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Constructing Fischer Matrices of an Extension Group from its Underlying Factor Group

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Abstract

Let $\overline{G} = P.G$ be a finite extension of a p-group P by a group G. Then it is well-known that the ordinary irreducible characters $\mathrm{Irr}(\overline{F})$ of the factor group $\overline{F} = \overline{G}/K$ can be lifted to $\overline{G}$, where $K \trianglelefteq \overline{G}$ is a characteristic subgroup of P. Therefore, it follows that a so-called Fischer matrix $\widehat{M(g)}$ of the factor group $\overline{F}$ is contained in the corresponding Fischer matrix $M(g)$ of $\overline{G}$. In this paper, a method called the *lifting of Fischer matrices technique* is used to construct $M(g)$ from $\widehat{M(g)}$. A maximal subgroup $2^{4+5}\cdot A_8$ of the Dempwolff group $2^5\cdot GL_5(2)$ will be used as an example to demonstrate this method.

Mathematics Subject Classification (2020): 20D10, 20D15, 20D20

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1 Introduction

Let K be a normal subgroup of a finite group extension $\overline{G} = P.G$ such that K is characteristic in the p-group P. It is a well-known fact that the ordinary irreducible characters $\mathrm{Irr}(\overline{F})$ of the factor group

$$\overline{F} = \overline{G}/K \simeq P_1.G$$

can be lifted to $\overline{G}$, where the set $\mathrm{Irr}(\overline{F})$ is identified with $\chi_i \in \mathrm{Irr}(\overline{G})$ such that $K \leqslant \ker(\chi_i)$. Therefore, for a class representative $g \in G$ the Fischer matrix $\widehat{M(g)}$ of $\overline{F}$ is embedded in the corresponding Fischer matrix $M(g)$ of $\overline{G}$ (see [8]).

In this paper, we will show how to construct $M(g)$ from $\widehat{M(g)}$ and call this method the *lifting of Fischer matrices technique*. If P is non-abelian and K the Frattini subgroup $\Phi(P)$ of P, then

$$P_1 = P/\Phi(P)$$

is an elementary abelian p-group. If this is the case, then the lifting of Fischer matrices technique is very powerful to construct the ordinary character table of $\overline{G}$ by the Fischer matrices technique. The lifting of Fischer matrices technique can be readily applied to the maximal subgroups of the finite simple groups (see [6]) with structures similar to the group $\overline{G}$ described above.

First the relationship between the conjugacy classes of $\overline{G}$, $\overline{F}$ and G will be discussed. A proof of a theorem concerning the relationship between the orders of conjugacy classes of P and $\overline{G}$ is given. The construction of the Fischer matrices $M(g)$ of $\overline{G}$ from the Fischer matrices $\widehat{M(g)}$ of $\overline{F}$ is outlined. The paper is concluded by an application of the lifting of Fischer matrices technique to a maximal subgroup $2^{4+5}\cdot A_8$ of the Dempwolff group $2^5\cdot GL_5(2)$ (see [7]).

Computations are carried out with the aid of the computer algebra systems MAGMA and GAP (see [4] and [24], respectively), and ATLAS notation [6] is mostly followed. For an update on recent developments around Fischer matrices, interested readers are referred to the papers [3], [20], [19], [18] and [21].

2 Relationship between the conjugacy classes of $\overline{G}$, $\overline{F}$ and G

Let $\overline{G} = P.G$ be a finite extension of a normal p-subgroup P of $\overline{G}$ by a group G. If K is a non-trivial characteristic subgroup of P then $K \lhd \overline{G}$. Hence we obtain the structures

$$\overline{G} = K.\overline{F} \qquad \text{and} \qquad \overline{F} = \overline{G}/K \simeq P_1.G,$$

where $P_1 \simeq P/K$. Moreover, the commutative diagram depicted in Figure 1 is associated with the structures $\overline{G}$, $\overline{F}$ and G, where η_1, η_2 and $\eta = \eta_2 \circ \eta_1$ are the natural homomorphisms from $\overline{G}$ onto $\overline{F}$, $\overline{F}$ onto G and $\overline{G}$ onto G, respectively.

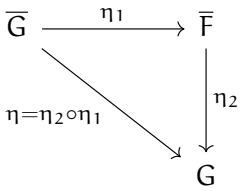


Figure 1

Let us consider the group $\overline{F}$, then G is identified with $\overline{F}/P_1$ under the map η_2. Moreover, under the map η_2, the pre-image of a conjugacy class $[P_1\overline{q}]$ in $\overline{F}/P_1$ is a union

$$\bigcup_{i=1}^{\widehat{c(g)}} [\overline{q_i}]$$

of say $\widehat{c(g)}$ conjugacy classes $[\overline{q_i}]$ in $\overline{F}$. Note that each coset $P_1\overline{q}$ can be identified with a $g \in G$ such that $\overline{q}$ is a lifting for g. Therefore, corresponding to a class representative $g \in G$ there is a set

$$\widehat{X(g)} = \{\overline{q_1} = \overline{q}, \overline{q_2}, \ldots, \overline{q_{\widehat{c(g)}}}\}$$

of representatives of conjugacy classes $[\overline{q_i}]$ of $\overline{F}$. Similarly, a pre-image of a class $[\overline{q_i}]$ of $\overline{F}$, with $\overline{q_i} \in \widehat{X(g)}$, under the map η_1 will be a union $\bigcup_{j=1}^{c(\overline{q_i})} [\overline{g_{i_j}}]$ of $c(\overline{q_i})$ classes $[\overline{g_{i_j}}]$ of $\overline{G}$. Note that $\overline{q_i} \in \widehat{X(g)}$ is identified with a coset $K\overline{g_i} \in \overline{G}/K \simeq \overline{F}$ where $\overline{g_i}$ is a lifting for $\overline{q_i}$ in $\overline{G}$. Hence a set

$$X(\overline{q_i}) = \{\overline{g_{i_1}} = \overline{g_i}, \overline{g_{i_2}}, \ldots, \overline{g_{i_{c(\overline{q_i})}}}\}$$

of representatives of conjugacy classes $[\overline{g_{i_j}}]$ is obtained from the coset $K\overline{g_i}$ (or $\overline{q_i} \in \overline{F}$). Since $\eta = \eta_2 \circ \eta_1$ (see Figure 1), it follows

that a pre-image for $g \in G$ under the map η is a set

$$\overline{X(g)} = \bigcup_{i=1}^{\widehat{c(g)}} X(\overline{q_i}) = X(\overline{q_1}) \bigcup X(\overline{q_2}) \bigcup \ldots \bigcup X(\overline{q_c})$$

$$= \Big\{ \overline{g_1}_1, \overline{g_1}_2, \ldots, \overline{g_1}_{c(\overline{q_1})}, \overline{g_2}_1, \overline{g_2}_2, \ldots, \overline{g_2}_{c(\overline{q_2})}, \ldots, \overline{g_{\widehat{c(g)}}}_1, \overline{g_{\widehat{c(g)}}}_2, \ldots, \overline{g_{\widehat{c(g)}}}_{c(\overline{q_c})} \Big\}.$$

where $q_c := q_{\widehat{c(g)}}$. Note that

$$\bigcup_{g \in [g]_G} \overline{X(g)}$$

contains the representatives of all the distinct conjugacy classes of $\overline{G}$. The so-called *coset analysis technique* due to Moori in [12] and [13] is the method of choice to compute the conjugacy classes of $\overline{G}$ and $\overline{F}$ if K and P_1 are abelian.

3 Class orders of an extension group $\overline{G} = P.G$

Lemma 3.1 *Let $\overline{G} = N.G$ be a finite extension of N by a group G. Then the order $o(N\overline{g})$ of a coset $N\overline{g}$ of N in $\overline{G}$ divides the order $o(\overline{g})$ of $\overline{g} \in \overline{G}$.*

Proof — Since

$$(N\overline{g})^{o(\overline{g})} = N\overline{g}^{o(\overline{g})} = N.$$

we obtain that $o(N\overline{g}) \mid o(\overline{g})$, which proves the claim. □

Let us note that, under the natural homomorphism $\eta : \overline{G} \longrightarrow \overline{G}/N$, where $\eta(\overline{g}) = N\overline{g}$ and $\ker(\eta) = N$, the group G is identified with the quotient group $\overline{G}/N$. Moreover, the coset $N\overline{g}$ is also identified with $g \in G$, where $\overline{g}$ is a lifting of g in $\overline{G}$. Hence $o(g) = o(N\overline{g})$ and, by Lemma 3.1, it follows that $o(g) \mid o(\overline{g})$.

Theorem 3.2 *Let $\overline{G} = P.G$ be a finite extension of a p-group P by G. Then, for $\overline{g} \in \overline{G}$ and $P\overline{g} \in \overline{G}/P$, the order*

$$o(\overline{g}) \in \{p^n o(P\overline{g}) \mid n = 0, 1, 2, 3, \ldots, v\},$$

where p^ν is the largest order of any element $\rho \in P$.

Proof — Let $\overline{G} = P.G$ be a finite extension of a p-group P by G. Let $P\overline{g}$ be a coset of P in $\overline{G}$ and $o(\overline{g})$ and $o(P\overline{g})$ the orders of $\overline{g} \in \overline{G}$ and $P\overline{g} \in \overline{G}/P$, respectively. It is well-known that the order of an element $\rho \in P$ is a power of the prime p. Let say that the orders of elements in P form the set $\{1, p^1, p^2, \ldots, p^\nu\}$, where ν is the largest power of the prime p. Then

$$(P\overline{g})^{o(P\overline{g})} = P\overline{g}^{o(P\overline{g})} = P.$$

Hence, $\overline{g}^{o(P\overline{g})} \in P$. Note that, if $\overline{g}^{o(P\overline{g})} = 1_P$, then $o(\overline{g}) \mid o(P\overline{g})$ and, by Lemma 3.1, $o(\overline{g}) = o(P\overline{g})$. For $\overline{g}^{o(P\overline{g})} \neq 1_P$ then $o(\overline{g}^{o(P\overline{g})}) = p^n$, for some $n \in \{1, 2, 3, \ldots, \nu\}$. So that

$$(\overline{g}^{o(P\overline{g})})^{p^n} = \overline{g}^{o(P\overline{g})p^n} = 1_P$$

and it follows that $o(\overline{g}) \mid p^n o(P\overline{g})$. It follows that $p^n o(P\overline{g}) = o(\overline{g})k$, for some $k \in \mathbb{Z}^+$. But, from Lemma 3.1, we have that $o(\overline{g}) = o(P\overline{g})k^*$, for some $k^* \in \mathbb{Z}^+$. Since $\overline{g}^{o(P\overline{g})} \neq 1_P$, it follows that $o(\overline{g}) \neq o(P\overline{g})$ and hence $k^* \neq 1$. From $o(\overline{g}) = o(P\overline{g})k^*$, we obtain

$$p^n o(P\overline{g}) = o(P\overline{g})k^*k$$

and this implies that $p^n = k^*k$. Notice that $k^* \in \{p^1, p^2, \ldots, p^\nu\}$ since $k^* \neq 1$. Let us take $n = 1$, then $p = k^*k$ and hence $k^* = p$ and $k = 1$. Thus $po(P\overline{g}) = o(\overline{g})$. If $n = 2$, then

$$p^2 = k^*k \Rightarrow k^* = p^2$$

and $k = 1$ or $k^* = k = p$. Consequently,

$$p^2 o(P\overline{g}) = o(\overline{g}) \qquad \text{or} \qquad p^2 o(P\overline{g}) = o(\overline{g})p \Rightarrow po(P\overline{g}) = o(\overline{g}).$$

For the case $n = 3$, we have that

$$p^3 = k^*k \Rightarrow k^* = p^3$$

and

$$k = 1, \quad k^* = p^2 \quad \text{and} \quad k = p \quad \text{or} \quad k^* = p \quad \text{and} \quad k = p^2.$$

Hence we obtain that

$$p^3o(P\overline{g}) = o(\overline{g}),\ p^3o(P\overline{g}) = o(\overline{g})p \Rightarrow p^2o(P\overline{g}) = o(\overline{g})$$

or

$$p^3o(P\overline{g}) = o(\overline{g})p^2 \Rightarrow po(P\overline{g}) = o(\overline{g}).$$

If we continue in the same manner up to $n = \nu$ then we obtain that

$$o(\overline{g}) \in \left\{po(P\overline{g}), p^2o(P\overline{g}), \ldots, p^{\nu-1}o(P\overline{g}), p^{\nu}o(P\overline{g})\right\}.$$

Therefore we can conclude that

$$o(\overline{g}) \in \left\{p^n o(P\overline{g}) \mid n = 0, 1, 2, 3, \ldots, \nu\right\},$$

which proves the claim. □

Since $G \simeq \overline{G}/P$ and also a coset $P\overline{g} \in \overline{G}/P$ is identified with $g \in G$, where $\overline{g}$ is a lifting of g, Corollary 3.3 below follows from Theorem 3.2.

Corollary 3.3 *Let* $\overline{G} = P.G$ *be a finite extension of a* p*-group* P *by* G*. For a lifting* $\overline{g} \in \overline{G}$ *of* $g \in G$*, the order* $o(\overline{g}) \in \left\{p^n o(g) \mid n = 0, 1, 2, 3, \ldots, \nu\right\}$.

4 Fischer matrices of $\overline{F}$

Suppose that $\overline{F}$ has t orbits on $\mathrm{Irr}(P_1)$ with corresponding inertia groups

$$\overline{H}_s = P_1.H_s = \left\{\overline{q} \in \overline{F} \mid \theta_s^{\overline{q}} = \theta_s\right\},$$

with $s = 1, 2, \ldots, t$, where $\theta_s \in \mathrm{Irr}(P_1)$ is an orbit representative and $\overline{H}_s/P_1 \simeq H_s \leqslant G$. Now each θ_s extends to a projective character

$$\psi_s \in \mathrm{Irr}\,\mathrm{Proj}(\overline{H_s}, \overline{\alpha}_s)$$

with factor set $\overline{\alpha}_s$, i.e.

$$\psi_s \downarrow_{P_1} = \theta_s.$$

Furthermore, an ordinary irreducible character $\chi = (\psi_s\overline{\beta})^{\overline{F}}$ of $\overline{F}$ is obtained by induction of $\psi_s\overline{\beta} \in \mathrm{Irr}(\overline{H_s})$ to $\overline{F}$. Since $\overline{\alpha}_s$ is constant on cosets of P_1 in $\overline{H}_s$, it can be identified as a factor set α_s of the inertia

factor H_s. Moreover, $\overline{\beta}$ is a lift for

$$\beta \in \operatorname{Irr}\operatorname{Proj}(H_s,\, \alpha_s^{-1})$$

to $\overline{H_s}$ and the set $\operatorname{Irr}\operatorname{Proj}(H_s,\, \alpha_s^{-1})$ can be obtained from the ordinary irreducible characters $\operatorname{Irr}(\overline{H_s})$ which contain θ_s as an irreducible constituent on their restriction to P_1. By Gallagher [11], we obtain

$$\operatorname{Irr}(\overline{F}) = \bigcup_{s=1}^{t} \left\{(\psi_s\overline{\beta})^{\overline{F}} \mid \beta \in \operatorname{Irr}\operatorname{Proj}(H_s,\, \alpha_s^{-1})\right\}.$$

Hence the set $\operatorname{Irr}(\overline{F})$ is partitioned into t blocks B_s with each block B_s corresponding to an inertia group $\overline{H_s}$. Observe that

$$|\operatorname{Irr}(\overline{F})| = |\operatorname{Irr}(H_1)| + |\operatorname{Irr}\operatorname{Proj}(H_2,\, \alpha_2^{-1})| + \ldots + |\operatorname{Irr}\operatorname{Proj}(H_t,\, \alpha_t^{-1})|.$$

We take $\overline{H_1} = \overline{F}$ and $H_1 = G$. We define the set

$$R(g) = \left\{(s, y_k) \mid 1 \leqslant s \leqslant t,\, H_s \cap [g] \neq \emptyset, 1 \leqslant k \leqslant r\right\},$$

where y_k, $k = 1, 2, \ldots, r$, are representatives of the α_s^{-1}-regular conjugacy classes $[y_k]$ of H_s that fuse into a class $[g]$ of $H_1 = G$.

Theorem 4.1 *Let y_{l_k} be representatives of the conjugacy classes of $\overline{H}_s$, where each y_{l_k} has y_k as an image under the homomorphism $\overline{H_s} \longrightarrow H_s$ whose kernel is P_1. Let also*

$$\widehat{X(g)} = \left\{\overline{q_1} = \overline{q}, \overline{q_2}, \ldots, \overline{q_{c\widehat{(g)}}}\right\}$$

be the set which contains the representatives of the classes of $\overline{F}$ coming from $P_1\overline{q}$, where $\overline{q}$ is a lifting of g. Then for $\overline{q_i} \in \widehat{X(g)}$, we have

$$(\psi_s\overline{\beta})^{\overline{F}}(\overline{q_i}) \;=\; \sum_{y_k:(s,y_k)\in R(g)} \left[\sum_{l}{}' \frac{|C_{\overline{F}}(\overline{q_i})|}{|C_{\overline{H_s}}(y_{l_k})|}\,\psi_s(y_{l_k})\right]\beta(y_k).$$

Proof — See [25] for the proof. □

A Fischer matrix $\widehat{M(g)} = \left(a^i_{(s,y_k)}\right)$ of $\overline{F}$ is then defined as

$$\left(a^i_{(s,y_k)}\right) = \left(\sum_l{}' \frac{|C_{\overline{F}}(\overline{q_i})|}{|C_{\overline{H_s}}(y_{l_k})|}\psi_s(y_{l_k})\right),$$

with columns indexed by $\widehat{X(g)}$, rows indexed by $R(g)$ and where $\sum'_l$ is the summation over all l for which y_{l_k} is conjugate to $\overline{q_i}$ in $\overline{F}$ (see [8]). The Fischer matrix $\widehat{M(g)}$ (see Figure 2) is partitioned row-wise into blocks $\widehat{M_s(g)}$, where each block corresponds to an inertia group $\overline{H_s}$.

	$\lvert C_{\overline{F}}(\overline{q_1})\rvert$	$\lvert C_{\overline{F}}(\overline{q_2})\rvert$	$\cdots$	$\lvert C_{\overline{F}}(\overline{q_{\widehat{c(g)}}})\rvert$
$\lvert C_G(g)\rvert$	1	1	$\cdots$	1
$\lvert C_{H_2}(y_1)\rvert$	$a^1_{(2,y_1)}$	$a^2_{(2,y_1)}$	$\cdots$	$a^{\widehat{c(g)}}_{(2,y_1)}$
$\lvert C_{H_2}(y_2)\rvert$	$a^1_{(2,y_2)}$	$a^2_{(2,y_2)}$	$\cdots$	$a^{\widehat{c(g)}}_{(2,y_2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lvert C_{H_s}(y_1)\rvert$	$a^1_{(s,y_1)}$	$a^2_{(s,y_1)}$	$\cdots$	$a^{\widehat{c(g)}}_{(s,y_1)}$
$\lvert C_{H_s}(y_2)\rvert$	$a^1_{(s,y_2)}$	$a^2_{(s,y_2)}$	$\cdots$	$a^{\widehat{c(g)}}_{(s,y_2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lvert C_{H_t}(y_1)\rvert$	$a^1_{(t,y_1)}$	$a^2_{(t,y_1)}$	$\cdots$	$a^{\widehat{c(g)}}_{(t,y_1)}$
$\lvert C_{H_t}(y_2)\rvert$	$a^1_{(t,y_2)}$	$a^2_{(t,y_2)}$	$\cdots$	$a^{\widehat{c(g)}}_{(t,y_2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	m_1	m_2	$\cdots$	$m_{\widehat{c(g)}}$

Figure 2: The Fischer Matrix $\widehat{M(g)}$

We write $|C_{\overline{F}}(\overline{q_i})|$, for each $\overline{q_i} \in \widehat{X(g)}$, at the top of the columns

of $\widehat{M(g)}$ and at the bottom we write

$$m_i = [C_{\overline{q}} : C_{\overline{F}}(\overline{q_i})] = |P_1|\frac{|C_G(g)|}{|C_{\overline{F}}(\overline{q_i})|}$$

and

$$C_{\overline{q}} = \{x \in \overline{F} | x(P_1\overline{q}) = (P_1\overline{q})x\} = P_1.C_G(g).$$

On the left of each row we write $|C_{H_s}(y_k)|$, where the α_s^{-1}-conjugacy classes $[y_k]$, $k = 1, 2, \ldots, r$, of an inertia factor H_s fuse into the conjugacy class $[g]$ of G. Since $|\widehat{X(g)}| = |R(g)|$ it follows that $\widehat{M(g)}$ is a square matrix of size $\widehat{c(g)}$. When there is no class fusion of an inertia factor H_s into a class $[g]$, the block $\widehat{M_s(g)}$ is omitted from $\widehat{M(g)}$.

Instead of directly using the formal definition above of a Fischer matrix $\widehat{M(g)}$, the arithmetical properties of $\widehat{M(g)}$ below are used to compute the entries of $\widehat{M(g)}$ (see [15]):

1. $a^i_{(1,g)} = 1$, for all $i = \{1, 2, \ldots, \widehat{c(g)}\}$.
2. $|\widehat{X(g)}| = |R(g)|$.
3. $\sum_{i=1}^{\widehat{c(g)}} m_i\, a^i_{(s,y_k)}\, \overline{a^i_{(s',y'_k)}} = \delta_{(s,y_k),(s',y'_k)}\, \frac{|C_G(g)|}{|C_{H_s}(y_k)|}\, |P_1|$.
4. $\sum_{(s,y_k)\in R(g)} a^i_{(s,y_k)}\, \overline{a^{i'}_{(s,y_k)}}\, |C_{H_s}(y_k)| = \delta_{ii'}\, |C_{\overline{F}}(\overline{q_i})|$.
5. $\widehat{M(g)}$ is square and nonsingular.

If P_1 is an elementary abelian p-group, we have the following additional properties of $\widehat{M(g)}$:

6. $a^1_{(s,y_k)} = \frac{|C_G(g)|}{|C_{H_s}(y_k)|}$.
7. $|a^1_{(s,y_k)}| \geqslant |a^i_{(s,y_k)}|$.
8. $a^i_{(s,y_k)} \equiv a^1_{(s,y_k)} (\mathrm{mod}\ p)$.

The partial character table of $\overline{F}$ on the classes

$$\widehat{X(g)} = \{\overline{q_1} = \overline{q}, \overline{q_2}, \ldots, \overline{q_{\widehat{c(g)}}}\}$$

is given by

$$\begin{bmatrix} C_1(g)\,M_1(g) \\ C_2(g)\,M_2(g) \\ \vdots \\ C_t(g)\,M_t(g) \end{bmatrix}.$$

The Fischer matrix $\widehat{M(g)}$ is divided into blocks $M_s(g)$ with each block corresponding to an inertia group $\overline{H}_s$ and $C_s(g)$ is the partial character table of H_s with factor set α_s^{-1} consisting of the columns corresponding to the α_s^{-1}-regular classes that fuse into $[g]$ in G. We obtain the characters of $\overline{F}$ by multiplying the relevant columns of the projective characters of H_s with factor set α_s^{-1} by the rows of $M(g)$.

5 Constructing the Fischer matrices $M(g)$ of $\overline{G}$ from the matrices $\widehat{M(g)}$ of $\overline{F}$

Since $\overline{F}$ is a factor group of $\overline{G}$, the set $\mathrm{Irr}(\overline{F})$ can be lifted to $\overline{G}$ and the lifts are equivalent to characters $\chi_d \in \mathrm{Irr}(\overline{G})$ such that $K \leqslant \mathrm{Ker}(\chi_d)$. Moreover, the characters $\chi_i \in \mathrm{Irr}(P)$ such that $K \leqslant \mathrm{Ker}(\chi_i)$ are the lifts of $\widehat{\chi_i} \in \mathrm{Irr}(P_1)$ to P. Since G has t orbits on $\mathrm{Irr}(P_1)$ it follows that G has also t orbits on the lifts of $\mathrm{Irr}(P_1)$ to P and their corresponding inertia factor groups H_s coincide. Suppose G has b orbits on the rest of the characters $\chi_i \in \mathrm{Irr}(P)$ that do not contain K in their kernels and with corresponding inertia factors $H_{t+1}, H_{t+2}, \ldots, H_{t+b}$.

The columns of a Fischer matrix $M(g)$ of $\overline{G}$ (see Figure 3) are indexed by the sizes of the centralizers $|C_{\overline{G}}(\overline{g_{ij}})|$ of the pre-images

$$\overline{g_{ij}} \in \overline{X(g)}$$

of the elements

$$\overline{q_i} \in \widehat{X(g)}.$$

Notice that if $\chi \in \mathrm{Irr}(\overline{G})$ is a lift for $\widehat{\chi} \in \mathrm{Irr}(\overline{F})$ to $\overline{G}$, then

$$\chi(\overline{g_{i_j}}) = \widehat{\chi}(K\overline{g_i}),$$

where $K\overline{g_i}$ is identified with $\overline{q_i} \in \widehat{X(g)}$. Therefore the columns $a^{i_j}_{(s,y_k)}$, with $1 \leqslant s \leqslant t$, of $M(g)$ (see Figure 3) corresponding to the pre-

images $\overline{g_{ij}}$ of $\overline{q_i} \in \widehat{X(g)}$ will be duplicates of the column $a^{i}_{(s,y_k)}$ of $\widehat{M(g)}$ (see Figure 2) which is indexed by $|C_{\overline{F}}(\overline{q_i})|$. For example, the columns $a^{1_j}_{(s,y_k)}$, with $1 \leqslant s \leqslant t$, (in Figure 3), corresponding to the class representatives

$$\overline{g_{1_j}} \in X(\overline{q_1}) \subseteq \overline{X(g)}, \quad \text{with} \quad j = 1, 2, \ldots, c(\overline{q_1})$$

are the duplicates of the column $a^{1}_{(s,y_k)}$ of $\widehat{M(g)}$ (see Figure 2) which is indexed by $|C_{\overline{F}}(\overline{q_1})|$, that is,

$$a^{1}_{(s,y_k)} = a^{1_j}_{(s,y_k)},$$

for $1 \leqslant s \leqslant t$, $1 \leqslant k \leqslant r$ and $j = 1, 2, \ldots, c(\overline{q_1})$.

The rows of $M(g)$ which are obtained from the inertia factors H_s, with $s = t+1, t+2, \ldots, t+b$, are displayed in Figure 3. If there is no class fusion from $H_{t+1}, H_{t+2}, \ldots, H_{t+b}$ into $[g]$ then we have that $M(g) = \widehat{M(g)}$, except possibly for changes in the class orders of $\overline{g_{i_j}} \in \overline{X(g)}$. Furthermore, the entries $a^{i_j}_{(s,y_k)}$ for the rows in the blocks $M_s(g)$, with $s = t+1, t+2, \ldots, t+b$, are obtained from the orthogonal row and column relations of the matrix $M(g)$ of $\overline{G}$. And in the case where there are more than one possibility for these entries, we can, for instance, restrict characters of a parent group of $\overline{G}$ to $\overline{G}$ to obtain the correct entries. The final shape of a $M(g)$ of $\overline{G}$ is depicted in Figure 3. Note that $M(g)$ is a $(\widehat{c(g)} + h) \times (\widehat{c(g)} + r_s)$ matrix, which was obtained by adding r_s columns of sizes $\widehat{c(g)} \times 1$ and h rows of sizes $1 \times (\widehat{c(g)} + r_s)$ to the $\widehat{c(g)} \times \widehat{c(g)}$ matrix $\widehat{M(g)}$ of $\overline{F}$. The number $r_s = |\overline{X(g)}| - |\widehat{X(g)}|$ and h is the total number of classes $[y_k]$ of H_s, with $s = t+1, t+2, \ldots, t+b$, which fuse into $[g]$, where it is noted that $r_s = h$. Of course, to easily construct $M(g)$ from $\widehat{M(g)}$, we do not want h to be very large. With the construction of $M(g)$ discussed above, we obtain Theorem 5.1 below.

Theorem 5.1 *For a class representative* $g \in G$, *the Fischer matrix* $\widehat{M(g)}$ *of the factor group* $\overline{F}$ *is embedded in the corresponding Fischer matrix* $M(g)$ *of* $\overline{G}$.

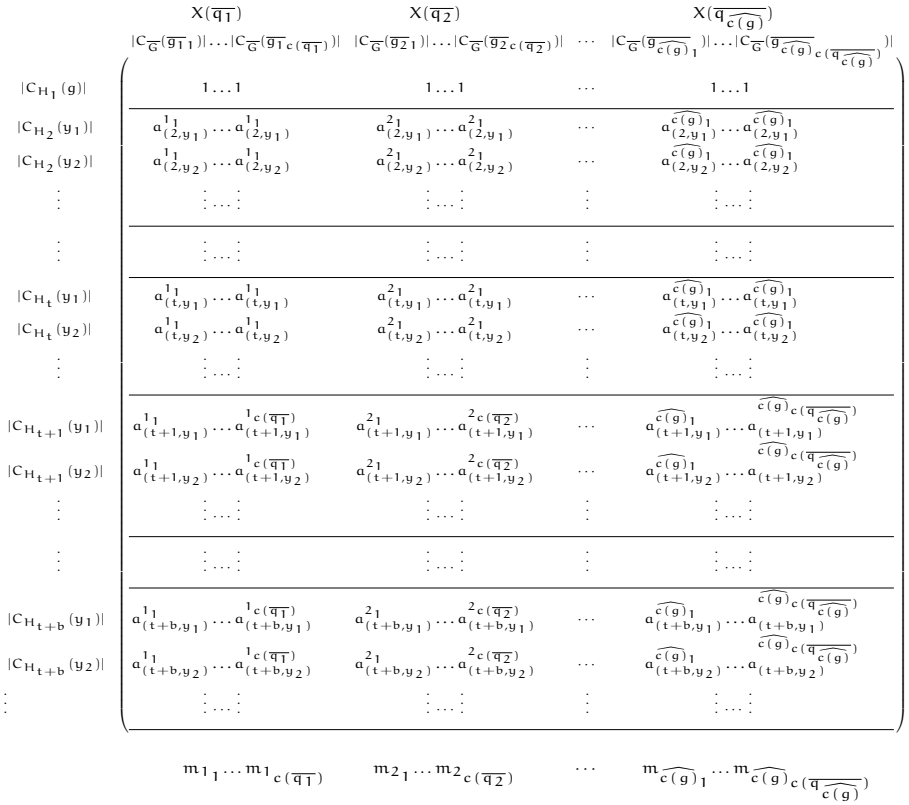


Figure 3: $\widehat{M(g)}$ of $\overline{F}$ lifted to $M(g)$ of $\overline{G}$

In a sense, we can say that the matrix $M(g)$ *is a lift for* $\widehat{M(g)}$ to $\overline{G}$. Having constructed the Fischer matrices $M(g)$ of $\overline{G}$ from the matrices $\widehat{M(g)}$ of $\overline{F}$ together with the fusion maps of the inertia factor groups $H_1, \ldots, H_s, H_{s+1}, \ldots, H_{s+b}$ into $H_1 = G$ and the sets

$$\operatorname{Irr}\operatorname{Proj}(H_s, \alpha_s^{-1}),$$

with $s = 1, 2, \ldots, t, t+1, t+2, \ldots, t+b$, the set $\operatorname{Irr}(\overline{G})$ can be fully assembled.

If P is non-abelian and we let K to be the Frattini subgroup $\Phi(P)$ of P then $P_1 = P/\Phi(P)$ is an elementary abelian p-group. In addition, if P_1 is a n-dimensional-module for G over the finite field $GF(p)$ then the technique of *lifting of Fischer matrices* of a factor group $\overline{F}$ to $\overline{G}$ is very powerful. Since P_1 is elementary abelian, we have extra

properties (see Section 4) at hand to compute with ease the entries of the matrices $\widehat{M(g)}$ of $\overline{F}$. In addition, a GAP routine in [5] was developed to compute possible candidates for a Fischer matrix of an extension N.G provided that the kernel N of the extension is an elementary abelian p-group and that N.G is not too large in size.

6 On a normalizer $\overline{G} = 2^{4+5}\cdot A_8$ of a radical subgroup 2^{4+5} in the Thompson group Th

The non-split extension group $D = 2^5\cdot GL_5(2)$, called the *Dempwolff group*, is the second largest maximal subgroup of the sporadic simple Thompson group Th (see [7]). The group D has two non-isomorphic maximal subgroups having the same structure $2^5\cdot(2^4 : GL_4(2))$ and order 10321920. The one group is the centralizer $C_D(2A)$ of elements in the class 2A of involutions in D whereas the other group is the normalizer $N_D(2^4)$ in D of a subgroup 2^4 of 2^5 in D. The Fischer matrices and associated ordinary character tables of $C_D(2A)$ and D were computed in [17] and [2], respectively. Note that $N_D(2^4)$ is also isomorphic to a normalizer $N_{Th}(2^{4+5})$ of a radical 2-subgroup 2^{4+5} in Th (see [27]) with structure $2^{4+5}\cdot A_8$. For the purpose of illustrating the *lifting of Fischer matrices* technique as discussed in Section 5, the structure $2^{4+5}\cdot A_8$ of $N_D(2^4)$ will be used.

6.1 Construction of $\overline{G}$ and $\overline{F}$

The Dempwolff group D is represented as a permutations on 7440 points in GAP, using Wilson's online ATLAS of Finite Group Representations [26]. The normalizer $N_D(2^4)$ in D of a subspace of dimension four in 2^5 is then constructed and has order 10321920. Next, the non-trivial normal subgroups of $N_D(2^4)$ are computed and there are four of them with structures 2^{4+5}, 2^5, 2^4 and 4^4. Hence four non-split structures

$$\overline{G} = 2^{4+5}\cdot A_8,\ \overline{G}_1 = 2^5\cdot(2^4 : GL_4(2)),\ \overline{G}_2 = 2^4\cdot(2^5\cdot GL_4(2))$$
$$\text{and}\quad \overline{G}_3 = 4^4\cdot(2.A_8)$$

exist for $N_D(2^4)$. In the remainder of this paper, we will turn our attention to the structure $\overline{G}$ which is a non-split extension of $P = 2^{4+5}$

by the alternating group $G = A_8 \simeq GL_4(2)$. Notice that P is a special 2-group of order 512, where the center $Z(P) = 2^4$ is a characteristic subgroup of P and therefore $Z(P)$ is normal in $\overline{G}$. Hence we can construct the factor group

$$\overline{F} = \overline{G}/Z(P) \simeq 2^5 \cdot A_8.$$

The group $\overline{F}$ is a two-fold cover of the maximal subgroup $2^4 : GL_4(2)$ of $GL_5(2)$. An isomorphic copy of $\overline{F}$ can be found as a subgroup in the double cover $2.Co_1$ of the Conway group Co_1. All of the structures of the groups above were identified with the aid of the ATLAS and the computer algebra systems MAGMA and GAP.

6.2 Action of $\overline{G}$ on P and Irr(P)

Let us observe that $\overline{G}$ has five orbits of lengths 1, 15, 16, 240 and 240 on P and hence by Brauer's theorem (see [9] or [10]) $N_D(2^4)$ has also five orbits on $Irr(P)$ with lengths 1, 1, 15, 15, and 120 with corresponding inertia factors

$$H_1 = H_2 = A_8,\ H_3 = H_4 = 2^3 : GL_3(2) \quad \text{and} \quad H_5 = 2^3 : (7 : 3).$$

Note that the orbits of $Irr(P)$ having lengths 1, 1, 15 and 15 contain the lifts of the linear characters of 2^5 to P. Therefore, A_8 has four orbits of lengths 1, 1, 15, 15 on $Irr(2^5)$ where the corresponding inertia factors coincide with

$$H_1 = H_2 = A_8 \quad \text{and} \quad H_3 = H_4 = 2^3 : GL_3(2).$$

6.3 The Conjugacy Classes and Fischer matrices of $\overline{F}$

Since P is a non-abelian 2-group we have less properties to use for the computation of the entries of the Fischer matrices of $\overline{G}$ than if the kernel of the extension is abelian (see Section 4). However, the process of finding the Fischer matrices of $\overline{G}$ can be made easier by constructing it from the Fischer matrices of the factor group $\overline{F}$ as explained in Section 5.

Let $\overline{F}$ be the non-split extension of $P_1 = 2^5$ by $G = A_8$, where P_1 is A_8-module of dimension five over $GF(2)$. Besides, P_1 is isomorphic to its dual space P_1^* as a five-dimensional module for A_8 over $GF(2)$. Hence G has four orbits of lengths 1, 1, 15 and 15 on both of P_1 and P_1^*. Note that P_1^* can be identified with the set $Irr(P_1)$. Hence, the inertia

factors corresponding to the four orbits of G are identified as

$$H_1 = H_2 = A_8 \qquad \text{and} \qquad H_3 = H_4 = 2^3 : GL_3(2).$$

By using similar techniques as in [1], [19] and [18], it can be shown that the irreducible projective character table of H_2 with factor set α_2^{-1} of order 2 (see Table 1) will be used in the construction of $\mathrm{Irr}(\overline{F})$. But these techniques are inconclusive to determine the type of characters which will be used for H_3 and H_4 in the construction of $\mathrm{Irr}(\overline{F})$ (see Table 6). These characters of H_3 and H_4 can be found from the ordinary irreducible characters of the corresponding inertia groups $\overline{H}_3$ and $\overline{H}_4$. The ordinary irreducible characters of $\overline{H}_3$ and $\overline{H}_4$ of interest to us are the ones which have the ordinary irreducible characters of P_1, contained in the two orbits of length 15, in their restriction to P_1. Using appropriate GAP commands, the desired character tables of H_3 and H_4 to be used in the construction of $\mathrm{Irr}(\overline{F})$ are *extracted* from the sets $\mathrm{Irr}(\overline{H}_3)$ and $\mathrm{Irr}(\overline{H}_4)$. It turns out that for H_3 we will use the set $\mathrm{Irr}(H_3)$ and for H_4 a set $\mathrm{Irr}\,\mathrm{Proj}(H_4, \alpha_4^{-1})$ with factor set α_4^{-1} of order 2 such that $|\mathrm{IrrProj}(H_4, \alpha_4^{-1})| = 5$. All the sets $\mathrm{Irr}\,\mathrm{Proj}(H_4, \alpha_j^{-1})$, with $j = 1, 2, 3, 4$, of irreducible projective characters with associated factor sets α_j^{-1} for the inertia factor H_4 are to be found in Table 4 of [22]. The desired set is listed in Table 2. Hence, the number $|\mathrm{Irr}(\overline{F})|$ of ordinary irreducible characters for $\overline{F}$ will be

$$\begin{aligned}|\mathrm{Irr}(\overline{F})| &= |\mathrm{Irr}(H_1)| + |\mathrm{IrrProj}(H_2, \alpha_2^{-1})| + |\mathrm{Irr}(H_3)| + |\mathrm{IrrProj}(H_4, \alpha_4^{-1})| \\ &= 14 + 9 + 11 + 5 = 39 = |[\overline{q}_i]_{\overline{F}}|,\end{aligned}$$

where $|[\overline{q}_i]_{\overline{F}}|$ is the number of classes of $\overline{F}$.

Considering the fusion of classes of H_1 and H_3 and the α_s^{-1}-regular classes of H_2 and H_4 into H_1, the sizes of the Fischer matrices $\widehat{M(g)}$ of $\overline{F}$ will vary from 1×1 to 4×4 matrices. Note that $\overline{F}$ is a subgroup of the two-fold cover $2.Co_1$ (see Table 3 for fusion map of $\overline{F}$ into $2.Co_1$) and making use of similar techniques as in [16] and [17] we compute the conjugacy classes and Fischer matrices $\widehat{M(g)}$ of $\overline{F}$ and are listed in Table 3 (columns 4 and 5) and Table 4 (columns 1 and 2), respectively. Since all the non-identity elements of P_1 are of order two, it follows by Theorem 3.2 and Corollary 3.3, that an element $\overline{q} \in \overline{F}$ which is a pre-image of an element $g \in G$ under the natural homomorphism $\eta_2 : \overline{F} \to G$ will have order $o(\overline{q}) \in \{o(g), 2o(g)\}$.

For example, a class in the set $\widehat{X(2A)} = \{2D, 2E, 4A\}$ of $\overline{F}$ in Table 3 coming from the class 2A of G has either class order 2 or 4. The parameters f_i and k in Table 3 are used in the computation of the centralizer order

$$|C_{\overline{F}}(\overline{q})| = k|C_G(g)|/f_i$$

of an element $\overline{q} \in \overline{F}$ (see [14] and [17]).

6.4 The Conjugacy Classes and Fischer matrices of $\overline{G}$

Following the lifting of Fischer matrices method described in Section 5, we will proceed to compute the Fischer matrices $M(g)$ of $\overline{G}$ by adding an appropriate number of rows and columns to the matrices $\widehat{M(g)}$ of $\overline{F}$ according to the number of classes $[y_k]$ of H_5 fusing into a class $[g]$ of A_8.

Let consider the natural epimorphisms

$$\eta_2 : \overline{F} \to G, \quad \eta_1 : \overline{G} \to \overline{F}, \quad \text{and} \quad \eta = \eta_2 \circ \eta_1 : \overline{G} \to G$$

(see Figure 1) for the groups $\overline{G} = 2^{4+5}\cdot A_8$ and $\overline{F} = 2^5\cdot A_8$, with

$$\ker(\eta_1) = K = Z(P) = 2^4, \ \ker(\eta_2) = P_1 = 2^5$$

$$\text{and} \quad \ker(\eta) = P = 2^{4+5}.$$

Then we obtain the sets

$$\widehat{X(g)},\ X(\overline{q_i}) \qquad \text{and} \qquad \overline{X(g)} = \bigcup_{i=1}^{\widehat{c(g)}} X(\overline{q_i})$$

as discussed in Section 2. As an example, let us consider the class 3B of G in Table 3 and we follow notation as in Section 2. Then under η_2 the set

$$\widehat{X(g)} = \{\overline{q_1} \in 3B, \overline{q_2} \in 6B, \overline{q_3} \in 6C, \overline{q_3} \in 6D\}$$

of class representatives $\overline{q_i} \in \overline{F}$ is obtained from a coset $P_1\overline{q}$, where the element $g \in 3B$ is identified with $P_1\overline{q}$ and we let $\overline{q_1} = \overline{q}$. As a known permutation representation of $\overline{G}$ is available, the pre-images $\eta_1^{-1}([\overline{q}]_{\overline{F}})$ of a conjugacy class $[\overline{q}]_{\overline{F}}$ in $\overline{F}$ under η_1, where $\overline{F}$ is identified with the factor group $\overline{G}/K$, are determined with the GAP commands

"`f:= NaturalHomomorhismByNormalSubgroup(`$\overline{G}$`,K)`",

"`F:=FusionConjugacyClassesOp(f)`".

So, we can determine the sets $X(\overline{q_i})$, $\overline{q_i} \in \widehat{X(g)}$, of class representatives of $\overline{G}$ coming from the cosets $K\overline{g_i}$, where $\overline{g_i}$ are liftings for the elements $\overline{q_i}$ to $\overline{G}$. Hence the sets

$$X(\overline{q_1}) = \{\overline{g_1}_1 \in 3B, \overline{g_1}_2 \in 6B\},\ X(\overline{q_2}) = \{\overline{g_2}_1 \in 6C\},$$

$$X(\overline{q_3}) = \{\overline{g_3}_1 \in 12A, \overline{g_3}_2 \in 12B\} \quad \text{and} \quad X(\overline{q_4}) = \{\overline{g_4}_1 \in 6D\}$$

of pre-images of $\overline{q_i} \in \widehat{X(g)}$ under η_1 are obtained. Since $\eta = \eta_2 \circ \eta_1$ it follows that the pre-images of a class representative $g \in 3B$ in G under η is the set

$$\overline{X(g)} = \bigcup_{i=1}^{\widehat{c(g)}=4} X(\overline{q_i}) = \{\overline{g_1}_1, \overline{g_1}_2, \overline{g_2}_1, \overline{g_3}_1, \overline{g_3}_2, \overline{g_4}_1\}.$$

Note that the elements of $\ker(\eta_1)$ have orders 1 and two and hence by Theorem 3.2 and Corollary 3.3, a pre-image $\overline{g} \in \overline{G}$ of a class representative $\overline{q} \in \overline{F}$ under η_1 has order

$$o(\overline{g}) \in \{o(\overline{q}), 2o(\overline{q})\}.$$

For example, the pre-images in $X(\overline{q_1})$ of $\overline{q_1} \in 3B$ have orders 3 and 6. Also, if we consider the order $o(\overline{g})$ of a pre-image $\overline{g} \in \overline{G}$ of a class representative $g \in G$ under η then $\overline{g}$ has order

$$o(\overline{g}) \in \{o(g), 2o(g), 4o(g)\}$$

by Theorem 3.2 and Corollary 3.3, since the elements of $\ker(\eta)$ are of orders, one, two and four. For example, the class representatives of $\overline{G}$ in $\overline{X(g)}$ coming from the class representative $g \in 3B$ of A_8 have orders 3, 6, and 12. The reader is referred to Table 3 for a summary of the pre-images (columns 4 and 7) of classes $[g]_G$ (in column 1) of G under the epimorphisms η_2 and η, respectively. Also, the pre-images (column 7) of classes $[\overline{q_i}]_{\overline{F}}$ (in column 4) of $\overline{F}$ under η_1 are summarized in Table 3.

From the discussion in Section 5 and the computations done in Subsection 6.3, the inertia factors H_1, H_2, H_3 and H_4 will contribute in to-

tal 39 ordinary irreducible characters to the set $\mathrm{Irr}(\overline{G})$. Since the Schur multiplier of H_5 is trivial, we will use the ordinary character table of H_5 in the construction of the set $\mathrm{Irr}(\overline{G})$. Hence

$$|\mathrm{Irr}(\overline{G})| = 39 + |\mathrm{Irr}(H_5)| = 47,$$

where $|\mathrm{Irr}(H_5)| = 8$. By Theorem 5.1 it follows that each Fischer matrix $\widehat{M(g)}$ of the factor group $\overline{F}$ is embedded in the corresponding Fischer matrix $M(g)$ of $\overline{G}$. Hence, the Fischer matrices of $\overline{F}$ will lay the platform from where the Fischer matrices of $\overline{G}$ will be built from. Following our discussion and notation in Section 5, it will be shown below how Fischer matrices $M(g)$ of $\overline{G}$ are built from the corresponding matrices $\widehat{M(g)}$ of $\overline{F}$. For this purpose, we will illustrate this process by hand of an example, where we show how the matrix $M(3B)$ (in Table 4) of $\overline{G}$ is built from the corresponding matrix $\widehat{M(3B)}$ of $\overline{F}$. The Fischer matrix $\widehat{M(3B)}$ which is associated with the class 3B of G is depicted in Figure 4.

$$\widehat{M(3B)} = \begin{array}{c} \\ |C_{H_1}(y_1 \in 3B)| \\ |C_{H_2}(y_1 \in 3B)| \\ |C_{H_3}(y_1 \in 3A)| \\ |C_{H_4}(y_1 \in 3A)| \end{array} \begin{array}{c} \begin{array}{cccc} |C_{\overline{F}}(\overline{q_1})| & |C_{\overline{F}}(\overline{q_2})| & |C_{\overline{F}}(\overline{q_3})| & |C_{\overline{F}}(\overline{q_4})| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 3 & -1 & -1 \\ 3 & -3 & -1 & 1 \end{pmatrix} \end{array}$$

Figure 4: $\widehat{M(3B)}$ of $\overline{F}$

The columns are indexed by the orders $|C_{\overline{F}}(\overline{q_i})|$ of the centralizers of class representatives $\overline{q_i} \in \widehat{X(g)}$, $g \in 3B$, and the rows are indexed by the orders of the centralizers $|C_{H_s}(y_1)|$ of class representatives y_k, with $k = 1$, of the inertia factors H_s, with $s = 1, 2, 3, 4$, which fuse into the class 3B of G.

Now, the matrix $M(3B)$ will be constructed from $\widehat{M(3B)}$ following the procedure in Section 5. The set $\overline{X(g)} = \bigcup_{i=1}^{c(g)=4} X(\overline{q_i})$ contains the pre-images of $\overline{q_i} \in \widehat{X(g)}$ under η_1 and will index the columns for $M(3B)$. Since

$$\widehat{\chi}(\overline{q_i}) = \chi(\overline{g_{ij}}),$$

for $\overline{g_{ij}} \in X(\overline{q_i})$, $\widehat{\chi} \in \mathrm{Irr}(\overline{F})$ and $\chi \in \mathrm{Irr}(\overline{G})$, the columns for $M(3B)$

corresponding to the blocks $M_s(3B)$, $i = 1,2,3,4$ (see Figure 5), will just be duplicates of the columns of the matrix $\widehat{M(3B)}$.

$$M(3B) = \begin{array}{c} \\ |C_{H_1}(y_1 \in 3B)| \\ |C_{H_2}(y_1 \in 3B)| \\ |C_{H_3}(y_1 \in 3A)| \\ |C_{H_4}(y_1 \in 3A)| \\ |C_{H_5}(y_1 \in 3A)| \\ |C_{H_5}(y_2 \in 3B)| \end{array} \begin{array}{c} \begin{array}{cccccc} \chi(\overline{q_1}) & & \chi(\overline{q_2}) & \chi(\overline{q_3}) & & \chi(\overline{q_4}) \\ |C_{\overline{G}}(\overline{g_{11}})| & |C_{\overline{G}}(\overline{g_{12}})| & |C_{\overline{G}}(\overline{g_{21}})| & |C_{\overline{G}}(\overline{g_{31}})| & |C_{\overline{G}}(\overline{g_{32}})| & |C_{\overline{G}}(\overline{g_{41}})| \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 3 & 3 & 3 & -1 & -1 & -1 \\ 3 & 3 & -3 & -1 & -1 & 1 \\ a & b & c & d & e & f \\ g & h & i & j & k & l \end{pmatrix} \end{array}$$

Figure 5: $M(3B)$ of $\overline{G}$

For example, the first column of $\widehat{M(3B)}$ labelled by $|C_{\overline{F}}(\overline{q_1})|$ will be duplicated for the columns of $M(3B)$ which are labelled by the sizes of the centralizers $|C_{\overline{G}}(\overline{g_{11}})|$, $|C_{\overline{G}}(\overline{g_{12}})|$ of the pre-images $\overline{g_{11}}, \overline{g_{12}}$ of $\overline{q_1} \in \widehat{X(g)}$ under η_1. Similarly, we have that the column of $\widehat{M(3B)}$ labelled by $|C_{\overline{F}}(\overline{q_3})|$ will be repeated for the columns of $M(3B)$ labelled by $|C_{\overline{G}}(\overline{g_{31}})|$ and $|C_{\overline{G}}(\overline{g_{32}})|$. Since both the elements $\overline{q_2}$ and $\overline{q_4}$ have only one pre-image under η_1, the columns of $\widehat{M(3B)}$ labelled by $|C_{\overline{F}}(\overline{q_2})|$ and $|C_{\overline{F}}(\overline{q_4})|$ will only be repeated once for $M(3B)$, i.e., the columns labelled by $|C_{\overline{G}}(\overline{g_{21}})|$ and $|C_{\overline{G}}(\overline{g_{41}})|$.

Furthermore, two classes 3A and 3B of the inertia factor H_5 fuse into the class 3B of G (see Table 5) and hence two more rows (rows 5 and 6 in Figure 5) will be added to complete the matrix $M(3B)$. The entries of rows five and six of $M(3B)$ are obtained by using the row and column orthogonality relations of Fischer matrices (see properties (3) and (4) in Section 4) and the desired matrix $M(3B)$ is found in Table 4. Similarly, all other Fischer matrices of $\overline{G}$ were computed and are listed in Table 4. Note that the entries of the blocks $M_s(3B)$ of $M(3B)$ corresponding to the inertia factors H_s, $s = 1,2,3,4$, are completely determined by the matrix $\widehat{M(3B)}$.

6.5 The Character Tables of $\overline{F}$ and $\overline{G}$

Let us notice that, using the sets $\mathrm{Irr}(H_1)$, $\mathrm{Irr}\,\mathrm{Proj}(H_2, \alpha^{-1})$ (see Table 1), $\mathrm{Irr}(H_3)$ and $\mathrm{Irr}\,\mathrm{Proj}(H_4, \alpha^{-1})$ (see Table 2), $\mathrm{Irr}(H_5)$, the fusion maps (see Table 5) of the inertia factors into H_1, Table 3 and Table 4, the set $\mathrm{Irr}(\overline{G})$ (see Table 6) is constructed according the Fischer matrices technique. The sets $\mathrm{Irr}(H_1)$, $\mathrm{Irr}(H_3)$ and $\mathrm{Irr}(H_5)$ can be found

in the GAP library. Consistency and accuracy checks of the character table of $\overline{G}$ have been carried with the aid of Programme E in [23] together with the computation of the class multiplication coefficients of the classes of $\overline{G}$. The set of ordinary irreducible characters of $\overline{G}$ will be partitioned into 5 blocks

$$\begin{aligned}
\triangle_1 &= \{\chi_j \mid \ 1 \leqslant j \leqslant 14\},\\
\triangle_2 &= \{\chi_j \mid 15 \leqslant j \leqslant 23\},\\
\triangle_3 &= \{\chi_j \mid 24 \leqslant j \leqslant 34\},\\
\triangle_4 &= \{\chi_j \mid 35 \leqslant j \leqslant 39\},\\
\triangle_5 &= \{\chi_j \mid 40 \leqslant j \leqslant 47\},
\end{aligned}$$

corresponding to the inertia factor groups H_1, H_2, H_3 H_4 and H_5, respectively, where $\chi_j \in \mathrm{Irr}(\overline{G})$.

Since $\overline{F}$ is a factor group $\overline{G}$ modulo 2^4, the ordinary irreducible characters of $\overline{F}$ are found in blocks $\triangle_1$ to $\triangle_4$ by deleting the columns of $\mathrm{Irr}(\overline{G})$ which correspond to the columns inserted into the matrices $\widehat{M(g)}$ to construct the matrices $M(g)$. As an example, $\widehat{M(3B)}$ has two columns inserted to obtain $M(3B)$. These inserted columns corresponds to the classes 6B and 12B of $\overline{G}$ and by deleting the entries of the columns for the blocks $\triangle_1$ to $\triangle_4$ of $\mathrm{Irr}(\overline{G})$ corresponding to these two classes, the partial ordinary character table of $\overline{F}$ coming form the class 3B of A_8 is obtained. The classes of $\overline{F}$ are listed in the second row of Table 6 and hence the complete ordinary character table of $\overline{F}$ can easily be read off from Table 6. The p-power maps of the classes of $\overline{G}$ are computed with the aid of Programme E and are listed in Table 6. In fact, the p-power maps of the classes of $\overline{F}$ are easily obtainable from the corresponding p-power maps of the classes of $\overline{G}$. Another advantage of using the lifting of Fischer matrices technique as presented in this paper, is that the embedded ordinary character table of the factor group $\overline{F}$ can be easily identified from the ordinary character table of $\overline{G}$.

Also, using the power maps of $\overline{G}$ and the Dempwolff group D, the permutation character $\chi(D|\overline{G}) = 1a + 31a$ of D on the classes of $\overline{G}$ and the restriction of some characters of small degrees of D to the set $\mathrm{Irr}(\overline{G})$, the fusion map of the classes of $\overline{G}$ into the classes of D is computed (see last column of Table 3).

7 Summary of tables

In this section, we present the tables referenced throughout the article.

Table 1: Projective character table of $H_2 = A_8$ with factor set α_2^{-1}

$[h]_{H_2}$	1A	2A	2B	3A	3B	4A	4B	5A	6A	6B	7A	7B	15A	15B
$\lvert C_{H_2}(h)\rvert$	20160	192	96	180	18	16	8	15	12	6	7	7	15	15
ϕ_1	8	0	0	−4	2	0	0	−2	0	0	1	1	1	1
ϕ_2	24	0	0	−6	0	0	0	−1	0	0	a	$\overline{a}$	−1	−1
ϕ_3	24	0	0	−6	0	0	0	−1	0	0	$\overline{a}$	a	−1	−1
ϕ_4	48	0	0	6	0	0	0	−2	0	0	−1	−1	1	1
ϕ_5	56	0	0	−4	−1	0	0	1	0	$\sqrt{3}i$	0	0	1	1
ϕ_6	56	0	0	−4	−1	0	0	1	0	$-\sqrt{3}i$	0	0	1	1
ϕ_7	56	0	0	2	2	0	0	1	0	0	0	0	b	$\overline{b}$
ϕ_8	56	0	0	2	2	0	0	1	0	0	0	0	$\overline{b}$	b
ϕ_9	64	0	0	4	−2	0	0	−1	0	0	1	1	−1	−1

where $a = -\frac{1}{2}(1 - i\sqrt{7})$ and $b = -\frac{1}{2}(1 - i\sqrt{15})$

Table 2: Projective character table of $H_4 = 2^3 : GL_3(2)$ with factor set α_4^{-1}

$[h]_{H_4}$	1A	2A	2B	2C	3A	4A	4B	4C	6A	7A	7B
$\lvert C_{H_4}(h)\rvert$	1344	192	32	32	6	16	8	8	6	7	7
ψ_1	8	0	0	0	2	0	0	0	0	1	1
ψ_2	8	0	0	0	−1	0	0	0	$\sqrt{3}i$	1	1
ψ_3	8	0	0	0	−1	0	0	0	$-\sqrt{3}i$	1	1
ψ_4	24	0	0	0	0	0	0	0	0	$\overline{c}$	c
ψ_5	24	0	0	0	0	0	0	0	0	c	$\overline{c}$

where $c = -\frac{1}{2}(1 + i\sqrt{7})$

Table 3: The conjugacy classes of $\overline{F}$ and $\overline{G}$

$[g]_G$	k	f_i	$[\overline{q}]_{\overline{F}}$	$\lvert C_{\overline{F}}(\overline{q})\rvert$	$\to [y]_{2.Co_1}$	$[\overline{g}]_{\overline{G}}$	$\lvert C_{\overline{G}}(\overline{g})\rvert$	$\to [y]_{2^5\cdot GL_5(2)}$
1A	32	$f_1 = 1$	1A	645120	1A	1A	10321920	1A
						2A	688128	2A
		$f_2 = 1$	2A	645120	2B	2B	645120	2A
		$f_3 = 15$	2B	43008	2C	2C	43008	2A
		$f_4 = 15$	2C	43008	2B	4A	43008	4A
2A	16	$f_1 = 2$	2D	1536	2B	4B	3072	4A
						2D	3072	2A
		$f_2 = 6$	2E	512	2D	4C	512	4B
		$f_3 = 8$	4A	384	4G	8A	384	8B
2B	8	$f_1 = 2$	4B	384	4E	4D	384	4B
		$f_2 = 6$	4C	128	4D	4E	128	4B
3A	2	$f_1 = 1$	6A	360	6I	6A	360	6B
		$f_2 = 1$	3A	360	3C	3A	360	3B
3B	8	$f_1 = 1$	3B	144	3B	3B	576	3A
						6B	192	6A
		$f_2 = 1$	6B	144	6K	6C	144	6A
		$f_3 = 3$	6C	48	6L	12A	96	12A
						12B	96	12A
		$f_4 = 3$	6D	48	6K	6D	48	6A
4A	8	$f_1 = 2$	4D	64	4E	4F	64	4A
		$f_2 = 2$	4E	64	4G	8B	64	8A
		$f_3 = 4$	4F	32	4H	8C	32	8A
4B	4	$f_1 = 2$	8A	16	8G	8D	16	8B
		$f_2 = 2$	8B	16	8F	8E	16	8B
5A	2	$f_1 = 1$	5A	30	5C	5A	30	5A
		$f_2 = 1$	10A	30	10H	10A	30	10A
6A	2	$f_1 = 2$	12A	12	12L	12C	12	12C
6B	4	$f_1 = 1$	6E	24	6K	12D	48	12C
						6E	48	6C
		$f_2 = 1$	6F	24	6K	12E	48	12C
						6F	48	6C
		$f_3 = 1$	12B	24	12P	24A	24	24A
		$f_4 = 1$	12C	24	12P	24B	24	24A
7A	4	$f_1 = 1$	7A	28	7B	7A	56	7A
						14A	56	14A
		$f_2 = 1$	14A	28	14C	28A	28	28A
		$f_3 = 1$	14B	28	14C	14B	28	14A
		$f_4 = 1$	14C	28	14D	14C	28	14A
7B	4	$f_1 = 1$	7B	28	7B	7B	56	7A
						14D	56	14D
		$f_2 = 1$	14D	28	14C	14E	28	14A
		$f_3 = 1$	14E	28	14D	14F	28	14A
		$f_4 = 1$	14F	28	14C	28B	28	28A
15A	2	$f_1 = 1$	15A	30	15E	15A	30	15A
		$f_2 = 1$	30A	30	30J	30A	30	30A
15B	2	$f_1 = 1$	15B	30	15E	15B	30	15B
		$f_2 = 1$	30B	30	30J	30B	30	30B

Table 4: The Fischer Matrices of $\overline{F}$ and $\overline{G}$

$\widehat{M(1A)}$	$\widehat{M(2A)}$	$M(1A)$	$M(2A)$
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 15 & 15 & -1 & -1 \\ 15 & -15 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 6 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 15 & 15 & 15 & -1 & -1 \\ 15 & 15 & -15 & 1 & -1 \\ 240 & -16 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 6 & 6 & -2 & 0 \\ 8 & -8 & 0 & 0 \end{pmatrix}$
$\widehat{M(2B)}$	$\widehat{M(3A)}$	$M(2B)$	$M(3A)$
$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$\widehat{M(3B)}$	$\widehat{M(4A)}$	$M(3B)$	$M(4A)$
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 3 & -1 & -1 \\ 3 & -3 & -1 & 1 \end{pmatrix}$	$M(4A) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 3 & 3 & 3 & -1 & -1 & -1 \\ 3 & 3 & -3 & -1 & -1 & 1 \\ 6 & -2 & 0 & 2 & -2 & 0 \\ 6 & -2 & 0 & -2 & 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$
$\widehat{M(4B)}$	$\widehat{M(5A)}$	$M(4B)$	$M(5A)$
$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$\widehat{M(6A)}$	$\widehat{M(6B)}$	$M(6B)$	$M(6A)$
$\begin{pmatrix} 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \end{pmatrix}$
$\widehat{M(7A)}$	$\widehat{M(7B)}$	$M(7A)$	$M(7B)$
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 2 & -2 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 2 & -2 & 0 & 0 & 0 \end{pmatrix}$
$\widehat{M(15A)}$	$\widehat{M(15B)}$	$M(15A)$	$M(15B)$
$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Table 5: The fusion maps of H_3 and H_5 into A_8

$[h]_{2^3:GL_3(2)} \longrightarrow$	$[g]_{A_8}$	$[h]_{2^3:GL_3(2)} \longrightarrow$	$[g]_{A_8}$
1A	1A	4B	4A
2A	2A	4C	4B
2B	2A	6A	6B
2C	2B	7A	7B
3A	3B	7B	7A
4A	4A		
$[h]_{2^3:(7:3)} \longrightarrow$	$[g]_{A_8}$	$[h]_{2^3:(7:3)} \longrightarrow$	$[g]_{A_8}$
1A	1A	6A	6B
2A	2A	6B	6B
3A	3B	7A	7B
3B	3B	7B	7A

Table 6: The character tables of $\overline{\mathrm{F}}$ and $\overline{\mathrm{G}}$

$[g]_{A_8}$			1A					2A			2B		3A	
$[\overline{q}]_{\overline{\mathrm{F}}}$	1A		2A	2B	2C	2D		2E	4A	4B	4C		6A	3A
$[\overline{g}]_{\overline{\mathrm{G}}}$	1A	2A	2B	2C	4A	4B	2D	4C	8A	4D	4E		6A	3A
p=2	1A	1A	1A	1A	2A	2A	1A	2A	4A	2B	2C		3A	3A
p=3	1A	2A	2B	2C	4A	4B	2D	4C	8A	4D	4E		2B	1A
p=5	1A	2A	2B	2C	4A	4B	2D	4C	8A	4D	4E		6A	3A
p=7	1A	2A	2B	2C	4A	4B	2D	4C	8A	4D	4E		6A	3A
χ_1	1	1	1	1	1	1	1	1	1	1	1		1	1
χ_2	7	7	7	7	7	−1	−1	−1	−1	3	3		4	4
χ_3	14	14	14	14	14	6	6	6	6	2	2		−1	−1
χ_4	20	20	20	20	20	4	4	4	4	4	4		5	5
χ_5	21	21	21	21	21	−3	−3	−3	−3	1	1		6	6
χ_6	21	21	21	21	21	−3	−3	−3	−3	1	1		−3	−3
χ_7	21	21	21	21	21	−3	−3	−3	−3	1	1		−3	−3
χ_8	28	28	28	28	28	−4	−4	−4	−4	4	4		1	1
χ_9	35	35	35	35	35	3	3	3	3	−5	−5		5	5
χ_{10}	45	45	45	45	45	−3	−3	−3	−3	−3	−3		0	0
χ_{11}	45	45	45	45	45	−3	−3	−3	−3	−3	−3		0	0
χ_{12}	56	56	56	56	56	8	8	8	8	0	0		−4	−4
χ_{13}	64	64	64	64	64	0	0	0	0	0	0		4	4
χ_{14}	70	70	70	70	70	−2	−2	−2	−2	2	2		−5	−5
χ_{15}	8	8	−8	−8	8	0	0	0	0	0	0		4	−4
χ_{16}	24	24	−24	−24	24	0	0	0	0	0	0		6	−6
χ_{17}	24	24	−24	−24	24	0	0	0	0	0	0		6	−6
χ_{18}	48	48	−48	−48	48	0	0	0	0	0	0		−6	6
χ_{19}	56	56	−56	−56	56	0	0	0	0	0	0		4	−4
χ_{20}	56	56	−56	−56	56	0	0	0	0	0	0		4	−4
χ_{21}	56	56	−56	−56	56	0	0	0	0	0	0		−2	2
χ_{22}	56	56	−56	−56	56	0	0	0	0	0	0		−2	2
χ_{23}	64	64	−64	−64	64	0	0	0	0	0	0		−4	4
χ_{24}	15	15	15	−1	−1	7	7	−1	−1	3	−1		0	0
χ_{25}	45	45	45	−3	−3	−3	−3	5	−3	−3	1		0	0
χ_{26}	45	45	45	−3	−3	−3	−3	5	−3	−3	1		0	0
χ_{27}	90	90	90	−6	−6	18	18	2	−6	6	−2		0	0
χ_{28}	105	105	105	−7	−7	−7	−7	1	1	9	−3		0	0
χ_{29}	105	105	105	−7	−7	1	1	9	−7	−3	1		0	0
χ_{30}	105	105	105	−7	−7	17	17	−7	1	−3	1		0	0
χ_{31}	120	120	120	−8	−8	8	8	8	−8	0	0		0	0
χ_{32}	210	210	210	−14	−14	10	10	−6	2	6	−2		0	0
χ_{33}	315	315	315	−21	−21	−21	−21	3	3	3	−1		0	0
χ_{34}	315	315	315	−21	−21	3	3	−5	3	−9	3		0	0
χ_{35}	120	120	−120	8	−8	0	0	0	0	0	0		0	0
χ_{36}	120	120	−120	8	−8	0	0	0	0	0	0		0	0
χ_{37}	120	120	−120	8	−8	0	0	0	0	0	0		0	0
χ_{38}	360	360	−360	24	−24	0	0	0	0	0	0		0	0
χ_{39}	360	360	−360	24	−24	0	0	0	0	0	0		0	0
χ_{40}	240	−16	0	0	0	8	−8	0	0	0	0		0	40
χ_{41}	240	−16	0	0	0	8	−8	0	0	0	0		0	0
χ_{42}	240	−16	0	0	0	8	−8	0	0	0	0		0	0
χ_{43}	720	−48	0	0	0	24	−24	0	0	0	0		0	0
χ_{44}	720	−48	0	0	0	24	−24	0	0	0	0		0	0
χ_{45}	1680	−112	0	0	0	−8	8	0	0	0	0		0	0
χ_{46}	1680	−112	0	0	0	−8	8	0	0	0	0		0	0
χ_{47}	1680	−112	0	0	0	−8	8	0	0	0	0		0	0

Table 6 (continued)

$[g]_{A_8}$	3B						4A				4B		5A	6A
$[\overline{q}]_{\overline{F}}$	3B		6B	6C		6D	4D	4E	4F	8A	8B	5A	10A	12A
$[\overline{g}]_{\overline{G}}$	3B	6B	6C	12A	12B	6D	8B	4F	8C	8D	8E	5A	10A	12C
p=2	3B	3B	3B	6B	6B	3B	4B	2D	4C	4D	4E	5A	5A	6A
p=3	1A	2A	2B	4A	4A	2C	8B	4F	8C	8D	8E	5A	10A	4D
p=5	3B	6B	6C	12B	12A	6D	8B	4F	8C	8D	8E	1A	2B	12C
p=7	3B	6B	6C	12A	12B	6D	8B	4F	8C	8D	8E	5A	10A	12C
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	−1	−1	−1	1	1	2	2	0
χ_3	2	2	2	2	2	2	2	2	2	0	0	−1	−1	−1
χ_4	−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	0	1
χ_5	0	0	0	0	0	0	1	1	1	−1	−1	1	1	−2
χ_6	0	0	0	0	0	0	1	1	1	−1	−1	1	1	1
χ_7	0	0	0	0	0	0	1	1	1	−1	−1	1	1	1
χ_8	1	1	1	1	1	1	0	0	0	0	0	−2	−2	1
χ_9	2	2	2	2	2	2	−1	−1	−1	−1	−1	0	0	1
χ_{10}	0	0	0	0	0	0	1	1	1	1	1	0	0	0
χ_{11}	0	0	0	0	0	0	1	1	1	1	1	0	0	0
χ_{12}	−1	−1	−1	−1	−1	−1	0	0	0	0	0	1	1	0
χ_{13}	−2	−2	−2	−2	−2	−2	0	0	0	0	0	−1	−1	0
χ_{14}	1	1	1	1	1	1	−2	−2	−2	0	0	0	0	−1
χ_{15}	2	2	−2	2	2	−2	0	0	0	0	0	−2	2	0
χ_{16}	0	0	0	0	0	0	0	0	0	0	0	−1	1	0
χ_{17}	0	0	0	0	0	0	0	0	0	0	0	−1	1	0
χ_{18}	0	0	0	0	0	0	0	0	0	0	0	−2	2	0
χ_{19}	−1	−1	1	−1	−1	1	0	0	0	0	0	1	−1	0
χ_{20}	−1	−1	1	−1	−1	1	0	0	0	0	0	1	−1	0
χ_{21}	2	2	−2	2	2	−2	0	0	0	0	0	1	−1	0
χ_{22}	2	2	−2	2	2	−2	0	0	0	0	0	1	−1	0
χ_{23}	−2	−2	2	−2	−2	2	0	0	0	0	0	−1	1	0
χ_{24}	3	3	3	−1	−1	−1	3	−1	−1	1	−1	0	0	0
χ_{25}	o	o	o	o	o	o	1	-3	1	1	-1	o	o	o
χ_{26}	0	0	0	0	0	0	1	−3	1	1	−1	0	0	0
χ_{27}	0	0	0	0	0	0	2	2	−2	0	0	0	0	0
χ_{28}	3	3	3	−1	−1	−1	−3	1	1	1	−1	0	0	0
χ_{29}	3	3	3	−1	−1	−1	−3	1	1	−1	1	0	0	0
χ_{30}	3	3	3	−1	−1	−1	1	−3	1	−1	1	0	0	0
χ_{31}	−3	−3	−3	1	1	1	0	0	0	0	0	0	0	0
χ_{32}	−3	−3	−3	1	1	1	−2	−2	2	0	0	0	0	0
χ_{33}	0	0	0	0	0	0	3	−1	−1	−1	1	0	0	0
χ_{34}	0	0	0	0	0	0	−1	3	−1	1	−1	0	0	0
χ_{35}	6	6	−6	−2	−2	2	0	0	0	0	0	0	0	0
χ_{36}	−3	−3	3	1	1	−1	0	0	0	0	0	0	0	0
χ_{37}	−3	−3	3	1	1	−1	0	0	0	0	0	0	0	0
χ_{38}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{39}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{40}	12	−4	0	0	0	0	0	0	0	0	0	0	0	0
χ_{41}	−6	2	0	A	−A	0	0	0	0	0	0	0	0	0
χ_{42}	−6	2	0	−A	A	0	0	0	0	0	0	0	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{44}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{45}	12	−4	0	0	0	0	0	0	0	0	0	0	0	0
χ_{46}	−6	2	0	A	−A	0	0	0	0	0	0	0	0	0
χ_{47}	−6	2	0	−A	A	0	0	0	0	0	0	0	0	0

$$A = -2\sqrt{3}i$$

Table 6 (continued)

$[g]_{A_8}$	6B						7A				
$[\overline{q}]_{\overline{F}}$	6E		6F		12B	12C	7A		14A	14B	14C
$[\overline{g}]_{\overline{G}}$	12D	6E	12E	6F	24A	24B	7A	14A	28A	14B	14C
p=2	6B	3B	6B	3B	12A	12B	7A	7A	14A	7A	7A
p=3	4B	2D	4B	2D	8A	8A	7B	14D	28B	14E	14F
p=5	12E	6F	12D	6E	24B	24A	7B	14D	28B	14E	14F
p=7	12D	6E	12E	6F	24A	24B	1A	2A	4A	2B	2C
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	−1	−1	−1	−1	−1	−1	0	0	0	0	0
χ_3	0	0	0	0	0	0	0	0	0	0	0
χ_4	1	1	1	1	1	1	−1	−1	−1	−1	−1
χ_5	0	0	0	0	0	0	0	0	0	0	0
χ_6	0	0	0	0	0	0	0	0	0	0	0
χ_7	0	0	0	0	0	0	0	0	0	0	0
χ_8	−1	−1	−1	−1	−1	−1	0	0	0	0	0
χ_9	0	0	0	0	0	0	0	0	0	0	0
χ_{10}	0	0	0	0	0	0	D	D	D	D	D
χ_{11}	0	0	0	0	0	0	$\overline{D}$	$\overline{D}$	$\overline{D}$	$\overline{D}$	$\overline{D}$
χ_{12}	−1	−1	−1	−1	−1	−1	0	0	0	0	0
χ_{13}	0	0	0	0	0	0	1	1	1	1	1
χ_{14}	1	1	1	1	1	1	0	0	0	0	0
χ_{15}	0	0	0	0	0	0	1	1	1	−1	−1
χ_{16}	0	0	0	0	0	0	D	D	D	−D	−D
χ_{17}	0	0	0	0	0	0	$\overline{D}$	$\overline{D}$	$\overline{D}$	-$\overline{D}$	−$\overline{D}$
χ_{18}	0	0	0	0	0	0	−1	−1	−1	1	1
χ_{19}	B	B	−B	−B	B	−B	0	0	0	0	0
χ_{20}	−B	−B	B	B	−B	B	0	0	0	0	0
χ_{21}	0	0	0	0	0	0	0	0	0	0	0
χ_{22}	0	0	0	0	0	0	0	0	0	0	0
χ_{23}	0	0	0	0	0	0	1	1	1	−1	−1
χ_{24}	1	1	1	1	−1	−1	1	1	−1	1	−1
χ_{25}	0	0	0	0	0	0	D	D	−D	D	−D
χ_{26}	0	0	0	0	0	0	$\overline{D}$	$\overline{D}$	−$\overline{D}$	$\overline{D}$	−$\overline{D}$
χ_{27}	0	0	0	0	0	0	−1	−1	1	−1	1
χ_{28}	−1	−1	−1	−1	1	1	0	0	0	0	0
χ_{29}	1	1	1	1	−1	−1	0	0	0	0	0
χ_{30}	−1	−1	−1	−1	1	1	0	0	0	0	0
χ_{31}	−1	−1	−1	−1	1	1	1	1	−1	1	−1
χ_{32}	1	1	1	1	−1	−1	0	0	0	0	0
χ_{33}	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	0	0	0	0	0	0	0	0	0	0	0
χ_{35}	0	0	0	0	0	0	1	1	−1	−1	1
χ_{36}	B	B	−B	−B	−B	B	1	1	−1	−1	1
χ_{37}	−B	−B	B	B	B	−B	1	1	−1	−1	1
χ_{38}	0	0	0	0	0	0	D	D	−D	−D	D
χ_{39}	0	0	0	0	0	0	$\overline{D}$	$\overline{D}$	−$\overline{D}$	−$\overline{D}$	$\overline{D}$
χ_{40}	2	−2	2	−2	0	0	2	−2	0	0	0
χ_{41}	C	−C	$\overline{C}$	−$\overline{C}$	0	0	2	−2	0	0	0
χ_{42}	$\overline{C}$	−$\overline{C}$	C	−C	0	0	2	−2	0	0	0
χ_{43}	0	0	0	0	0	0	E	−E	0	0	0
χ_{44}	0	0	0	0	0	0	$\overline{E}$	−$\overline{E}$	0	0	0
χ_{45}	−2	2	−2	2	0	0	0	0	0	0	0
χ_{46}	−C	C	−$\overline{C}$	$\overline{C}$	0	0	0	0	0	0	0
χ_{47}	−$\overline{C}$	$\overline{C}$	−C	C	0	0	0	0	0	0	0

$B = -\sqrt{3}i$, $C = -1 - \sqrt{3}i$, $D = \frac{-1-\sqrt{7}i}{2}$, $E = -1 + \sqrt{7}i$

Table 6 (continued)

$[g]_{A_8}$			7B					15A		15B
$[\overline{q}]_{\overline{F}}$	7B		14D	14E	14F	15A	30A	15B	30B	
$[\overline{g}]_{\overline{G}}$	7B	14D	14E	14F	28B	15A	30A	15B	30B	
p=2	7B	7B	7B	7B	14D	15A	15A	15B	15B	
p=3	7A	14A	14B	14C	28A	5A	10A	5A	10A	
p=5	7A	14A	14B	14C	28A	5A	10A	5A	10A	
p=7	1A	2A	2B	2C	4A	15B	30B	15A	30A	
χ_1	1	1	1	1	1	1	1	1	1	
χ_2	0	0	0	0	0	−1	−1	−1	−1	
χ_3	0	0	0	0	0	−1	−1	−1	−1	
χ_4	−1	−1	−1	−1	−1	0	0	0	0	
χ_5	0	0	0	0	0	1	1	1	1	
χ_6	0	0	0	0	0	F	F	$\overline{F}$	$\overline{F}$	
χ_7	0	0	0	0	0	$\overline{F}$	$\overline{F}$	F	F	
χ_8	0	0	0	0	0	1	1	1	1	
χ_9	0	0	0	0	0	0	0	0	0	
χ_{10}	$\overline{D}$	$\overline{D}$	$\overline{D}$	$\overline{D}$	$\overline{D}$	0	0	0	0	
χ_{11}	D	D	D	D	D	0	0	0	0	
χ_{12}	0	0	0	0	0	1	1	1	1	
χ_{13}	1	1	1	1	1	−1	−1	−1	−1	
χ_{14}	0	0	0	0	0	0	0	0	0	
χ_{15}	1	1	−1	−1	1	1	−1	1	−1	
χ_{16}	$\overline{D}$	$\overline{D}$	$-\overline{D}$	$-\overline{D}$	$\overline{D}$	−1	1	−1	1	
χ_{17}	D	D	−D	−D	D	−1	1	−1	1	
χ_{18}	−1	−1	1	1	1	1	−1	1	−1	
χ_{19}	0	0	0	0	0	1	−1	1	−1	
χ_{20}	0	0	0	0	0	1	−1	1	−1	
χ_{21}	0	0	0	0	0	F	−F	$\overline{F}$	-$\overline{F}$	
χ_{22}	0	0	0	0	0	$\overline{F}$	-$\overline{F}$	F	−F	
χ_{23}	1	1	−1	−1	1	−1	1	−1	1	
χ_{24}	1	1	1	−1	−1	0	0	0	0	
χ_{25}	$\overline{D}$	$\overline{D}$	$\overline{D}$	$-\overline{D}$	$-\overline{D}$	0	0	0	0	
χ_{26}	D	D	D	−D	−D	0	0	0	0	
χ_{27}	−1	−1	−1	1	1	0	0	0	0	
χ_{28}	0	0	0	0	0	0	0	0	0	
χ_{29}	0	0	0	0	0	0	0	0	0	
χ_{30}	0	0	0	0	0	0	0	0	0	
χ_{31}	1	1	1	−1	−1	0	0	0	0	
χ_{32}	0	0	0	0	0	0	0	0	0	
χ_{33}	0	0	0	0	0	0	0	0	0	
χ_{34}	0	0	0	0	0	0	0	0	0	
χ_{35}	1	1	−1	1	−1	0	0	0	0	
χ_{36}	1	1	−1	1	−1	0	0	0	0	
χ_{37}	1	1	−1	1	−1	0	0	0	0	
χ_{38}	$\overline{D}$	$\overline{D}$	$-\overline{D}$	$\overline{D}$	$-\overline{D}$	0	0	0	0	
χ_{39}	D	D	−D	D	−D	0	0	0	0	
χ_{40}	2	−2	0	0	0	0	0	0	0	
χ_{41}	2	−2	0	0	0	0	0	0	0	
χ_{42}	2	−2	0	0	0	0	0	0	0	
χ_{43}	$\overline{E}$	$-\overline{E}$	0	0	0	0	0	0	0	
χ_{44}	E	−E	0	0	0	0	0	0	0	
χ_{45}	0	0	0	0	0	0	0	0	0	
χ_{46}	0	0	0	0	0	0	0	0	0	
χ_{47}	0	0	0	0	0	0	0	0	0	

$D = \frac{-1-\sqrt{7}i}{2}$, $E = -1+\sqrt{7}i$, $F = \frac{-1-\sqrt{15}i}{2}$

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