_n \quad \text{because } g \notin N_G(H_n) \tag{5.4}$$

and

$$H_{n-1}g = H_{n-1} \quad \text{because } g \in H_{n-1}. \tag{5.5}$$

By (5.5),

$$H_1H_2 \dots H_n = H_1H_2 \dots (H_{n-1}g)H_n = H_1H_2 \dots H_{n-1}(gH_ng^{-1})g.$$

Accordingly, $G = H_1H_2 \dots H_n$ is rewritten as

$$G = H_1H_2 \dots H_{n-1}(gH_ng^{-1})g.$$

Multiplying this by g^{-1} from the right yields a factorization

$$G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1}).$$

Note that this construction works only for a factorization but not for a sect — in general

$$H_1 H_2 \dots H_n \neq H_1 H_2 \dots H_{n-1}(gH_n g^{-1}).$$

The factorization

$$G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1})$$

is said to be obtained from $G = H_1 H_2 \dots H_n$ by (right) end conjugation.

Claim 5.6 *The factorization* $G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1})$ *is not equivalent to the factorization* $G = H_1 H_2 \dots H_n$.

Proof — Let us proceed by contradiction to prove the statement. Setting $H'_n := gH_n g^{-1}$, suppose that

$$G = H_1 H_2 \dots H_n \quad \text{and} \quad G = H_1 H_2 \dots H_{n-1} H'_n$$

are equivalent, namely, there exists a mutation from $H_1 H_2 \dots H_n$ to $H_1 H_2 \dots H_{n-1} H'_n$, which permutes the factors of $H_1 H_2 \dots H_n$, say by a permutation $\sigma \in \mathfrak{S}_n$. Then

$$H_{\sigma(1)} H_{\sigma(2)} \dots H_{\sigma(n)} \equiv H_1 H_2 \dots H_{n-1} H'_n \qquad \text{(identical)},$$

that is,

$$\begin{cases} H_{\sigma(i)} = H_i \ \ (i = 1, 2, \dots, n-1), \\ H_{\sigma(n)} = H'_n. \end{cases} \tag{5.6}$$

Now, for the number n, let k be the smallest positive integer such that $\sigma^k(n) = n$ and consider the σ-orbit

$$\sigma(n), \sigma^2(n), \dots, \sigma^k(n) = n.$$

For brevity, set

$$i_1 := \sigma(n),\ i_2 := \sigma(i_1)\,(= \sigma^2(n)),$$
$$i_3 := \sigma(i_2)\,(= \sigma^3(n)),\ \ldots, i_k := \sigma(i_{k-1})\,(= \sigma^k(n) = n).$$

Then from (5.6), we have

$$\begin{cases} \text{(a)}\ \ H_{i_2} = H_{i_1},\ H_{i_3} = H_{i_2}, \ldots, H_n = H_{i_{k-1}}, \\ \text{(b)}\ \ H_{i_1} = H_n'. \end{cases} \tag{5.7}$$

From (a), $H_{i_1} = H_n$. This with (b) gives $H_n = H_n'$, which contradicts (5.4). □

In Claim 5.6, the factorization $G = H_1H_2\ldots H_{n-1}\,(gH_ng^{-1})$ is not necessarily irredundant, but still the following assertion holds.

Claim 5.7 *No irredundantization of* $G = H_1H_2\ldots H_{n-1}\,(gH_ng^{-1})$ *is equivalent to the factorization* $G = H_1H_2\ldots H_n$.

Proof — In the case that $G = H_1H_2\ldots H_{n-1}\,(gH_ng^{-1})$ is irredundant, the assertion is just Claim 5.6. In the case that

$$G = H_1H_2\ldots H_{n-1}\,(gH_ng^{-1})$$

is redundant, the length of its irredundantization is shorter than n, thus the irredundantization is not equivalent to $G = H_1H_2\ldots H_n$, hence the claim follows. □

While $G = H_1H_2\ldots H_n$ is irredundant,

$$G = H_1H_2\ldots H_{n-1}(gH_ng^{-1})$$

is not necessarily irredundant. When it is redundant, we shall investigate which factor is (ir)redundant.

Claim 5.8 *In* $G = H_1H_2\ldots H_{n-1}\,(gH_ng^{-1})$, *then the following are satisfied*:

(1) *The factor* gH_ng^{-1} *is irredundant.*

(2) *In the case* $n \geqslant 3$, *the factors* H_i $(1 \leqslant i \leqslant n-2)$ *are irredundant while* H_{n-1} *is possibly redundant* (*see* Example 5.10 *below*).

Proof — Note first that

$$H_1H_2\dots H_n = H_1H_2\dots H_{n-1}(gH_ng^{-1}) \tag{5.8}$$

as both are equal to G.

(1) If gH_ng^{-1} is redundant in $H_1H_2\dots H_{n-1}$ (gH_ng^{-1}), then

$$H_1H_2\dots H_{n-1}(gH_ng^{-1}) = H_1H_2\dots H_{n-1}.$$

Combining this with (5.8) yields $H_1H_2\dots H_n = H_1H_2\dots H_{n-1}$, which is contradictory to the irredundancy of $H_1H_2\dots H_n$. So we get that gH_ng^{-1} is irredundant in $H_1H_2\dots H_{n-1}$ (gH_ng^{-1}).

(2) In the case that $n \geqslant 3$, if some H_i $(1 \leqslant i \leqslant n-2)$ is redundant in $H_1H_2\dots H_{n-1}$ (gH_ng^{-1}), then

$$H_1H_2\dots H_{n-1}(gH_ng^{-1}) = H_1H_2\dots \breve{H}_i\dots H_{n-1}(gH_ng^{-1}).$$

Multiplying this by g from the right yields

$$H_1H_2\dots H_{n-1}gH_n = H_1H_2\dots \breve{H}_i\dots H_{n-1}gH_n. \tag{5.9}$$

Here, $H_{n-1}g = H_{n-1}$ (see (5.5)), hence

$$H_1H_2\dots H_{n-1}H_n = H_1H_2\dots \breve{H}_i\dots H_{n-1}H_n.$$

This however contradicts the irredundancy of $H_1H_2\dots H_n$. So we obtain that H_i $(1 \leqslant i \leqslant n-2)$ is irredundant in

$$H_1H_2\dots H_{n-1}(gH_ng^{-1}).$$

Therefore, the claim follows. □

Remark 5.9 The above argument in the proof of (2) of Theorem 5.8 does not work for H_{n-1}. Indeed, in the case of H_{n-1}, (5.9) is replaced with

$$H_1H_2\dots H_{n-1}gH_n = H_1H_2\dots H_{n-2}gH_n,$$

and to the right hand side, $H_{n-1}g = H_{n-1}$ is not applicable.

Consider the case that $n \geqslant 3$. Then in $H_1H_2\dots H_{n-1}$ (gH_ng^{-1}),

the factor H_{n-1} may be redundant, i.e.

$$H_1H_2\dots H_{n-1}(gH_ng^{-1}) = H_1H_2\dots H_{n-2}(gH_ng^{-1}). \tag{5.10}$$

Then

$$G = H_1H_2\dots H_{n-1}(gH_ng^{-1}) = H_1H_2\dots H_{n-2}(gH_ng^{-1}).$$

Multiplying this by g from the right yields

$$G = H_1H_2\dots H_{n-1}gH_n = H_1H_2\dots H_{n-2}gH_n.$$

Here $g \in H_{n-1}$ (see (5.5)), so this is rewritten as

$$G = H_1H_2\dots H_{n-1}H_n = H_1H_2\dots H_{n-2}gH_n.$$

Let us now give an example.

Example 5.10 Let us consider the dihedral group D_4 (of order 8) that is generated by two elements a and g of orders 4 and 2 with a relation $gag^{-1} = a^{-1}$. Let $H_1 := \langle g\rangle$ and $H_2 := \langle ag\rangle$ be cyclic subgroups of order 2 generated by g and ag, respectively. Then we have that $gH_2g^{-1} \neq H_2$, since $g(ag)g^{-1} = ga = a^{-1}g \notin H_2$ and

$$D_4 = H_1H_2H_1H_2$$

both of which are of order 8. Here $H_1H_2H_1H_2$ is irredundant. In particular,

$$H_1H_2\check{H}_1H_2 \subsetneq H_1H_2H_1H_2,$$

that is,

$$H_1H_2H_2\,(= H_1H_2) \subsetneq H_1H_2H_1H_2.$$

However by computation, $H_1H_2gH_2 = H_1H_2H_1H_2$, thus

$$H_1H_2(gH_2g^{-1}) = H_1H_2H_1(gH_2g^{-1}),$$

where we used $g \in H_1$.

From Claim 5.8, in the case $n \geqslant 3$, if $H_1H_2\dots H_{n-1}(gH_ng^{-1})$ is redundant, then it is so only at H_{n-1}. In irredundantizing

$$H_1H_2\dots H_{n-1}(gH_ng^{-1}),$$

we thus first remove H_{n-1}:

$$H_1H_2\dots H_{n-1}(gH_ng^{-1}) = H_1H_2\dots H_{n-2}(gH_ng^{-1}). \tag{5.11}$$

We point out that a further removal from $H_1H_2\dots H_{n-2}\ (gH_ng^{-1})$ is unnecessary — the irredundantization indeed terminates at the first step. Otherwise, we have that another factor is removable in the second step, but then from the independence of the order of an irredundantization [17, Proposition 2.21]. this factor is removable at the first step. This is impossible by Claim 5.8. The following assertion is thus confirmed.

Claim 5.11 *If* $H_1H_2\dots H_{n-1}(gH_ng^{-1})$ *is not irredundant, its irredundantization is unique and given by the removal of* H_{n-1}*, i.e.*

$$H_1H_2\dots H_{n-2}(gH_ng^{-1}).$$

Actually, note that the case $n = 3$ is special — it does not occur that $H_{n-1}\ (= H_2)$ is redundant. In fact, the following result holds.

Lemma 5.12 *Let* $H_1H_2H_3$ *be a sect of a group* G*. If* $H_1H_2H_3$ *is irredundant, then* $H_1gH_3 \subsetneq H_1H_2H_3$ *for any* $g \in H_2$*; so*

$$H_1(gH_3g^{-1}) \subsetneq H_1H_2(gH_3g^{-1}),$$

that is, H_2 *is irredundant in* $H_1H_2(gH_3g^{-1})$*.*

Proof — First from the theory of double cosets,

$$H_1H_2H_3 = \bigcup_{h_2\in H_2} H_1h_2H_3$$

is rewritten as a disjoint union

$$H_1H_2H_3 = \coprod_{g\in\Lambda} H_1gH_3, \tag{5.12}$$

where $\Lambda\ (\subset H_2)$ is a set of representatives of double cosets H_1gH_3 with $g \in H_2$. Now by assumption, $H_1H_2H_3$ is irredundant at H_2, i.e.

$$H_1H_2H_3 \supsetneq H_1H_3.$$

Namely, when $g = 1$, the double coset H_1H_3 is a proper subset

of $H_1H_2H_3$. This, together with (5.12) implies that any other double coset H_1gH_3 ($g \in \Lambda$) is also a proper subset of $H_1H_2H_3$, that is,

$$H_1H_2H_3 \supsetneq H_1gH_3.$$

Hence, the claim follows. □

We summarize the above results as follows.

Claim 5.13 *The case that*

$$H_1H_2 \dots H_{n-1}(gH_ng^{-1})$$

is redundant occurs only when $n \geqslant 4$ *and only at* H_{n-1}. *Moreover, in this case*

$$H_1H_2 \dots H_{n-2}(gH_ng^{-1})$$

is a unique irredundantization of $H_1H_2 \dots H_{n-1}(gH_ng^{-1})$.

We now proceed to the next case.

Case 2 $H_1H_2 \neq H_2H_1$.

Then, by Lemma 5.4 (2), $H_2 \not\subset N_G(H_1)$. Take $g \in H_2 \setminus N_G(H_1)$. Then $g^{-1} \in H_2 \setminus N_G(H_1)$, and as in Case 1, while replacing the roles of g and g^{-1}, we may rewrite $G = H_1H_2 \dots H_n$ as

$$G = g^{-1}(gH_1g^{-1})H_2 \dots H_n.$$

Multiplying this by g from the left yields

$$G = (gH_1g^{-1})H_2 \dots H_n.$$

This factorization is said to be obtained from $G = H_1H_2 \dots H_n$ by (left) end conjugation. As for Claim 5.6 (resp. Claim 5.7), we can show that this factorization (resp. its irredundantization) is not equivalent to $G = H_1H_2 \dots H_n$. Moreover, as for Claim 5.13, the redundant case of $(gH_1g^{-1})H_2 \dots H_n$ occurs only when $n \geqslant 4$ and only at H_2, and its irredundantization is unique and given by $(gH_1g^{-1})H_3H_4 \dots H_n$.

We summarize the results in *Case 1* and *Case 2* as follows.

Proposition 5.14 *If a factorization* $G = H_1H_2 \dots H_n$ *of a group* G *is semitight, i.e.*

(i) $H_{n-1}H_n \neq H_nH_{n-1}$ *or*

(ii) $H_1H_2 \neq H_2H_1$,

then there exists an inequivalent factorization of the group G *obtained from* $G = H_1H_2 \dots H_n$ *by end conjugation. In fact, the following are satisfied*:

(1) *Depending on whether*

(i) $H_{n-1}H_n \neq H_nH_{n-1}$ *or*

(ii) $H_1H_2 \neq H_2H_1$,

take $g \in H_{n-1} \setminus N_G(H_n)$ *or* $g \in H_2 \setminus N_G(H_1)$, (*see* Lemma 5.4), *then the inequivalent factorization is given by*

$$\begin{aligned} &\text{(i) } G = H_1H_2 \dots H_{n-1}(gH_ng^{-1}) \quad \textit{or} \\ &\text{(ii) } G = (gH_1g^{-1})H_2H_3 \dots H_n. \end{aligned} \tag{5.13}$$

(2) *An irredundantization of* (5.13) *is also inequivalent to*

$$G = H_1H_2 \dots H_n,$$

Here, the redundant case of (5.13) *occurs only when* $n \geqslant 4$ *and only at*

(i) H_{n-1} *or*

(ii) H_2,

and an irredundantization of (5.13) *is unique and given by*

$$\begin{aligned} &\text{(i)}_{\text{irred}} \quad G = H_1H_2 \dots H_{n-2}(gH_ng^{-1}) \ \textit{or} \\ &\text{(ii)}_{\text{irred}} \quad G = (gH_1g^{-1})H_3H_4 \dots H_n. \end{aligned}$$

For the convenience of later discussion, let us give a simplified version of the above result.

Corollary 5.15 *If a group* G *has a semitight factorization*

$$G = H_1H_2 \dots H_n,$$

i.e.

$$H_1H_2 \neq H_2H_1 \quad \textit{or} \quad H_{n-1}H_n \neq H_nH_{n-1},$$

then G *admits an inequivalent factorization, and its irredundantization gives an irredundant factorization of* G *inequivalent to* $G = H_1H_2 \dots H_n$.

5.2 Application to prime factorizations

Let $H_1H_2\dots H_n$ be a sect of a group G. Then its conjugate by $g \in G$ is rewritten as follows:

$$\begin{aligned} gH_1H_2\dots H_ng^{-1} &= (gH_1g^{-1})(gH_2g^{-1})\dots(gH_ng^{-1}) \\ &= H_1^gH_2^g\dots H_n^g \end{aligned} \tag{5.14}$$

where we set $H_i^g := gH_ig^{-1}$.

Lemma 5.16 *Irredundancy of sects of a group* G *is preserved under the conjugation action of* G.

Proof — Let $H_1H_2\dots H_n$ be an irredundant sect. We show that its conjugate $H_1^gH_2^g\dots H_n^g$ for any $g \in G$ is irredundant. Otherwise,

$$H_1^gH_2^g\dots H_n^g = H_1^gH_2^g\dots \widetilde{H_i^g}\dots H_n^g \tag{5.15}$$

for some i and $g \in G$. The conjugation of this by g^{-1} yields

$$H_1H_2\dots H_n = H_1H_2\dots \breve{H}_i\dots H_n,$$

which contradicts the irredundancy of $H_1H_2\dots H_n$. □

Note next the following result.

Lemma 5.17 *Atomicity of a subgroup of a group* G *is preserved under the conjugation action of* G. *Namely, a subgroup* H *of* G *is atomic if and only if* H^g $(:= gHg^{-1})$ *for* $g \in G$ *is atomic.*

Proof — "$\Longrightarrow$" We show this implication by contradiction. If H^g is not atomic, then it admits a nontrivial irredundant factorization, say

$$H^g = K_1K_2\dots K_m$$

with $m \geqslant 2$. The conjugation of this by g^{-1} yields

$$H = K_1^{g^{-1}}K_2^{g^{-1}}\dots K_m^{g^{-1}},$$

which is irredundant by Lemma 5.16, so gives a nontrivial irredundant factorization of H. This contradicts the atomicity of H. Hence the claim is shown.

"$\Longleftarrow$" This implication is shown similarly to "$\Longrightarrow$", by interchanging the roles of H and H^g and the roles of g^{-1} and g. □

Recall that a sect $H_1H_2\dots H_n$ is called an atomic sect if every factor H_i $(i = 1, 2, \dots, n)$ is an atomic group (see Definition 4.29).

Definition 5.18 A factorization $G = H_1H_2\dots H_n$ of a group G is called an *atomic factorization* if $H_1H_2\dots H_n$ is an atomic sect.

Remark 5.19 While an atomic factorization does not split at all — its factors are unsplittable — some factors may be removable, i.e. redundant. Thus an atomic factorization is not necessarily a prime factorization — it is a prime factorization only when it is irredundant.

Let us observe that, in Proposition 5.14, if the irredundant factorization $G = H_1H_2\dots H_n$ is atomic, then the factorization

$$G = H_1H_2\dots H_{n-1}(gH_ng^{-1}) \quad \big(\text{resp. } G = (gH_1g^{-1})H_2H_3\dots H_n\big)$$

is atomic by Lemma 5.17. In particular, if $G = H_1H_2\dots H_n$ is a prime factorization, then

$$G = H_1H_2\dots H_{n-1}(gH_ng^{-1}) \quad \big(\text{resp. } G = (gH_1g^{-1})H_2H_3\dots H_n\big)$$

is atomic. If this factorization is irredundant, then this is a prime factorization. Based on this, Proposition 5.14 is specialized to the following result.

Proposition 5.20 *Let* $G = H_1H_2\dots H_n$ *be a prime factorization of a group* G. *If this is semitight,* i.e.

(i) $H_{n-1}H_n \neq H_nH_{n-1}$ *or*

(ii) $H_1H_2 \neq H_2H_1$,

then there exists an inequivalent prime factorization of the group G *obtained from* $G = H_1H_2\dots H_n$ *by end conjugation. In fact, depending on whether* (i) *or* (ii), *take* $g \in H_{n-1} \setminus N_G(H_n)$ *or* $g \in H_2 \setminus N_G(H_1)$ (*see* Lemma 5.4), *then it is given by an irredundantization of*

$$\begin{aligned} &\text{(i) } G = H_1\dots H_{n-1}(gH_ng^{-1}) \quad \textit{or} \\ &\text{(ii) } G = (gH_1g^{-1})H_2\dots H_n. \end{aligned} \tag{5.16}$$

Here the redundant case of (5.16) *occurs only when* $n \geqslant 4$ *and only at*

(i) H_{n-1} *or*

(ii) H_2,

and an irredundantization of (5.16) *is unique and given by*

$$\begin{aligned}&(\mathrm{i})_{\mathrm{irred}}\ G = H_1 \dots H_{n-2}(gH_n g^{-1}) \quad \textit{or}\\ &(\mathrm{ii})_{\mathrm{irred}}\ G = (gH_1 g^{-1})H_3 \dots H_n.\end{aligned}$$

Definition 5.21 A factorization $G = H_1 H_2 \dots H_n$ of a group G is called a *primary factorization* if every factor H_i $(i = 1, 2, \dots, l)$ is a cyclic group of prime power order.

Now, consider a finite group G. Then, by [17, Theorem 3.16], G has a prime factorization, say $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$, where each $\mathfrak{p}_i$ is a cyclic subgroup of prime power order of G. If $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ is semitight, then Proposition 5.20 is specialized as follows.

Theorem 5.22 *If a finite group* G *has a semitight prime factorization given by* $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$, *i.e.*

(i) $\mathfrak{p}_{n-1}\mathfrak{p}_n \neq \mathfrak{p}_n\mathfrak{p}_{n-1}$ *or*

(ii) $\mathfrak{p}_1\mathfrak{p}_2 \neq \mathfrak{p}_2\mathfrak{p}_1$,

then there exists an inequivalent prime factorization of the group G *obtained from* $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ *by end conjugation. In fact, depending on whether* (i) *or* (ii), *take*

$$g \in \mathfrak{p}_{n-1} \setminus N_G(\mathfrak{p}_n) \qquad \textit{or} \qquad g \in \mathfrak{p}_2 \setminus N_G(\mathfrak{p}_1)$$

(*see* Lemma 5.4), *then the inequivalent prime factorization is given by an irredundantization of*

$$\begin{aligned}&(\mathrm{i})\ G = \mathfrak{p}_1 \dots \mathfrak{p}_{n-1}(g\mathfrak{p}_n g^{-1}) \quad \textit{or}\\ &(\mathrm{ii})\ G = (g\mathfrak{p}_1 g^{-1})\mathfrak{p}_2 \dots \mathfrak{p}_n.\end{aligned} \tag{5.17}$$

Here the redundant case of (5.17) *occurs only when* $n \geqslant 4$ *and only at*

(i) $\mathfrak{p}_{n-1}$ *or*

(ii) $\mathfrak{p}_2$,

and an irredundantization of (5.17) *is unique and given by*

$$(\mathrm{i})_{\mathrm{irred}}\ G = \mathfrak{p}_1 \dots \mathfrak{p}_{n-2}(g\mathfrak{p}_n g^{-1}) \quad \textit{or}$$
$$(\mathrm{ii})_{\mathrm{irred}}\ G = (g\mathfrak{p}_1 g^{-1})\mathfrak{p}_3 \dots \mathfrak{p}_n.$$

5.3 Counting inequivalent factorizations obtained by end conjugation

Let $G = H_1 H_2 \dots H_n$ be an irredundant (resp. prime) factorization of a group G such that $H_{n-1}H_n \neq H_n H_{n-1}$ or $H_1 H_2 \neq H_2 H_1$. We may then construct its inequivalent irredundant (resp. prime) factorization by end conjugation, i.e. conjugation of H_n or H_1 by Proposition 5.20 (resp. Theorem 5.22). We count the number of mutually inequivalent factorizations obtained in this way. We separate into two cases depending on whether $H_{n-1}H_n \neq H_n H_{n-1}$ or $H_1 H_2 \neq H_2 H_1$.

Case $H_{n-1}H_n \neq H_n H_{n-1}$.

Let us recall that $H_{n-1} \setminus N_G(H_n)$ is nonempty and, in addition, for $g \in H_{n-1} \setminus N_G(H_n)$, two factorizations

$$G = H_1 H_2 \dots H_n \quad \text{and} \quad G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1})$$

are not equivalent (see Proposition 5.14 (1)). Similarly, we obtain that, for another $g' \in H_{n-1} \setminus N_G(H_n)$, two factorizations

$$G = H_1 H_2 \dots H_n \quad \text{and} \quad G = H_1 H_2 \dots H_{n-1}(g'H_n(g')^{-1})$$

are not equivalent. In notation:

- $G = H_1 H_2 \dots H_n \not\sim G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1})$.
- $G = H_1 H_2 \dots H_n \not\sim G = H_1 H_2 \dots H_{n-1}(g'H_n(g')^{-1})$.

Claim 5.23 *If* $gH_n g^{-1} \neq g'H_n(g')^{-1}$, *then*

$$G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1}) \not\sim G = H_1 H_2 \dots H_{n-1}(g'H_n(g')^{-1}).$$

Proof — Set

$$K_n := gH_n g^{-1} \quad \text{and} \quad k := g'g^{-1} \in H_{n-1}.$$

Then $kK_nk^{-1} = g'H_n(g')^{-1}$. It thus suffices to show that

$$G = H_1H_2\dots H_{n-1}K_n \not\sim G = H_1H_2\dots H_{n-1}(kK_nk^{-1}). \tag{5.18}$$

Here, since $gH_ng^{-1} \neq g'H_n(g')^{-1}$, we obtain that $K_n \neq kK_nk^{-1}$. Consequently,

$$k \notin N_G(K_n).$$

Note also that from $k := g'g^{-1}$ and $g, g' \in H_{n-1}$, we have $k \in H_{n-1}$. Hence,

$$k \in H_{n-1} \setminus N_G(K_n).$$

Thus (5.18) holds by Proposition 5.14 (1). □

From Claim 5.23, the following equivalence holds:

$$\begin{aligned} G &= H_1H_2\dots H_{n-1}(gH_ng^{-1}) \\ &\sim G = H_1H_2\dots H_{n-1}(g'H_n(g')^{-1}) \end{aligned} \tag{5.19}$$

if and only if $gH_ng^{-1} = g'H_n(g')^{-1}$.

Now, let us consider all factorizations obtained by end conjugations of H_n:

$$\{G = H_1H_2\dots H_{n-1}(gH_ng^{-1}) : g \in H_{n-1} \setminus N_G(H_n)\}. \tag{5.20}$$

Here, the factors gH_ng^{-1}, with $g \in H_{n-1} \setminus N_G(H_n)$, may not be different — possibly

$$gH_ng^{-1} = g'H_n(g')^{-1}$$

for distinct $g, g' \in H_{n-1} \setminus N_G(H_n)$. Note that:

$$\begin{aligned} gH_ng^{-1} = g'H_n(g')^{-1} &\iff (g')^{-1}gH_n((g')^{-1}g)^{-1} = H_n \\ &\iff (g')^{-1}g \in N_G(H_n). \end{aligned} \tag{5.21}$$

On the other hand, as $g, g' \in H_{n-1}$, we always have $(g')^{-1}g \in H_{n-1}$. Thus

$$gH_ng^{-1} = g'H_n(g')^{-1} \iff (g')^{-1}g \in H_{n-1} \cap N_G(H_n). \tag{5.22}$$

This equivalence combined with the paragraph with (5.19) yields the

following assertion.

Claim 5.24 *For $g, g' \in H_{n-1} \setminus N_G(H_n)$, the following equivalence holds*:

$$G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1}) \sim G = H_1 H_2 \dots H_{n-1}(g'H_n(g')^{-1}) \iff (g')^{-1}g \in H_{n-1} \cap N_G(H_n).$$

Hence among the factorizations $G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1})$ with

$$g \in H_{n-1} \setminus N_G(H_n),$$

mutually inequivalent ones are given by letting g run over representatives of $H_{n-1}/(H_{n-1} \cap N_G(H_n))$. We formalize this result as follows.

Proposition 5.25 *Let $G = H_1 H_2 \dots H_n$ be an irredundant factorization of a group G with $H_{n-1}H_n \neq H_n H_{n-1}$. Then among the factorizations*

$$G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1})$$

with $g \in H_{n-1} \setminus N_G(H_n)$, the inequivalent ones are given by letting g run over representatives of

$$H_{n-1}/(H_{n-1} \cap N_G(H_n)).$$

Consequently, if H_{n-1} is finite, the number of inequivalent ones is

$$\frac{|H_{n-1}|}{|H_{n-1} \cap N_G(H_n)|}.$$

Remark 5.26 In the special case that $H_{n-1} \cap N_G(H_n) = \{1\}$, the inequivalent factorizations in question are given by

$$G = H_1 H_2 \dots (gH_n g^{-1})$$

with $g \in H_{n-1}$. The number of them is $|H_{n-1}|$.

The above results are carried over to the irredundant factorizations obtained by irredundantizing the factorizations

$$G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1}) \tag{5.23}$$

with $g \in H_{n-1} \setminus N_G(H_n)$. Recall that the irredundantization of (5.23)

is given as follows (see Proposition 5.14 (2)):

$$\begin{cases} G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1}) & \text{if (5.23) is irredundant,} \\ G = H_1 H_2 \dots H_{n-2}(gH_n g^{-1}) & \text{otherwise.} \end{cases} \tag{5.24}$$

Claim 5.27 *Let* $g, g' \in H_{n-1} \setminus N_G(H_n)$. *For two factorizations*

$$G = H_1 H_2 \dots H_{n-1}(gH_n g^{-1}) \tag{5.25}$$

$$G = H_1 H_2 \dots H_{n-1}(g'H_n(g')^{-1}) \tag{5.26}$$

their irredundantizations are equivalent if and only if

$$gH_n g^{-1} = g'H_n(g')^{-1}.$$

Proof — "$\Longleftarrow$" is trivial.

"$\Longrightarrow$" It suffices to show that if $gH_n g^{-1} \neq g'H_n(g')^{-1}$, then the irredundantizations of (5.25) and (5.26) are not equivalent. We consider three cases separately:

Case 1 Both (5.25) and (5.26) are irredundant.

Then the assertion follows from the paragraph with (5.19).

Case 2 Either (5.25) or (5.26) is redundant.

Without loss of generality, we may assume that (5.25) is redundant. Then by (5.24) above, the irredundantization of (5.25) is given by

$$G = H_1 H_2 \dots H_{n-2}(gH_n g^{-1}).$$

This and (5.26) are not equivalent because their lengths are different. *Note*: Here $gH_n g^{-1} \neq g'H_n(g')^{-1}$ is not used in the proof, but if $gH_n g^{-1} = g'H_n(g')^{-1}$, then (5.25) and (5.26) coincide, and Case 2 does not occur.)

Case 3 Both (5.25) and (5.26) are redundant.

Then their irredundantizations are

$$G = H_1 H_2 \dots H_{n-2}(gH_n g^{-1})$$

and

$$G = H_1 H_2 \dots H_{n-2}(g'H_n(g')^{-1}).$$

Setting $K_n := gH_ng^{-1}$ and $k := g'g^{-1}$, rewrite these as

$$G = H_1H_2\dots H_{n-2}K_n \quad \text{and} \quad G = H_1H_2\dots H_{n-2}(kK_nk^{-1}). \tag{5.27}$$

Here, noting that $K_n \neq kK_nk^{-1}$, we may apply to the factorizations in (5.27) the same argument used in the proof of Claim 5.6, confirming that they are not equivalent. □

Claim 5.27 combined with (5.22) yields the following result.

Claim 5.28 *The irredundantizations of two factorizations*

$$G = H_1H_2\dots H_{n-1}(gH_ng^{-1})$$

and

$$G = H_1H_2\dots H_{n-1}(g'H_n(g')^{-1})$$

are equivalent if and only if $(g')^{-1}g \in H_{n-1} \cap N_G(H_n)$.

We may thus sharpen the statement of Proposition 5.25, that is, "*the factorizations* $G = H_1H_2\dots H_{n-1}(gH_ng^{-1})$ $\big(g \in H_{n-1} \setminus N_G(H_n)\big)$" may be replaced with "*the irredundant factorizations obtained by their irredundantizations*" as follows.

Proposition 5.29 *Let* $G = H_1H_2\dots H_n$ *be an irredundant factorization of a group* G *with* $H_{n-1}H_n \neq H_nH_{n-1}$. *Then among the irredundant factorizations that are obtained by irredundantizing*

$$G = H_1H_2\dots H_{n-1}(gH_ng^{-1}) \quad \big(g \in H_{n-1} \setminus N_G(H_n)\big),$$

the inequivalent ones are given by letting g *run over representatives of*

$$H_{n-1}/(H_{n-1} \cap N_G(H_n)).$$

Consequently, if H_{n-1} *is finite, the number of inequivalent ones is*

$$\frac{|H_{n-1}|}{|H_{n-1} \cap N_G(H_n)|}.$$

We next consider the other case.

Case $H_1H_2 \neq H_2H_1$.

As for Proposition 5.25, we can show the following result.

Proposition 5.30 *Let* $G = H_1H_2 \dots H_n$ *be an irredundant factorization of a group* G *with* $H_1H_2 \neq H_2H_1$. *Then among the factorizations*

$$G = (gH_1g^{-1})H_2H_3 \dots H_n$$

with $g \in H_2 \setminus N_G(H_1)$, *the inequivalent ones are given by letting* g *run over representatives of* $H_2/(H_2 \cap N_G(H_1))$. *Consequently, if* H_2 *is finite, the number of inequivalent ones is*

$$\frac{|H_2|}{|H_2 \cap N_G(H_1)|}.$$

Remark 5.31 In the special case that $H_2 \cap N_G(H_1) = \{1\}$, the inequivalent factorizations in question are given by

$$G = (gH_1g^{-1})H_2H_3 \dots H_n$$

with $g \in H_2$. The number of them is $|H_2|$.

As for Proposition 5.29, we can show the following — the irredundantization version of Proposition 5.30.

Proposition 5.32 *Let* $G = H_1H_2 \dots H_n$ *be an irredundant factorization of a group* G *with* $H_1H_2 \neq H_2H_1$. *Then among the irredundant factorizations obtained by irredundantizing*

$$G = (gH_1g^{-1})H_2H_3 \dots H_n$$

with $g \in H_2 \setminus N_G(H_1)$, *the inequivalent ones are given by letting* g *run over representatives of*

$$H_2/(H_2 \cap N_G(H_1)).$$

Consequently, if H_2 *is finite, the number of inequivalent ones is*

$$\frac{|H_2|}{|H_2 \cap N_G(H_1)|}.$$

6 Construction of inequivalent factorizations by partial conjugation

Let G be a group. In Section 5, for a *semitight* irredundant (resp. prime) factorization $G = H_1H_2 \dots H_n$ ($n \geqslant 2$), i.e.

$$H_1H_2 \neq H_2H_1 \quad \text{or} \quad H_{n-1}H_n \neq H_nH_{n-1},$$

we constructed inequivalent irredundant (resp. prime) factorizations of G by end conjugations, i.e. conjugations of H_1 or H_n. For more details, see Proposition 5.20 (resp. see Theorem 5.22). In this section, we consider the remaining case that

$$H_1H_2 = H_2H_1 \quad \text{and} \quad H_{n-1}H_n = H_nH_{n-1}.$$

A sect satisfying this condition is said to be *soft* — a weaker notion *semisoft* is defined by the condition that

$$H_1H_2 = H_2H_1 \quad \text{or} \quad H_{n-1}H_n = H_nH_{n-1}.$$

We will construct for semisoft (in particular, soft) factorizations, their inequivalent factorizations by partial conjugations, which are more general than end conjugations. We begin with preparation.

Definition 6.1 For a sect $\mathcal{H} := H_1H_2 \dots H_n$ ($n \geqslant 2$):

(1) If $H_1H_2 = H_2H_1$, then the *left permutative part* of $\mathcal{H}$ is the subsect $H_1H_2 \dots H_l$ ($2 \leqslant l \leqslant n$) of $\mathcal{H}$ with l the maximum such that $H_iH_j = H_jH_i$ for any $i, j \in \{1, 2, \dots, l\}$. It is denoted by $\mathcal{L}$ as below:

$$\underbrace{H_1H_2 \dots H_l}_{\mathcal{L}} \dots H_n.$$

If $H_1H_2 \neq H_2H_1$, then $\mathcal{L}$ is empty.

(2) If $H_{n-1}H_n = H_nH_{n-1}$, then the *right permutative part* of $\mathcal{H}$ is the subsect $H_r \dots H_{n-1}H_n$ ($1 \leqslant r \leqslant n-1$) of $\mathcal{H}$ with r the minimum such that $H_iH_j = H_jH_i$ for any $i, j \in \{r, r+1, \dots, n\}$. It is denoted by $\mathcal{R}$:

$$H_1H_2 \dots \underbrace{H_r \dots H_{n-1}H_n}_{\mathcal{R}}.$$

If $H_{n-1}H_n \neq H_nH_{n-1}$, then $\mathcal{R}$ is empty.

Remark 6.2 If $\mathcal{H} := H_1H_2 \dots H_n$ itself is a permutative sect, i.e.

$$H_iH_j = H_jH_i$$

for any $i, j \in \{1, 2, \dots, n\}$, then

$$\mathcal{L} = \mathcal{R} = \mathcal{H}.$$

We intend to eventually construct from *semisoft* prime factorizations of nonabelian finite groups, their inequivalent prime factorizations. This is already done for the simplest case, i.e. *permutative* prime factorizations (see Theorem 3.12). *In what follows, we consider the other case that a semisoft factorization* $G = H_1H_2 \dots H_n$ $(n \geqslant 2)$ *is nonpermutative, where* G *is a* (*not necessarily finite*) *group — yet* G *is limited:* G *must be nonabelian, as* $H_1H_2 \dots H_n$ *is nonpermutative.*

Convention 6.3 Let $\mathcal{H} := H_1H_2 \dots H_n$ be a nonpermutative sect.

(1) *Case* $\mathcal{L} \neq \emptyset$.

The left permutative part $\mathcal{L} = H_1H_2 \dots H_l$ of $\mathcal{H}$ has, by definition, a factor H_i such that

$$H_{l+1}H_i \neq H_iH_{l+1}.$$

As $\mathcal{L}$ is permutative, H_i may be shifted to the rightmost of $\mathcal{L}$. This is obtained by shifting is a mutation and does not change the equivalence class of $\mathcal{H}$. *Renumbering the factors of* $\mathcal{L}$ *if necessary, we assume that* $i = l$, *so* $H_lH_{l+1} \neq H_{l+1}H_l$.

(2) *Case* $\mathcal{R} \neq \emptyset$.

The right permutative part $\mathcal{R} = H_r \dots H_{n-1}H_n$ of $\mathcal{H}$ has, by definition, a factor H_j such that

$$H_{r-1}H_j \neq H_jH_{r-1}.$$

As $\mathcal{R}$ is permutative, H_j may be shifted to the leftmost of $\mathcal{R}$ This is obtained by shifting is a mutation and does not change the equivalence class of $\mathcal{H}$. *Renumbering the factors of* $\mathcal{R}$ *if necessary, we assume that* $j = r$, *so* $H_{r-1}H_r \neq H_rH_{r-1}$.

Definition 6.4 Let $\mathcal{H} := H_1H_2 \dots H_n$ be a nonpermutative sect.

(1) As in Convention 6.3 (1), for the left permutative part

$$\mathcal{L} = H_1 H_2 \dots H_l$$

of $\mathcal{H}$ if any, we assume that

$$H_l H_{l+1} \neq H_{l+1} H_l.$$

Then H_l is called the *buffer* of $\mathcal{L}$ (or the *left buffer*).

(2) As in Convention 6.3 (2), for the right permutative part

$$\mathcal{R} = H_r \dots H_{n-1} H_n$$

of $\mathcal{H}$ if any, we assume that

$$H_{r-1} H_r \neq H_r H_{r-1}.$$

Then H_r is called the *buffer* of $\mathcal{R}$ (or the *right buffer*).

Recall that a sect $H_1 H_2 \dots H_n$ is *semisoft* if

$$H_1 H_2 = H_2 H_1 \quad \text{or} \quad H_{n-1} H_n = H_n H_{n-1}.$$

We separately treat the cases $H_1 H_2 = H_2 H_1$ and $H_{n-1} H_n = H_n H_{n-1}$.

6.1 Case $H_{n-1} H_n = H_n H_{n-1}$: Construction from the right permutative part

Let

$$\mathcal{H} := H_1 H_2 \dots H_n \quad (n \geqslant 2)$$

be an irredundant sect with $H_{n-1} H_n = H_n H_{n-1}$ (right-soft). Let

$$\mathcal{R} := H_r H_{r+1} \dots H_n \quad (1 \leqslant r < n)$$

be its right permutative part; this is irredundant because it is a subsect of irredundant $\mathcal{H}$ (see Lemma 4.2). The permutativity and irredundancy of $\mathcal{R}$ then imply that its factors $H_r, H_{r+1}, \dots, H_n$ are distinct (see Lemma 4.1 (2)). For later use, we formalize this as follows:

Lemma 6.5 *For an irredundant sect* $H_1 H_2 \dots H_n$*, the following hold*:

(1) *Its right permutative part* $\mathcal{R} := H_r H_{r+1} \dots H_n$ *is irredundant.*

(2) *Moreover the factors* $H_r, H_{r+1}, \dots, H_n$ *of* $\mathcal{R}$ *are distinct.*

For a sect $\mathcal{H} := H_1 H_2 \dots H_n$ ($n \geqslant 2$) with right permutative part

$$\mathcal{R} := H_r H_{r+1} \dots H_n$$

such that $1 < r < n$, note the following equivalences:

(R1) $r < n \iff H_{n-1} H_n = H_n H_{n-1}$. In particular, $\text{length}(\mathcal{R}) \geqslant 2$.

(R2) $1 < r \iff \mathcal{R}$ is a *proper* subsect of $\mathcal{H} \iff \mathcal{H}$ is nonpermutative. In this case, $n = \text{length}(\mathcal{H}) \geqslant 3$. Indeed, by the second statement in the equivalences, we have that

$$\text{length}(\mathcal{H}) > \text{length}(\mathcal{R}) \geqslant 2.$$

Based on (R1) and (R2), we adopt the following convention.

Convention 6.6 Let us observe that, as an alternative to saying "$\mathcal{H} := H_1 H_2 \dots H_n$ ($n \geqslant 2$) *is a nonpermutative sect with*

$$H_{n-1} H_n = H_n H_{n-1}$$

and let $\mathcal{R} := H_r H_{r+1} \dots H_n$ *be its right permutative part*", we typically say that "$\mathcal{H} := H_1 H_2 \dots H_n$ *is a sect with right permutative part* $\mathcal{R} := H_r H_{r+1} \dots H_n$ ($1 < r < n$)".

Let now $G = H_1 H_2 \dots H_n$ be an irredundant factorization of a group G with right permutative part

$$\mathcal{R} := H_r H_{r+1} \dots H_n \quad (1 < r < n).$$

Here, G must be nonabelian, otherwise $\mathcal{H}$ is permutative, which contradicts (R2). *In the sequel, without loss of generality we only consider the case that* G *is nonabelian.* We assume that $\mathcal{R}$ is finite, and regard $\mathcal{R}$ as a finite group (see Lemma 4.1 (1)). We separate into two cases depending on whether $\mathcal{R}$ is cyclic or not.

Case R.1 $\mathcal{R}$ is a noncyclic finite group.

Recall that $\mathcal{R} = H_r H_{r+1} \dots H_n$ is irredundant (see Lemma 6.5 (1)). Here, if every factor of $\mathcal{R}$ is primary, i.e. a cyclic group of prime power order — this is always the case if $G = H_1 H_2 \dots H_n$ itself is a prime factorization — then $\mathcal{R} = H_r H_{r+1} \dots H_n$ is a prime factorization. In this case, $\mathcal{R} = H_r H_{r+1} \dots H_n$ is a permutative prime factorization of

the noncyclic finite group $\mathcal{R}$. Then $\mathcal{R}$ admits an inequivalent prime factorization (see Theorem 3.12), say $\mathcal{R} = K_r K_{r+1} \dots K_m$. Noting that

$$H_r H_{r+1} \dots H_n = K_r K_{r+1} \dots K_m \, (= \mathcal{R}), \tag{6.1}$$

we may replace

$$H_r H_{r+1} \dots H_n$$

in $G = H_1 H_2 \dots H_n$ with

$$K_r K_{r+1} \dots K_m$$

and hence we obtain that

$$G = H_1 H_2 \dots H_{r-1} K_r K_{r+1} \dots K_m. \tag{6.2}$$

This factorization is not necessarily irredundant. Irredundantizing this gives an irredundant factorization of G inequivalent to

$$G = H_1 H_2 \dots H_n$$

(see Proposition 4.44). Moreover, we have the following assertion.

Claim 6.7 *If the factorization* $G = H_1 H_2 \dots H_n$ *is prime, then* (6.2) *is atomic. Consequently, an irredundantization of* (6.2) *gives a prime factorization of* G.

Proof — It suffices to show that any factors H_i and K_j appearing in (6.2) are atomic. Since $G = H_1 H_2 \dots H_n$ is a prime factorization, any factor H_i is atomic. Since

$$\mathcal{R} = K_r K_{r+1} \dots K_m$$

is a prime factorization, any factor K_j is primary, so atomic (*caution*: as G is not necessarily finite, H_i may not be primary). □

We formalize the above results as follows.

Proposition 6.8 (Noncyclic $\mathcal{R}$ case) *Let* $G = H_1 H_2 \dots H_n$ *be an irredundant factorization of a* (*not necessarily finite*) *nonabelian group* G *with right permutative part*

$$\mathcal{R} = H_r H_{r+1} \dots H_n \quad (1 < r < n)$$

such that $\mathcal{R}$ *is finite noncyclic* (*as a group*) *and* $\mathcal{R} = H_r H_{r+1} \dots H_n$ *is its prime factorization. Then, the following statements are satisfied*:

(i) *There exists a prime factorization of* $\mathcal{R}$ *inequivalent to*

$$\mathcal{R} = H_r H_{r+1} \dots H_n$$

(ii) *Write the prime factorization in* (i) *as* $\mathcal{R} = K_r K_{r+1} \dots K_m$. *Then any irredundantization of*

$$G = H_1 H_2 \dots H_{r-1} K_r K_{r+1} \dots K_m$$

gives an irredundant factorization of G *inequivalent to*

$$G = H_1 H_2 \dots H_n$$

(iii) *In the case that* $G = H_1 H_2 \dots H_n$ *itself is a prime factorization, the inequivalent irredundant factorization of* G *obtained in* (ii) *is a prime factorization of* G; *so* G *is* not *a unique factorization group.*

For ease of reference in what follows, we extract and restate Proposition 6.8 (iii).

Corollary 6.9 *Let* $G = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_n$ *be a prime factorization of a* (*not necessarily finite*) *nonabelian group* G *with right permutative part*

$$\mathcal{R} = \mathfrak{p}_r \mathfrak{p}_{r+1} \dots \mathfrak{p}_n \quad (1 < r < n)$$

such that $\mathcal{R}$ *is noncyclic finite* (*as a group*). *Then there exists a prime factorization of* G *inequivalent to*

$$G = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_n;$$

so G *is not a unique factorization group.*

Before proceeding, note the following (see [6, Theorem 3.6]).

Lemma 6.10 *Subgroups of a cyclic finite group are uniquely determined by their orders.*

Next, before we go into Case R.2 ($\mathcal{R}$ is a *cyclic* finite group), we show the following technical result used there.

Lemma 6.11 *The following statements are satisfied*:

(1) *If* $K_1K_2\dots K_l$ $(l \geqslant 2)$ *is an irredundant sect of a cyclic finite group, then the orders of* $K_1, K_2, \dots, K_l$ *are different.*

(2) *Let* Z *be a cyclic finite subgroup of a group* G *and let* $K_1K_2\dots K_l$ $(l \geqslant 2)$ *be an irredundant sect of* Z*. Then the following statements hold*:

(2.a) $K_i \neq gK_jg^{-1}$ *for any distinct* $i, j \in \{1, 2, \dots, l\}$ *and any* $g \in G$.

(2.b) *If* $K_i \neq gK_ig^{-1}$ *for some* $i \in \{1, 2, \dots, l\}$ *and some* $g \in G$, *then* $gZg^{-1} \neq Z$.

Proof — (1) If the orders of two factors K_i and K_j $(i \neq j)$ are equal, then $K_i = K_j$ (see Lemma 6.10). This however contradicts the fact that the factors of an irredundant sect of an abelian group are distinct (see [17, Lemma 2.12]).

(2.a) If $K_i = gK_jg^{-1}$ for some distinct i, j and $g \in G$, then in particular the orders of K_i and K_j are equal, which contradicts (1).

(2.b) If $gZg^{-1} = Z$, then since the left hand side contains gK_ig^{-1} and the right hand side contains K_i, the cyclic finite group Z contains both gK_ig^{-1} and K_i. Here the orders of gK_ig^{-1} and K_i are equal, thus, by Lemma 6.10,

$$K_i = gK_ig^{-1},$$

which contradicts the assumption $K_i \neq gK_ig^{-1}$. □

Case R.2 $\mathcal{R}$ is a cyclic finite group.

Recall the notation:

$$\mathcal{H} := H_1H_2\dots H_n$$

is an irredundant sect with right permutative part

$$\mathcal{R} := H_rH_{r+1}\dots H_n \quad (1 < r < n).$$

In the present case that $\mathcal{R}$ is a cyclic finite group, together with the fact that $\mathcal{R} = H_rH_{r+1}\dots H_n$ is irredundant (see Lemma 6.5 (1)), the following result holds by Lemma 6.11 (1).

Claim 6.12 *The orders of* $H_r, H_{r+1}, \dots, H_n$ *are distinct.*

Permuting the factors $H_r, H_{r+1}, \dots, H_n$ of $\mathcal{R}$ if necessary, we assume that

$$H_{r-1}H_r \neq H_rH_{r-1}$$

(as in Convention 6.3 (2)). It follows that, by Lemma 5.4 (1), there exists $g \in H_{r-1} \setminus N_G(H_r)$. Note the following:

$$gH_rg^{-1} \neq H_r \quad \text{from } g \notin N_G(H_r). \tag{6.3}$$

$$H_{r-1}g = H_{r-1} \quad \text{from } g \in H_{r-1}. \tag{6.4}$$

Moreover the following result holds.

Claim 6.13 *The following statements are satisfied*:

(1) $H_i \neq gH_jg^{-1}$ *for any distinct* $i, j \in \{r, r+1, \dots, n\}$.

(2) $g\mathcal{R}g^{-1} \neq \mathcal{R}$.

Proof — (1) It follows by Lemma 6.11 (2.a).

(2) Observing that, by (6.3), $gH_rg^{-1} \neq H_r$, the statement follows from Lemma 6.11 (2.b). □

By (6.4), we rewrite $G = H_1H_2 \dots H_n$ as follows.

$$\begin{aligned} G &= H_1H_2 \dots (H_{r-1}g)H_rH_{r+1} \dots H_n \\ &= H_1H_2 \dots H_{r-1}(gH_rg^{-1})(gH_{r+1}g^{-1}) \dots (gH_ng^{-1})g. \end{aligned} \tag{6.5}$$

Multiplying

$$G = H_1H_2 \dots H_{r-1}(gH_rg^{-1})(gH_{r+1}g^{-1}) \dots (gH_ng^{-1})g$$

by g^{-1} from the right yields

$$G = H_1H_2 \dots H_{r-1}(gH_rg^{-1})(gH_{r+1}g^{-1}) \dots (gH_ng^{-1}). \tag{6.6}$$

Definition 6.14 The factorization (6.6) is called a *partial conjugate* of $G = H_1H_2 \dots H_n$.

For brevity we set

$$H_i' := gH_ig^{-1} \quad (i = r, r+1, \dots, n).$$

Then (6.6) is rewritten as

$$G = H_1H_2 \dots H_{r-1}H_r'H_{r+1}' \dots H_n'. \tag{6.7}$$

Claim 6.15 *We have that* $H_s \neq H_r'$ *for any* $s \in \{r, r+1, \dots, n\}$.

Proof — If $s = r$, this is (6.3). If $s \neq r$, this is Claim 6.13 (1). □

For ease of the subsequent discussion, let us restate Claim 6.15 as below.

Claim 6.16 *If* $H_s = H'_r$, *then* $s \in \{1, 2, \ldots, r-1\}$.

For simplicity, write $G = H_1 H_2 \ldots H_n$ as

$$G = H_1 H_2 \ldots H_{r-1} \mathcal{R},$$

where

$$\mathcal{R} := H_r H_{r+1} \ldots H_n \quad (1 < r < n)$$

is the right permutative part of $H_1 H_2 \ldots H_n$.

Similarly, write $G = H_1 H_2 \ldots H_{r-1} H'_r H'_{r+1} \ldots H'_n$ as

$$G = H_1 H_2 \ldots H_{r-1} \mathcal{R}',$$

where however

$$\mathcal{R}' := H'_r H'_{r+1} \ldots H'_n \tag{6.8}$$

is generally not the right permutative part $\mathcal{S}$ of

$$H_1 H_2 \ldots H_{r-1} H'_r H'_{r+1} \ldots H'_n$$

but merely a subsect of $\mathcal{S}$: Taking the conjugation by g of

$$H_i H_j = H_j H_i \quad (i, j \in \{r, r+1, \ldots, n\})$$

yields

$$H'_i H'_j = H'_j H'_i \quad (i, j \in \{r, r+1, \ldots, n\}),$$

but besides, possibly

$$H_{r-1} H'_j = H'_j H_{r-1}$$

for any $j \in \{r, r+1, \ldots, n\}$, in which case $H_{r-1} \in \mathcal{S}$ but $H_{r-1} \notin \mathcal{R}'$, so $\mathcal{R}' \subsetneq \mathcal{S}$. Note next the following result.

Claim 6.17 *The following statements are satisfied*:

(1) $\mathcal{R}' = g\mathcal{R}g^{-1}$.

(2) $\mathcal{R} \neq \mathcal{R}'$.

Proof — (1) The assertion is confirmed as follows:

$$\begin{aligned}\mathcal{R}' &= (gH_rg^{-1})(gH_{r+1}g^{-1})\dots(gH_ng^{-1})\\ &= g(H_rH_{r+1}\dots H_n)g^{-1} = g\mathcal{R}g^{-1}.\end{aligned}$$

(2) The statement follows from $\mathcal{R}' = g\mathcal{R}g^{-1}$ in (1) and $g\mathcal{R}g^{-1} \neq \mathcal{R}$ in Claim 6.13 (2). □

Although $G = H_1H_2\dots H_{r-1}\mathcal{R}$ is irredundant by assumption, the same does not necessarily hold for $H_1H_2\dots H_{r-1}\mathcal{R}'$.

Remark 6.18 An irredundantization of $G = H_1H_2\dots H_{r-1}\mathcal{R}'$ is not necessarily of the form

$$G = H_1H_2\dots H_{r-1}\overset{\vee}{\mathcal{R}}',$$

where $\overset{\vee}{\mathcal{R}}'$ is a subsect of $\mathcal{R}'$ obtained by removing some (not all) factors from $\mathcal{R}'$. In fact, since $\mathcal{R} \neq \mathcal{R}'$ (see Claim 6.17 (2)), Lemma 4.7 (replacing $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{R}$ and $\mathcal{R}'$) is not applicable (see also Remark 4.8 (3)).

We now show the following assertion.

Claim 6.19 *No irredundantization of the factorization*

$$G = H_1H_2\dots H_{r-1}\mathcal{R}'\,(= H_1H_2\dots H_{r-1}H'_rH'_{r+1}\dots H'_n)$$

is equivalent to $G = H_1H_2\dots H_{r-1}\mathcal{R}\,(= H_1H_2\dots H_n)$.

Proof — In the case that

$$G = H_1H_2\dots H_{r-1}H'_rH'_{r+1}\dots H'_n$$

is not irredundant, the assertion is trivial, because the length of its irredundantization is shorter than n and so the irredundantization cannot be equivalent to

$$G = H_1H_2\dots H_n.$$

Next, in the case that

$$G = H_1H_2\dots H_{r-1}H'_rH'_{r+1}\dots H'_n$$

is irredundant, we show the assertion by contradiction. Suppose that

$$G = H_1 H_2 \dots H_n \quad \text{and} \quad G = H_1 H_2 \dots H_{r-1} H'_r H'_{r+1} \dots H'_n$$

are equivalent. Therefore, there exists a mutation from $H_1 H_2 \dots H_n$ to $H_1 H_2 \dots H_{r-1} H'_r H'_{r+1} \dots H'_n$, which permutes the factors of

$$H_1 H_2 \dots H_n,$$

say, by a permutation $\sigma \in \mathfrak{S}_n$. Then

$$H_{\sigma(1)} H_{\sigma(2)} \dots H_{\sigma(n)} \equiv H_1 H_2 \dots H_{r-1} H'_r H'_{r+1} \dots H'_n \quad \text{(identical)},$$

that is,

$$\begin{aligned} &H_{\sigma(1)} = H_1, H_{\sigma(2)} = H_2, \dots, H_{\sigma(r-1)} = H_{r-1}, \\ &H_{\sigma(r)} = H'_r, H_{\sigma(r+1)} = H'_{r+1}, \dots, H_{\sigma(n)} = H'_n. \end{aligned} \tag{6.9}$$

Or simply

$$H_{\sigma(i)} = \begin{cases} H_i & (i = 1, 2, \dots, r-1) \qquad \text{(i)}, \\ H'_i & (i = r, r+1, \dots, n) \qquad \text{(ii)}. \end{cases} \tag{6.10}$$

Let k be the smallest positive integer such that $\sigma^k(r) = r$ and consider the σ-orbit of r:

$$\sigma(r), \sigma^2(r), \dots, \sigma^k(r) = r. \tag{6.11}$$

From (6.10) (ii), we have

$$H_{\sigma(r)} = H'_r. \tag{6.12}$$

Here, $\sigma(r) \in \{1, 2, \dots, r-1\}$ by Claim 6.16. Then, from (6.10) (i), we have $H_{\sigma(\sigma(r))} = H_{\sigma(r)}$, that is,

$$H_{\sigma^2(r)} = H_{\sigma(r)}. \tag{6.13}$$

We claim that $\sigma^2(r) \in \{1, 2, \dots, r-1\}$. Indeed, by combining equalities (6.12) and (6.13) yields

$$H_{\sigma^2(r)} = H'_r. \tag{6.14}$$

Thus $\sigma^2(r) \in \{1, 2, \ldots, r-1\}$ by Claim 6.16. Then, from (6.10) (i), we have $H_{\sigma(\sigma^2(r))} = H_{\sigma^2(r)}$, that is, $H_{\sigma^3(r)} = H_{\sigma^2(r)}$. Combining this with (6.14) we obtain that

$$H_{\sigma^3(r)} = H'_r.$$

Therefore $\sigma^3(r) \in \{1, 2, \ldots, r-1\}$ by Claim 6.16. Repeating this argument, we reach

$$H_{\sigma^k(r)} = H'_r.$$

Here, $\sigma^k(r) = r$, so $H_r = H'_r$. This contradicts Claim 6.15. □

We formalize the above results as follows.

Proposition 6.20 (Cyclic $\mathcal{R}$ case) *Let* $G = H_1 H_2 \ldots H_n$ *be an irredundant factorization of a* (*not necessarily finite*) *nonabelian group* G *with right permutative part*

$$\mathcal{R} = H_r H_{r+1} \ldots H_n \quad (1 < r < n)$$

such that $\mathcal{R}$ *is finite cyclic* (*as a group*) *and without loss of generality we assume that* $H_{r-1}H_r \neq H_r H_{r-1}$ (*see* Convention 6.3 (2)). *Then the following statements are satisfied*:

(i) $H_{r-1} \setminus N_G(H_r)$ *is nonempty and for* $g \in H_{r-1} \setminus N_G(H_r)$, *and*

$$G = H_1 H_2 \ldots H_{r-1}(gH_r g^{-1})(gH_{r+1}g^{-1}) \ldots (gH_n g^{-1}). \quad (6.15)$$

(ii) *Any irredundantization of* (6.15) *is an irredundant factorization inequivalent to* $G = H_1 H_2 \ldots H_n$.

(iii) *In the case that* $G = H_1 H_2 \ldots H_n$ *is a prime factorization, the inequivalent irredundant factorization obtained in* (ii) *is a prime factorization; so* G *is not a unique factorization group.*

Remark 6.21 If $\mathcal{R}$ is a cyclic group of order m with prime factorization $m = p_1^{m_1} p_2^{m_2} \ldots p_u^{m_u}$, then by the fundamental theorem of finite abelian groups, we obtain that $\mathcal{R} = \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \ldots \times \mathbb{Z}_{p_u^{m_u}}$, so

$$\mathcal{R} = \mathbb{Z}_{p_1^{m_1}} \mathbb{Z}_{p_2^{m_2}} \ldots \mathbb{Z}_{p_u^{m_u}}.$$

This is a unique prime factorization of $\mathcal{R}$ (see Theorem 6.43). Therefore, in Proposition 6.20, the prime factorization $\mathcal{R} = H_r H_{r+1} \ldots H_n$

must coincide with $\mathcal{R} = \mathbb{Z}_{p_1^{m_1}}\mathbb{Z}_{p_2^{m_2}}\dots\mathbb{Z}_{p_u^{m_u}}$, that is, $u = n-r+1$ and possibly after reordering,

$$H_{r-1+i} = \mathbb{Z}_{p_i^{m_i}} \quad (i = 1, 2, \dots, u).$$

For the ease of later discussion, we extract Proposition 6.20 (iii) and restate it as follows.

Corollary 6.22 *Let* $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n$ *be a prime factorization of a* (*not necessarily finite*) *nonabelian group* G *with right permutative part*

$$\mathcal{R} := \mathfrak{p}_r\mathfrak{p}_{r+1}\dots\mathfrak{p}_n \quad (1 < r < n)$$

such that $\mathcal{R}$ *is finite cyclic* (*as a group*). *Then there exists a prime factorization of* G *inequivalent to* $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n$; *so* G *is not a unique factorization group.*

Summary for the case $H_{n-1}H_n = H_nH_{n-1}$

Let $G = H_1H_2\dots H_n$ $(n \geqslant 2)$ be a nonpermutative irredundant factorization of a (not necessarily finite) nonabelian group G with

$$H_{n-1}H_n = H_nH_{n-1}.$$

Hence, it has a nonempty right permutative part

$$\mathcal{R} := H_rH_{r+1}\dots H_n \quad (1 < r < n).$$

Suppose that $\mathcal{R}$ is finite. Then no matter whether $\mathcal{R}$ is cyclic or not (as a group), Proposition 6.20 (ii) or Proposition 6.8 (ii) ensures that G admits an irredundant factorization inequivalent to $G = H_1H_2\dots H_n$. Yet in the case of noncyclic $\mathcal{R}$, it is required that

$$\mathcal{R} = H_rH_{r+1}\dots H_n$$

is a prime factorization, accordingly the statement unifying the cyclic case and the noncyclic case is given as follows.

Theorem 6.23 *Let* $G = H_1H_2\dots H_n$ $(n \geqslant 2)$ *be a nonpermutative irredundant factorization of a* (*not necessarily finite*) *nonabelian group* G

with $H_{n-1}H_n = H_nH_{n-1}$ *such that its right permutative part*

$$\mathcal{R} = H_rH_{r+1}\dots H_n \quad (1 < r < n)$$

is finite and a prime factorization. Then there exists an irredundant factorization of G *inequivalent to* $G = H_1H_2\dots H_n$.

If $G = H_1H_2\dots H_n$ itself is a prime factorization, then

$$\mathcal{R} = H_rH_{r+1}\dots H_n$$

is automatically a prime factorization (see Lemma 4.30). Hence the following holds.

Corollary 6.24 *Let* $G = H_1H_2\dots H_n$ $(n \geqslant 2)$ *be a nonpermutative prime factorization of a* (*not necessarily finite*) *nonabelian group* G *such that* $H_{n-1}H_n = H_nH_{n-1}$ *such that its right permutative part is finite. Then there exists an irredundant factorization of* G *that is inequivalent to* $G = H_1H_2\dots H_n$.

If moreover G is finite, the finiteness of the right permutative part is trivially satisfied, and the following holds (we slightly change notation):

Corollary 6.25 *Let* $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n$ $(n \geqslant 2)$ *be a nonpermutative prime factorization of a finite nonabelian group* G *with*

$$\mathfrak{p}_{n-1}\mathfrak{p}_n = \mathfrak{p}_n\mathfrak{p}_{n-1}.$$

Then there exists an irredundant factorization of G *inequivalent to*

$$G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n.$$

Here, the conclusion "*there exists an irredundant factorization of* G *inequivalent to* $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n$" is actually sharpened to "*there exists a prime factorization of* G *inequivalent to* $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n$", by Corollary 6.22 or Corollary 6.9 depending on whether $\mathcal{R}$ is cyclic or not. In addition, now since G is finite, even if $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n$ is permutative, G admits an inequivalent prime factorization by Theorem 3.12 which states: *If a finite noncyclic group has a permutative prime factorization, then it has another* (*inequivalent*) *prime factorization* — note that in the present case, G is assumed to be nonabelian, so G is noncyclic. Hence, the nonpermutativity condition of $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n$ may be dropped, and the following result holds.

Theorem 6.26 *Let* $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n$ $(n \geqslant 2)$ *be a prime factorization of a finite nonabelian group* G *with* $\mathfrak{p}_{n-1}\mathfrak{p}_n = \mathfrak{p}_n\mathfrak{p}_{n-1}$. *Then there exists a prime factorization of* G *inequivalent to* $G = \mathfrak{p}_1\mathfrak{p}_2\dots\mathfrak{p}_n$; *so* G *is not a unique factorization group.*

6.2 Case $H_1H_2 = H_2H_1$: Construction from the left permutative part

Let $H_1H_2\dots H_n$ $(n \geqslant 2)$ be an irredundant sect such that

$$H_1H_2 = H_2H_1,$$

i.e. left-soft. Let $\mathcal{L} = H_1H_2\dots H_l$ $(2 \leqslant l \leqslant n)$ be its left permutative part, namely the maximal subsect such that for any $i, j \in \{1, 2, \dots, l\}$, the equality $H_iH_j = H_jH_i$ holds. As for Lemma 6.5, the following result holds (the proof is the same).

Lemma 6.27 *The following statements are satisfied*:

(1) *The left permutative part* $\mathcal{L} = H_1H_2\dots H_l$ *of an irredundant sect*

$$H_1H_2\dots H_n$$

is irredundant.

(2) *The factors* $H_1, H_2, \dots, H_l$ *are distinct.*

Let G be a (not necessarily finite) nonabelian group. Suppose that

$$G = H_1H_2\dots H_n$$

is an irredundant left-soft factorization, i.e. $H_1H_2 = H_2H_1$. We can then show the corresponding results to the right-soft case

$$H_{n-1}H_n = H_{n-1}H_n$$

in the same way, or more economically, using the reverse factorization as follows: First take the inverse of $G = H_1H_2\dots H_n$, that is,

$$G^{-1} = H_n^{-1}\dots H_2^{-1}H_1^{-1}.$$

Noting that

$$G^{-1} = G \quad \text{and} \quad H_i^{-1} = H_i$$

by the invertibility of groups, we have $G = H_n \dots H_2 H_1$, which is the reverse factorization of $G = H_1 H_2 \dots H_n$. In this way, the left-soft case turns into the right-soft case, and vice versa.

Before proceeding, we shall closely look at the one-to-one correspondence between factorizations of G and their reverse factorizations:

$$G = H_1 H_2 \dots H_n \longleftrightarrow G = H_n \dots H_2 H_1. \tag{6.16}$$

Claim 6.28 *Under the correspondence* (6.16)*, irredundancy and primeness are preserved.*

Proof — The preservation of irredundancy: We alternatively show the preservation of redundancy, that is, $H_1 H_2 \dots H_n$ is redundant if and only if $H_n \dots H_2 H_1$ is redundant. We first show "$\Longrightarrow$". It suffices to show that if a factor H_i is redundant in $H_1 H_2 \dots H_n$, i.e.

$$H_1 H_2 \dots H_n = H_1 H_2 \dots \check{H}_i \dots H_n, \tag{6.17}$$

then H_i is also redundant in $H_n \dots H_2 H_1$. This is confirmed by taking the reverse (6.17):

$$H_n \dots H_2 H_1 = H_n \dots \check{H}_i \dots H_2 H_1. \tag{6.18}$$

We can show "$\Longleftarrow$" by reversing the above argument.

Once we showed the preservation of irredundancy, the preservation of primeness is immediate, because if every factor of $H_1 H_2 \dots H_n$ is atomic, then every factor of $H_n \dots H_2 H_1$ is atomic. □

The following assertion moreover holds.

Claim 6.29 *Under the correspondence* (6.16)*, equivalences between factorizations are preserved*: *For factorizations*

$$G = H_1 H_2 \dots H_n \quad \textit{and} \quad G = K_1 K_2 \dots K_m,$$

we have that

$$\begin{aligned} &G = H_1 H_2 \dots H_n \sim G = K_1 K_2 \dots K_m \\ \Longleftrightarrow\ &G = H_n \dots H_2 H_1 \sim G = K_m \dots K_2 K_1. \end{aligned}$$

Proof — In fact there is a one-to-one correspondence between mutations of sects and mutations of reverse sects: First recall that for a

sect

$$H_1H_2\dots\underbrace{H_iH_{i+1}\dots H_j}_{\mathcal{A}}\dots H_n$$

with a regular subsect $\mathcal{A}$, a mutation along $\mathcal{A}$ is given by

$$\mu:\ H_1H_2\dots\underbrace{H_iH_{i+1}\dots H_j}_{\mathcal{A}}\dots H_n \longrightarrow H_1H_2\dots\underbrace{H_j\dots H_{i+1}H_i}_{\mathcal{A}^{\mathrm{rev}}}\dots H_n,$$

where $\mathcal{A}^{\mathrm{rev}}$ denotes the reverse of $\mathcal{A}$. In turn the reverse mutation of μ is given by

$$\begin{aligned}\mu^{\mathrm{rev}}: H_n\dots\underbrace{H_j\dots H_{i+1}H_i}_{\mathcal{A}^{\mathrm{rev}}}\dots H_2H_1&\\ \longrightarrow H_1H_2\dots\underbrace{H_iH_{i+1}\dots H_j}_{\mathcal{A}}\dots H_n.&\end{aligned}$$

Therefore the claim follows. □

Let us summarize Claim 6.28 and Claim 6.29 as follows.

Lemma 6.30 *Two (resp. irredundant or prime) factorizations of a group* G *are equivalent if and only if their reverse (resp. irredundant or prime) factorizations of* G *are equivalent.*

Now we return to the previous context: Let G be a nonabelian group, $G = H_1H_2\dots H_n$ ($n \geqslant 2$) be an irredundant left-soft factorization, i.e. $H_1H_2 = H_2H_1$, and let

$$\mathcal{L} := H_1H_2\dots H_l$$

be its left permutative part. We summarize the results corresponding to those in *Case R.1* and *Case R.2* of the right-soft case. We separately state the results depending on whether $\mathcal{L}$ is cyclic or not.

Case L.1 $\mathcal{L}$ is a noncyclic finite group.

Proposition 6.31 (Noncyclic $\mathcal{L}$ case) *Let* $G = H_1H_2\dots H_n$ *be an irredundant factorization of a (not necessarily finite) nonabelian group* G *with left permutative part* $\mathcal{L} = H_1H_2\dots H_l$ ($1 < l < n$) *such that* $\mathcal{L}$ *is finite noncyclic (as a group) and* $\mathcal{L} = H_1H_2\dots H_l$ *is its prime factorization. Then the following statement hold*:

(i) *There exists a nontrivial prime factorization of* $\mathcal{L}$ *inequivalent to*

$$\mathcal{L} = H_1 H_2 \dots H_l.$$

(ii) *Write the prime factorization in* (i) *as* $\mathcal{L} = K_1 K_2 \dots K_m$. *Then any irredundantization of*

$$G = K_1 K_2 \dots K_m H_{l+1} H_{l+2} \dots H_n$$

gives an irredundant factorization of the group G *that is inequivalent to* $G = H_1 H_2 \dots H_n$.

(iii) *In the case that* $G = H_1 H_2 \dots H_n$ *itself is a prime factorization, the inequivalent irredundant factorization obtained in* (ii) *is a prime factorization of* G; *so* G *is not a unique factorization group.*

For ease of later discussion, we extract Proposition 6.31 (iii) and restate it as follows.

Corollary 6.32 *Let* $G = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_n$ *be a prime factorization of a* (*not necessarily finite*) *nonabelian group* G *with left permutative part*

$$\mathcal{L} := \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_l \quad (1 < l < n)$$

such that $\mathcal{L}$ *is finite noncyclic* (*as a group*). *Then there exists a prime factorization of* G *inequivalent to* $G = \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_n$; *so* G *is not a unique factorization group.*

Case L.2 $\mathcal{L}$ is a cyclic finite group.

Proposition 6.33 (Cyclic $\mathcal{L}$ case) *Let* $G = H_1 H_2 \dots H_n$ *be an irredundant factorization of a* (*not necessarily finite*) *nonabelian group* G *with left permutative part*

$$\mathcal{L} = H_1 H_2 \dots H_l \quad (1 < l < n)$$

such that $\mathcal{L}$ *is finite cyclic* (*as a group*) *and without loss of generality we assume that* $H_l H_{l+1} \neq H_{l+1} H_l$ (*see* Convention 6.3 (1)). *Then the following statements are satisfied*:

(i) $H_{l+1} \setminus N_G(H_l)$ *is nonempty and for* $g \in H_{l+1} \setminus N_G(H_l)$,

$$G = (gH_1g^{-1})(gH_2g^{-1}) \dots (gH_lg^{-1}) H_{l+1} H_{l+2} \dots H_n. \quad (6.19)$$

(ii) *Any irredundantization of* (6.19) *is an irredundant factorization inequivalent to* $G = H_1H_2 \dots H_n$.

(iii) *In the case that* $G = H_1H_2 \dots H_n$ *is a prime factorization, the inequivalent irredundant factorization obtained in* (ii) *is a prime factorization of* G*; so* G *is not a unique factorization group.*

For ease of later discussion, we extract Proposition 6.33 (iii) and restate it as follows.

Corollary 6.34 *Let* $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ *be a prime factorization of a* (*not necessarily finite*) *nonabelian group* G *with left permutative part*

$$\mathcal{L} := \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_l \quad (1 < l < n)$$

such that $\mathcal{L}$ *is finite cyclic* (*as a group*)*. Then there exists an inequivalent prime factorization of* G*; so* G *is not a unique factorization group.*

Summary for the case $H_1H_2 = H_2H_1$

We shall mention the statements corresponding to those appearing in Summary for the case $H_{n-1}H_n = H_nH_{n-1}$.

Theorem 6.35 *Let* $G = H_1H_2 \dots H_n$ $(n \geqslant 2)$ *be a nonpermutative irredundant factorization of a* (*not necessarily finite*) *nonabelian group* G *with* $H_1H_2 = H_2H_1$ *such that its left permutative part*

$$\mathcal{L} = H_1H_2 \dots H_l \quad (1 < l < n)$$

is finite and a prime factorization. Then there exists an irredundant factorization of G *inequivalent to* $G = H_1H_2 \dots H_n$.

In the above, by replacing the irredundancy assumption for

$$G = H_1H_2 \dots H_n$$

with the primeness assumption for it yields the following result.

Corollary 6.36 *Let* $G = H_1H_2 \dots H_n$ $(n \geqslant 2)$ *be a nonpermutative prime factorization of a* (*not necessarily finite*) *nonabelian group* G *with*

$$H_1H_2 = H_2H_1$$

such that its left permutative part is finite. Then there exists an irredundant factorization of G *inequivalent to* $G = H_1H_2 \dots H_n$.

Corollary 6.37 *Let* $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ $(n \geqslant 2)$ *be a nonpermutative prime factorization of a finite nonabelian group* G *with* $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1$. *Then there exists an irredundant factorization of* G *inequivalent to* $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$.

The corresponding statement to Theorem 6.26 is as follows.

Theorem 6.38 *Let* $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ $(n \geqslant 2)$ *be a prime factorization of a* finite *nonabelian group* G *with* $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1$. *Then there exists a prime factorization of* G *inequivalent to* $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$; *so* G *is not a unique factorization group.*

6.3 Consequences

Before giving consequences of the results obtained so far, note the following result.

Lemma 6.39 *For an atomic sect of a group, its image under any mutation is also an atomic sect — consequently an atomic sect and a non-atomic sect cannot be equivalent.*

Proof — This is because a mutation merely permutes factors. □

The following is an immediate consequence (the factorization version) of Lemma 6.39.

Corollary 6.40 *For two distinct factorizations of a group, if one is an atomic factorization and the other is not, then they are not equivalent. In particular if one is a prime factorization and the other is not, then they are not equivalent.*

We now show the following result.

Theorem 6.41 *For an irredundant factorization* $G = H_1H_2 \dots H_n$ $(n \geqslant 2)$ *of a finite nonabelian group* G, *the following hold*:

(1) *If* $H_1H_2 \neq H_2H_1$ *or* $H_{n-1}H_n \neq H_nH_{n-1}$, *then there exists an irredundant factorization of* G *inequivalent to* $G = H_1H_2 \dots H_n$.

(2) *If* $H_1H_2 \dots H_n$ *is nonpermutative and*

$$H_1H_2 = H_2H_1 \quad \textit{or} \quad H_{n-1}H_n = H_nH_{n-1},$$

then there exists an irredundant factorization of G *that is inequivalent to* $G = H_1H_2 \dots H_n$.

Proof — (1) The claim follows by Corollary 5.15.

(2) Let us the statement by separating into two cases depending on whether $G = H_1H_2 \dots H_n$ is a prime factorization or not.

Case 1 $G = H_1H_2 \dots H_n$ is a prime factorization.

Then depending on whether

$$H_1H_2 = H_2H_1 \quad \text{or} \quad H_{n-1}H_n = H_nH_{n-1},$$

the assertion follows from Corollary 6.37 or Corollary 6.25.

Case 2 $G = H_1H_2 \dots H_n$ is not a prime factorization.

Let us recall that any finite group admits a prime factorization (see [17, Theorem 3.16]). There thus exists a prime (in particular irredundant) factorization $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_m$. This factorization is not equivalent to $G = H_1H_2 \dots H_n$ by Corollary 6.40, so it is the desired factorization. □

We next summarize results concerning prime factorizations for finite groups. First, any finite group G admits at least one prime factorization $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ (see [17, Theorem 3.16]). For nonabelian G, the following holds:

(P) If $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ is permutative, then there exists a prime factorization of G inequivalent to it (see Theorem 3.12).

(NP) If $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ is nonpermutative, then the following hold:

(i) No matter whether $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1$ or $\mathfrak{p}_1\mathfrak{p}_2 \neq \mathfrak{p}_2\mathfrak{p}_1$, there exists a prime factorization of G inequivalent to

$$G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$$

(see Theorem 6.38 or Theorem 5.22).

(ii) No matter whether

$$\mathfrak{p}_{n-1}\mathfrak{p}_n = \mathfrak{p}_n\mathfrak{p}_{n-1} \quad \text{or} \quad \mathfrak{p}_{n-1}\mathfrak{p}_n \neq \mathfrak{p}_n\mathfrak{p}_{n-1},$$

there exists a prime factorization of the group G that is inequivalent to $G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n$ (see Theorem 6.26 or Theorem 5.22).

Therefore, in any case, G has a prime factorization inequivalent to

$$G = \mathfrak{p}_1\mathfrak{p}_2 \dots \mathfrak{p}_n,$$

and the following result is established.

Theorem 6.42 *Any finite nonabelian group has at least two* (*inequivalent*) *prime factorizations; so it is not a unique factorization group.*

On the other hand, by Theorem 2.13, a finite abelian group is a unique factorization group if and only if it is cyclic. Here for a cyclic group G of order n with prime factorization $n = p_1^{n_1} p_2^{n_2} \dots p_l^{n_l}$, its unique prime factorization is

$$G = \mathbb{Z}_{p_1^{n_1}} \mathbb{Z}_{p_2^{n_2}} \dots \mathbb{Z}_{p_l^{n_l}}.$$

Combining this with Theorem 6.42 yields the following result.

Theorem 6.43 *A finite group G is a unique factorization group if and only if G is cyclic. Here for a cyclic group G of order n with prime factorization $n = p_1^{n_1} p_2^{n_2} \dots p_l^{n_l}$, its unique prime factorization is*

$$G = \mathbb{Z}_{p_1^{n_1}} \mathbb{Z}_{p_2^{n_2}} \dots \mathbb{Z}_{p_l^{n_l}}.$$

Remark 6.44 The number of prime factorizations of a finite group is finite (see [17, Theorem 3.19]).

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