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Otto H. Kegel (1934 – 2025)

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Otto Kegel was a very well known mathematician. He made significant contributions to mathematics in particular to the finite and locally finite groups. He possessed a characteristic that not many successful mathematicians share. He had a complete personality. He was delightful company for discussing mathematics and life in general. He was a very kind, generous and supportive group theorist. As A. Zalesski said: "He played a significant role in establishing scientific relations between Soviet and western group theorists".

Otto H. Kegel was born in Bethlehem, Pennsylvania (USA), the second of three sons to German parents on the 20th of July 1934.

In 1938, the family returned to Berlin, Germany. At the end of the Second World War, he was living on the small island Pellworm in the North Sea, with his mother and younger brother. To get a better education, he was sent to a boarding school in 1947. When the family moved to Frankfurt am Main in 1951, he continued school and graduated from high school in Frankfurt am Main in 1954. After finishing school, he decided to study mathematics at Johann Wolfgang Goethe University, Frankfurt am Main. His widespread interests also led him to study Persian, play chess, and visit many lectures in other fields. He spent 1956/57 studying in Paris. From Frankfurt, his Professor Reinhold Baer sent him as an assistant to the Oberwolfach Research Institute for Mathematics, a place he was very attached to, in the Black Forest. In 1961, he got his PhD from the Johann Wolfgang Goethe University in Frankfurt. His thesis is entitled "Vertauschbarkeit von Untergruppen und Kompositionsstruktur in endlichen Gruppen" ("Commutability of Subgroups and Composition Structure in Finite Groups") was written under the supervision of Professor Reinhold Baer, who was a very important mentor and supporter to him. In 1961, he married Waltraut. Together, they traveled to Chicago for three months in 1961 and for a year in 1963/64, during which Otto worked at the University of Chicago.

He completed his habilitation in Frankfurt in 1966. Then he worked as a professor in Frankfurt, Köln, and Tübingen, where his daughter Hannah was born. In 1967/68, he and his family accompanied Reinhold Baer to the University of New Mexico "Las Cruces" (USA). From there, he went to Queen Mary College, University of London, where he stayed from 1968 to 1975, first as a reader and, from 1970, as a professor. In 1975, he became full professor at University of Freiburg (Albert Ludwig University), Germany, where he was teaching until 1999. Otto Kegel was a renowned group theorist. He was coeditor of the Archiv für Mathematik from 1975 to 1989, of the Journal and Proceedings of the London Mathematical Society from 1970 to 1972, of Communications in Algebra from 1974 to 1985, and of the Georgian Mathematical Journal. In 1962, he gave a lecture at the International Congress of Mathematicians in Stockholm ("Locally Finite Groups with a Partition").

His book "Locally Finite Groups," co-authored with Bertram Wehrfritz, is a well known classical book on locally finite groups. As Brian Hartley mentions in the review of the book, the purpose of writing the book was "directed mainly towards the kind of material needed for the proof that locally finite groups satisfying the minimal condi-

tion are abelian-by-finite, but a brief and up-to-date account is given of recent work in many areas other than those treated in detail. In this way, and by means of its excellent bibliography, the book gives a good picture of the present state of locally finite group theory".

He took part in many international conferences and organized many meetings; [15] and [16] are just two examples of conferences which have proceedings.

From Freiburg, Oberwolfach was nearby, and there he organized and participated in many meetings. In particular, the "Locally Finite Groups Workshop" in 1996 in Oberwolfach was very successful. As deputy director, he stepped in and led the MFO (Oberwolfach Research Institute for Mathematics) for two years. He very much enjoyed the international contacts and inspiration there.

After retiring in 1999, Otto continued to actively work in mathematics, participating in many conferences and collaborating with many colleagues. According to the Mathematics Genealogy Project database, Otto Kegel has 22 students and 52 descendants.

His home was always open to guests from all over the world. Otto Kegel passed away on his birthday 20 July 2025 at the age of 91.

Now, in group theory, one of the well-known theorems of Kegel from his early career is called the Kegel–Wielandt Theorem, see [12] and [25]. A group G is the product of its subgroups A and B whenever $G = AB$, where $AB = \{ab : a \in A \text{ and } b \in B\}$.

Theorem 1 (see [1], Theorem 2.4.3) *If the finite group $G = AB$ is the product of two nilpotent subgroups A and B, then G is soluble.*

In the above theorem, if we remove the finiteness condition on the group G, then one may ask, what conditions on A and B must be imposed in order to get similar results as above. There are many works of this type in the literature.

After the classification of finite simple groups, the study of the locally finite simple groups are considered. The classification of linear locally finite simple groups is completed by Belyaev [2], Borovik [3], Hartley–Shute [8], and Thomas [24].

Theorem 2 (see [6], Theorem 1.2) *Every locally finite simple group which is not finite but has a faithful representation as a linear group in finite dimension over a field is isomorphic to a group $\Phi(K)$ of Lie type, where K is an infinite, locally finite field, that is, an infinite subfield of $\overline{F_p}$, for some prime p.*

There are of course non-linear simple locally finite groups which still need to be classified. There are many works on this classification, see [7].

Given an infinite locally finite simple group, the following strong theorem is very useful to reduce to the countable case.

Theorem 3 (see [17], Theorem 4.4) *The infinite group* G *is simple if and only if it has a local system* Σ *consisting of countably infinite simple subgroups of* G.

The above theorem is attributed to P. Hall by Kegel in [11]. Then we have another important theorem of Kegel which helps us to use the classification of finite simple groups.

Theorem 4 (see [17], Theorem 4.5) *If* G *is a countably infinite, locally finite, simple group, then there exists a strictly ascending sequence* G_n *of finite subgroups of* G *with* $G = \bigcup_{n=1}^{\infty} G_n$ *such that for each natural number* n *there exists a maximal normal subgroup* M_{n+1} *of* G_{n+1} *satisfying* $G_n \cap M_{n+1} = \{1\}$.

The generalization of the above theorem to uncountable locally finite simple groups is also proved afterwards. In the above theorem the sequence $\mathcal{K} = (G_i, M_i)_{i\in\mathbb{N}}$ is called a *Kegel sequence* of G. The maximal normal subgroups M_i are called *Kegel kernels* and the simple groups G_i/M_i are called *Kegel simple sections.*

By this theorem of Kegel, we may be able to use the classification of finite simple groups. In order to be able to give you some feeling about this theorem and how to use it, we give the following application of it. A long-standing open question asks, whether there exists a simple, locally finite minimal non-FC-group? This was a question in Kourovka Notebook (see [19], Question 5.1).

Recall that a group G is called a *minimal non-FC-group*, if every proper subgroup of G is an FC-group but the group G itself is not an FC-group. The negative answer to this question was given in [20] by Kuzucuoğlu and Phillips in (1989). The main ingredients of the proof of this result were the aforementioned theorems of P. Hall and Kegel. First, by P. Hall's theorem, if such a simple locally finite minimal non-FC-group exists, then it must be countable. Then by the theorem of Kegel it must have a Kegel sequence. Then using properties of minimal non-FC groups, the structure of FC-groups, the classification of finite simple groups and centralizer of elements in simple groups, it was possible to determine that such a simple group does not exist.

As mentioned above, one of the reasons to write the book "Locally Finite Groups" was to prove that locally finite groups satisfying the minimal condition on subgroups are abelian-by-finite. The following theorem of Shunkov appears in the book.

Theorem 5 (see [17], Theorem 5.8) *For the locally finite group* G*, the following properties are equivalent*:

(1) G *is a Chernikov group*;

(2) G *satisfies the minimal condition on subgroups*;

(3) *The centralizer of every non-identity element of* G *satisfies the minimal condition on subgroups*;

(4) *Every abelian subgroup of* G *satisfies the minimal condition on subgroups.*

The special case of this result, where the 2-subgroups of G are all finite, is contained in [18].

Kegel's paper "Four Lectures on Sylow theory in locally finite groups" (see [11]) is another important paper to mention here. The general question is the following: "Which theorems of finite groups can be extended to infinite locally finite groups?" A special case of the above question is the following: "Can we extend the Sylow theorems to locally finite groups by ignoring the numerical parts of Sylow's theorems?" One of the relevant examples of countable locally finite simple groups is P. Hall's universal locally finite group U, which is constructed as a direct limit of right regular representations of finite symmetric groups G_i into $\mathrm{Sym}(G_i)$ for $i \in \mathbb{N}$ (see [5]). K. Hickin proved that every countably infinite locally finite p-group can be embedded into U as a maximal p-subgroup of U. So conjugacy of Sylow subgroups is not true for Hall's universal group U. In particular, an isomorphic copy of the infinite elementary abelian p-group and an isomorphic copy of the infinite locally cyclic p-group, C_{p^∞}, are maximal p-subgroups of U.

A locally finite group G *satisfies the Sylow theorem for the prime* p if all maximal p-subgroups of G are conjugate in G. Moreover, G *satisfies the strong Sylow theorem for* p if all its subgroups also satisfy the Sylow theorem for the prime p. Thus, locally finite groups satisfying the strong Sylow theorem for p must have some restricted properties. There are some results in this direction in [11].

Kegel always has an interest in the construction of groups. One of the important subjects he was interested in is the κ-existentially closed groups and the construction of such groups.

Let κ be an infinite cardinal. A group G is called κ-*existentially closed* if every system of less than κ equations and in-equations with coefficients in G has a solution in G provided that it has a solution in a group containing G. Clearly inspired from algebraically closed fields, κ-existentially closed groups were first defined by W.R. Scott in [23]. In particular, $\aleph_0$-existentially closed groups have been studied by many authors using a great variety of techniques. Kegel published in the Ischia Group Theory Proceedings [13] an important plan to study κ-existentially closed groups. In [14], we have constructed an explicit example of a κ-existentially closed group for an uncountable strong inaccessible cardinal κ. The construction was a transfinite direct limit of symmetric groups in which the previous one is embedded with the right regular representation into the next one (this accounts for the successor cardinals), and for limit cardinals we take the union of such embeddings. In [14], we also proved that if for an uncountable cardinal κ, a κ-existentially closed group of cardinality κ exists, then it is unique up to isomorphisms. Such a theorem is not true for countable existentially closed groups; by a construction of B.H. Neumann, there are uncountably many countable two generated pairwise non-isomorphic such groups (see [22]). Existence of κ-existentially closed groups of cardinality κ for each infinite regular cardinal κ is proved in [9].

Following Kegel [13], a group U is called a *universal group* in the class of all groups, if every group H with $|H| \leqslant |U|$ is isomorphic to a subgroup of U. Kegel constructed examples of universal groups of cardinality κ for limit cardinals κ in [13], Theorem C, and asked the following question: For which cardinals κ do there exist universal groups of cardinality κ? In [10], this question is answered by Kaya, Kuzucuoğlu, Longobardi, and Maj using the Generalized Continuum Hypothesis.

Theorem 6 *For any uncountable cardinal κ, there exist universal groups of cardinality κ.*

A very popular subject, recently studied by many mathematicians, is the Gruenberg–Kegel graph of a finite group. Recall that the *Gruenberg–Kegel graph* of a group is the undirected graph whose vertices are those primes which occur as the order of an element of the group, and distinct vertices p, q are joined by an edge whenever the group

has an element of order pq. The Gruenberg–Kegel graph of a group reflects interesting properties of the group. Natalia Maslova is one of the colleagues that works on Gruenberg–Kegel graphs. For those who are interested in Gruenberg–Kegel graphs, we suggest the following two important works of Maslova: [4] and [21].

Working with Kegel was fruitful, as for each project we worked with him on, either he had some history about the problem or about the author of the problem. His whole life was devoted to mathematics. We will miss him very much.

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