

Advances in Group Theory and Applications
© 2025 AGTA - www.advgrouptheory.com/journal
21 (2025), pp. 111–123 ISSN: 2499-1287
DOI: 10.32037/agta-2025-006

Sum of Element Orders of Finite Linear Groups of Degree Two

S.N. Banisadr — H. Refaghat — Y. Marefat — B. Vakili

(*Received July 11, 2023; Accepted Mar. 12, 2024 — Communicated by F. de Giovanni*)

Abstract

Let G be a finite group of order n and $\psi(G) = \sum_{g \in G} o(g)$, the sum of its element orders. In this paper, we present a computational method to find the ψ value of finite groups and provide explicit formulas for the sum of element orders of finite linear groups of degree two.

Mathematics Subject Classification (2020): 20-08, 20D60

Keywords: finite group; element order; conjugacy class; linear group

1 Introduction

Recently, the sum of element orders of finite groups has been used to study the structure of groups. Let G be a finite group of order n and $\psi(G) = \sum_{g \in G} o(g)$, the sum of its element orders. In [2], Amiri et al. proved that the maximum value of ψ on the class of groups of order n will occur at the cyclic group C_n, namely, a cyclic group of order n can be characterized by the order n and the value ψ. In [3], authors turned their attention to the minimum value of G, and they showed that among all nilpotent groups of order n, the minimum value of ψ is attained by groups with all Sylow subgroups of prime exponent. They also proved that: If G is a nilpotent group of order n and there are non-nilpotent groups of order n, then there exists a non-nilpotent group K of order n satisfying $\psi(K) < \psi(G)$. The minimal

value of ψ on finite groups of the same order was also investigated in non-abelian simple groups. For simple groups of small orders, the ψ value is the unique minimum of the values of ψ for groups of the corresponding orders. Hence, the authors in [3] posed the following conjecture: Let S be a simple group. If G is a non-simple group of order $|S|$, then $\psi(S) < \psi(G)$.

In [10], Jafarian Amiri proved that $PSL(2,5)$ and $PSL(2,7)$ are uniquely determined by their orders and the sum of the element orders, and confirmed the conjecture for $PSL(2,5)$ and $PSL(2,7)$. But Marefat et al. showed in [13] that the minimality conjecture of $\psi(G)$ in simple groups is not correct, considering the groups $PSL(2,64)$ and $C_3 \times C_3 \times Sz(8)$. Since the counterexample to the conjecture was given by a non-solvable group, the authors in [8] stated the following new conjecture: If S is a simple group and G is a solvable group satisfying $|G| = |S|$, then $\psi(S) < \psi(G)$. In [4], Baniasad Azad and Khosravi proved that if $p \in \{11, 13, 17, 19, 23, 29, 37, 61\}$, then $PSL(2,p)$ is uniquely determined by its order and the sum of element orders. In [11], Jahani et al. showed that this conjecture is also incorrect, considering the groups $C_{65} \rtimes (C_{12} \times C_2) \times (C_2^3 \rtimes C_7) \rtimes C_3$ and $PSL(2,64)$.

In this paper, by studying the ψ value of some finite groups, we determine the exact value of ψ in some linear groups. We state and prove our main results and provide their codes in the Appendix, written in GAP [7], and MAGMA [5].

Theorem 1 *Let $q = p^\alpha$ for some prime p and some positive integer α. Then $\psi(GL_2(q))$ equals*

$$\frac{q}{2}\big((q+1)\psi(C_{q-1}^2) + (q-1)\psi(C_{q^2-1})\big) + (q^2-1)(p-1)\psi(C_{q-1}).$$

Theorem 2 *Suppose p is an odd prime and $q = p^m$ for some positive integer m. Then $\psi(SL_2(q))$ equals*

$$\frac{q}{2}\big((q+1)\psi(C_{q-1}) + (q-1)\psi(C_{q+1})\big) + 3(p-1)(q^2-1).$$

Theorem 3 *If $q = 2^m$ for some positive integer m, then*

$$\psi(L_2(q)) = \frac{q}{2}\Big((q+1)\psi(C_{q-1}) + (q-1)\psi(C_{q+1})\Big) + q^2 - 1\ .$$

2 Preliminaries

Throughout this paper we assume that p is an arbitrary prime number, q is a prime power of p and m, n and α are positive integers.

Let

$$G = 1^G \cup g_2^G \cup \ldots \cup g_k^G$$

be a finite group of order n. If $S \subseteq G$, then we write $\psi(S) = \sum_{x \in S} o(x)$. We define the class function $\psi_c : G \longrightarrow \mathbb{Z}^+$ as follows:

$$\psi_c(g) = o(g)|g^G|\,,$$

where $|g^G|$ is the size of conjugacy class g^G for $g \in G$. It is clear that

$$\psi(G) = \sum_{i=1}^{k} \psi_c(g_i).$$

If g is a self-centralizing (s.c.) element of G, that is, $\langle g \rangle = C_G(g)$, then $\psi_c(g) = o(g)|g^G| = |G|$. Hence the maximum value of ψ_c, on the elements of G, is $|G|$ and occurs on self-centralizing elements. A group in which all non-trivial elements are self-centralizing is called *whole self-centralizing* (*w.s.c.*) *group*. A finite group G is a w.s.c. group if and only if $\psi(G) = (k-1)|G| + 1$. For example, all of the Frobenius groups $F_{p,q}$, where p and q are primes, are w.s.c. groups.

Lemma 4 *Suppose that* $F_{p,q} = C_p \rtimes C_q$, *where* p *and* q *are two prime numbers and* $q|(p-1)$. *Then* $\psi(F_{p,q}) = (k-1)|F_{p,q}| + 1$.

Proof — Note that $G = F_{p,q}$ has trivial center, otherwise $G/Z(G)$ would be cyclic and G would be abelian. Thus every element of order q has its centralizer of order q and thus has p elements in its conjugacy class. By the same argument, an element of order p has q elements in its conjugacy class. Let r and s be the number of conjugacy classes of order p and q, respectively. We therefore have

$$\psi(F_{p,q}) = 1 + p(qr) + q(ps) = 1 + pq(s+r) = pq\big(\tfrac{p-1}{q} + q - 1\big) + 1$$

and we are done. □

For

$$C_p^\alpha = C_p \times C_p \times \ldots \times C_p, \quad (\alpha \text{ times})$$

we have $\psi(C_p^\alpha) = p(p^\alpha - 1) + 1$. In Lemma 5 we describe the ψ value of all finite cyclic groups.

Lemma 5 *Let $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, where p_i $(1 \leqslant i \leqslant r)$ are different primes, and C_n be the cyclic group of order n. Then*

$$\psi(C_n) = \prod_{i=1}^{r} \frac{p_i^{2\alpha_i+1} + 1}{p_i + 1}, \quad \psi(D_{2n}) = \psi(C_n) + 2n,$$

$$\psi(Q_{2^n}) = \psi(C_{2^{n-1}}) + 2^{n+1},$$

and $\psi(T_{4n}) = \psi(C_{2n}) + 8n$.

Proof — Suppose $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, where p_i $(1 \leqslant i \leqslant r)$ are different primes. For the result related to $\psi(C_n)$ see [6, Theorem 1.4], or also [9, Lemma 2.9].

The dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = 1\ ,\ bab = a^{-1} \rangle$ is the semidirect product of the cyclic group $\langle a \rangle$ of order n with the cyclic group $\langle b \rangle$ of order 2. Hence, it is the union of a set of n elements of order two and the cyclic group $\langle a \rangle$. Similarly, the quaternion group

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^4 = 1\ ,\ b^{-1}ab = a^{-1} \rangle$$

is the union of a set of 2^{n-1} elements of order four and the cyclic group $\langle a \rangle$. Finally, dicyclic group

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2\ ,\ b^{-1}ab = a^{-1} \rangle$$

is the union of a set of $2n$ elements of order four and the cyclic group $\langle a \rangle$ of order $2n$. □

The ψ value of $C_n^2 = C_n \times C_n$ is essential to prove Theorem 1.

Theorem 6 *Let n be a positive integer and write $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, where p_i, $1 \leqslant i \leqslant r$ are different primes. Then*

$$\psi(C_n{}^m) = \prod_{i=1}^{r} \frac{p_i^{(m+1)\alpha+1}(p_i^m - 1) + p_i - 1}{p_i^{m+1} - 1}.$$

Proof — See [14]. □

Next, we determine the ψ value of $GL_2(q)$, $SL_2(q)$ and $L_2(2^n)$. We will use the notation and the results of [12, Chapter 28], for the

group $GL_2(q)$, and of [1], for the group $SL_2(q)$.

3 Proofs

Proof of the Theorem 1 — Suppose that $q = p^\alpha$ is a power of a prime p and that $\mathbb{F}_q$ is the field of order q. The general linear group $G = GL_2(q)$ has $q(q-1)^2(q+1)$ elements partitioned into q^2-1 conjugacy classes. There are four families of conjugacy classes in G. The matrices

$$sI = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}, \quad s \in \mathbb{F}_q^*$$

are representatives of $q-1$ different conjugacy classes, each of size 1. Since $\mathbb{F}_q^*$ is a cyclic group of order $q-1$, then $\psi(\mathbb{F}_q^*) = \psi(C_{q-1})$. Suppose that $A = \{sI | s \in \mathbb{F}_q^*\}$. Then $\psi(A) = \psi(\mathbb{F}_q^*) = \psi(C_{q-1})$.

The matrices

$$u_s = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix}, \quad s \in \mathbb{F}_q^*$$

are representatives of $q-1$ different conjugacy classes, and each of these conjugacy class contains q^2-1 elements. Suppose that

$$B = \bigcup \{u_s^G | s \in \mathbb{F}_q^*\}.$$

Since $o(u_s) = p \cdot o(s)$, then we have

$$\begin{aligned} \psi(B) &= \sum_{s \in \mathbb{F}_q^*} \psi_c(u_s) = \sum_{s \in \mathbb{F}_q^*} o(u_s)|u_s^G| \\ &= (q^2-1) \sum_{s \in \mathbb{F}_q^*} p \cdot o(s) = (q^2-1)p\psi(\mathbb{F}_q^*). \end{aligned}$$

Therefore $\psi(B) = (q^2-1)p\psi(C_{q-1})$.

The matrices

$$d_{s,t} = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}, \quad s,t \in \mathbb{F}_q{}^*, \ s \neq t$$

are representatives of $(q-1)(q-2)/2$ different conjugacy classes, and each of these conjugacy classes contains q^2+q elements. The family of conjugacy class representatives $d_{s,t}$ is indexed by unordered

pairs $\{s,t\}$ of distinct elements of $\mathbb{F}_q^*$. Since $d_{s,t}^m = d_{s^m,t^m}$, the order of $d_{s,t}$ in $GL_2(q)$ is equal to the order of (s,t) in $\mathbb{F}_q^* \times \mathbb{F}_q^*$. Thus we have

$$\begin{aligned}\psi(C) &= \frac{1}{2}\psi(\mathbb{F}_q^* \times \mathbb{F}_q^* - \{(s,s)|s \in \mathbb{F}_q^*\}) \cdot (q^2+q) \\ &= \frac{1}{2}(\psi(\mathbb{F}_q^{*^2}) - \psi(\mathbb{F}_q^*))(q^2+q).\end{aligned}$$

Therefore $\psi(C) = \frac{1}{2}(q^2+q)(\psi(C_{q-1}^2) - \psi(C_{q-1}))$.

The matrices

$$\nu_r = \begin{pmatrix} 0 & 1 \\ -r^{1+q} & r+r^q \end{pmatrix}, \quad r \in \mathbb{F}_{q^2}^* \backslash \mathbb{F}_q^*$$

are representatives of $(q^2-q)/2$ different conjugacy classes, and each of these conjugacy class contains (q^2-q) elements from $\mathbb{F}_{q^2}^* \backslash \mathbb{F}_q^*$. Write $\mathbb{F}_{q^2}^* = \langle \epsilon \rangle$. The eigenvalues of ν_ϵ^i and ν_ϵ^{iq} are ϵ^i and ϵ^{iq}. If $\epsilon^i \neq \epsilon^{iq}$, then ν_ϵ^i and ν_ϵ^{iq} must be conjugate to ν_{ϵ^i}, see [12, Proposition 28.6]. Hence the order of ν_r in $GL_2(q)$ is equal to the order of r in $\mathbb{F}_{q^2}^*$. Thus

$$\begin{aligned}\psi(D) = \frac{1}{2}\sum_{r \in \mathbb{F}_{q^2}^* \backslash \mathbb{F}_q^*} \psi_c(\nu_r) &= \frac{1}{2}(q^2-q)\sum_{r \in \mathbb{F}_{q^2}^* \backslash \mathbb{F}_q^*} o(r) \\ &= \frac{1}{2}(q^2-q)\psi(\mathbb{F}_{q^2}^* \backslash \mathbb{F}_q^*).\end{aligned}$$

Therefore $\psi(D) = \frac{1}{2}(q^2-q)(\psi(C_{q^2-1}) - \psi(C_{q-1}))$.

Finally,

$$\begin{aligned}\psi(GL_2(q)) = (1+(q^2-1)p - q^2)\psi(C_{q-1}) \\ + \frac{1}{2}((q^2+q)\psi(C_{q-1}^2) + (q^2-q)\psi(C_{q^2-1})).\end{aligned}$$

The proof is complete. □

Proof of the Theorem 2 — We know that for $q = 3$, $\psi(SL_2(3)) = 99$. Suppose that $q = p^m$ and $q > 3$. Since $\mathbb{F}_q^*$ is a cyclic group of order $q-1$, it has a unique involution $t = -1$. The special linear

group $SL_2(q)$ has $q(q-1)(q+1)$ elements partitioned in $q+4$ conjugacy classes. The matrices I and $-I$ are representatives of two conjugacy classes, each of size 1. Each of the two conjugacy classes of u_1 and u_{-1} relative to $GL_2(q)$ splits into two conjugacy classes of size $(q^2-1)/2$ in $SL_2(q)$. The order of their elements is either p or $2p$, and the total number of these elements is $2(q^2-1)$. The matrices $d_{s_i,s_i^{-1}}$ ($s_i \in \mathbb{F}_q^*$ and $s_i \neq t$) are representatives of $(q-3)/2$ conjugacy classes, and each of these conjugacy classes contains q^2+q elements. The set $\mathbb{F}_q^*\backslash\{1,t\}$ can be partitioned into subsets $\{s_i, s_i^{-1}\}$ for $1 \leqslant i \leqslant (q-3)/2$. Suppose that

$$C = \bigcup \{ d_{s_i,s_i^{-1}}^G \mid 1 \leqslant i \leqslant \frac{q-3}{2} \}.$$

The family of conjugacy class representatives $d_{s_i,s_i^{-1}}$ is indexed by unordered pairs $\{s_i, s_i^{-1}\}$ of elements of $\mathbb{F}_q^*\backslash\{1,t\}$. It follows that

$$\psi(C) = \sum_{i=1}^{\frac{q-3}{2}} \psi_c(d_{s_i,s_i^{-1}}) = (q^2+q) \sum_{i=1}^{\frac{q-3}{2}} o(d_{s_i,s_i^{-1}}) = q(q+1) \sum_{i=1}^{\frac{q-3}{2}} o(s_i).$$

Therefore $\psi(C) = \frac{1}{2}(q^2+q)(\psi(C_{q-1}) - 3)$.

The matrices v_r ($r \in \mathbb{F}_{q^2}^*\backslash\mathbb{F}_q^*$ and $r^{1+q} = 1$) are representatives of $(q-1)/2$ conjugacy classes, and each of these conjugacy classes contains q^2-q elements. Let D be the union of all these classes. Since $\mathbb{F}_{q^2}^* = \langle \epsilon \rangle$ is a cyclic group of order q^2-1, it has a unique subgroup of order $q+1$, namely the subgroup $\langle \epsilon^{q-1} \rangle$. Since q is odd, it is clear that $\langle \epsilon^{q-1} \rangle \backslash \{1,-1\}$ is a subset of $\mathbb{F}_{q^2}^*\backslash\mathbb{F}_q^*$. The set $\langle \epsilon^{q-1} \rangle \backslash \{1,-1\}$ can be partitioned into subsets $\{r_i, r_i^{-1}\}$ for $1 \leqslant i \leqslant (q-1)/2$. Hence

$$D = \bigcup \{ v_{r_i}^G \mid 1 \leqslant i \leqslant \frac{q-1}{2} \},$$

and

$$\psi(D) = \sum_{i=1}^{\frac{q-1}{2}} \psi_c(v_{r_i}) = \sum_{i=1}^{\frac{q-1}{2}} o(v_{r_i}) |v_{r_i}^G| = (q^2-q) \sum_{i=1}^{\frac{q-1}{2}} o(v_{r_i}) \, .$$

Therefore $\psi(D) = \frac{1}{2}(q^2 - q)(\psi(C_{q+1}) - 3)$.

Finally,

$$\psi(SL_2(q)) = 3 + (p + 2p)(q^2 - 1) + \frac{1}{2}(q^2+q)(\psi(C_{q-1}) - 3) + \frac{1}{2}(q^2 - q)(\psi(C_{q+1}) - 3).$$

The proof is complete. □

Proof of the Theorem 3 — If $m = 1$, then $\psi(L_2(2)) = \psi(S_3) = 13$. Suppose that $q = 2^m$ and $m > 1$. The finite simple group $L_2(q)$ has $q^3 - q$ elements partitioned in $q + 1$ conjugacy classes. The first class is the class of the identity element I. The matrix u_1 represents a class consisting of $q^2 - 1$ elements of order 2.

There are $(q-2)/2$ conjugacy classes of elements with representatives $d_{s,s^{-1}}$, indexed by unordered pairs $\{s, s^{-1}\}$ of elements from $\mathbb{F}_{q^2}^* \backslash \mathbb{F}_q^*$. Each such element has a centralizer of order $q - 1$. Thus the size of each of these classes is $q(q + 1)$. Let C be the union of all these classes.

Since $\mathbb{F}_q^*$ is a cyclic group of order $q - 1$, $\psi(\mathbb{F}_q^* \backslash \{1\}) = \psi(C_{q-1}) - 1$. The set $\mathbb{F}_q \backslash \mathbb{F}_2 = \mathbb{F}_q^* \backslash \{1\}$ can be partitioned into subsets $\{s_i, s_i^{-1}\}$ for $1 \leqslant i \leqslant (q-2)/2$, therefore

$$C = \bigcup \{ d^{G}_{s_i, s_i^{-1}} | 1 \leqslant i \leqslant \frac{q-2}{2} \}$$

and

$$\psi(C) = \sum_{i=1}^{\frac{q-2}{2}} \psi_c(d_{s_i, s_i^{-1}}) = \sum_{i=1}^{\frac{q-2}{2}} o(d_{s_i, s_i^{-1}}) | d^{G}_{s_i, s_i^{-1}} = q(q+1) \sum_{i=1}^{\frac{q-2}{2}} o(d_{s_i, s_i^{-1}}).$$

Since $d^m_{s_i, s_i^{-1}} = d_{s_i^m, s_i^{-m}}$, the order of $d_{s_i, s_i^{-1}}$ in $L_2(q)$ is equal to the

order of s_i in $\mathbb{F}_q^*$. Therefore we have

$$\psi(C) = q(q+1)\sum_{i=1}^{\frac{q-2}{2}} o(s_i)\ .$$

Now, $d_{s_i,s_i^{-1}}$ and $d_{s_i^{-1},s_i}$ are conjugate, thus they are in the same class, it follows that

$$\psi(C) = (q^2+q)\frac{1}{2}(\psi(C_{q-1})-1).$$

There are $q/2$ conjugacy classes of elements with representatives ν_r, indexed by unordered pairs $\{r, r^{-1}\}$ of elements from $\mathbb{F}_{q^2}^*\backslash\mathbb{F}_q^*$ such that $r^{1+q} = 1$. Each of such elements has a centralizer of order $q+1$, thus the size of its class is $q(q-1)$. Let D be the union of all these classes. Similar to the proof of the Corollary 2, $\langle\epsilon^{q-1}\rangle$ is the unique subgroup of order $q+1$ of $\mathbb{F}_{q^2}^* = \langle\epsilon\rangle$. Clearly, $\langle\epsilon^{q-1}\rangle\backslash\{1\}$ is a subset of $\mathbb{F}_{q^2}^*\backslash\mathbb{F}_q^*$. The set $\langle\epsilon^{q-1}\rangle\backslash\{1\}$ can be partitioned into subsets $\{r_i, r_i^{-1}\}$ for $1 \leqslant i \leqslant \frac{q}{2}$. Therefore

$$D = \bigcup\{\nu_{r_i}^G | 1 \leqslant i \leqslant \frac{q}{2}\},$$

and

$$\psi(D) = \sum_{i=1}^{\frac{q}{2}} \psi_c(\nu_{r_i}) = \sum_{i=1}^{\frac{q}{2}} o(\nu_{r_i})|\nu_{r_i}^G| = q(q-1)\sum_{i=1}^{\frac{q}{2}} o(\nu_{r_i})\ .$$

Since ν_{r_i} and $\nu_{r_i^{-1}}$ are conjugate they are in the same class, it follows that

$$\psi(D) = (q^2-q).\frac{1}{2}(\psi(C_{q+1})-1).$$

Consequently, the total sum of orders is equal to

$$\psi(L_2(q)) = 1 + 2(q^2-1) + \frac{1}{2}(\psi(C_{q-1})-1)(q^2+q) + \frac{1}{2}(\psi(C_{q+1})-1)(q^2-q).$$

The proof is complete. □

4 Appendix

*. A program for $\psi(C_n)$

```
sum:=function(n)
    local f, p, s i;
      f:=Factors(n);
      s:=1;
          for i in Set(f) do
             p:=(i^(2*Number(f, x -> x=i)+1)+1)/(i+1);
             s:=p*s;
          od;
     Print(s,"\n");
 end;;
```

*. A program for $\psi(GL_2(q))$

```
sum:=function(q)
    local f, g, h, i, j, k, r, s, t, a, b , c, d, p;
      h:=Factors(q);
      f:=Factors(q^2-1);
      g:=Factors(q-1);
      r:=1;; s:=1;; t:=1;;
          for i in Set(f) do
             a:=(i^(2*Number(f, x -> x=i)+1)+1)/(i+1);
             r:=a*r;
          od;
          for j in Set(g) do
             b:=(j^(2*Number(g, x -> x=j)+1)+1)/(j+1);
             s:=b*s;
          od;
          for k in Set(g) do
             c:=(k^(3*Number(g, x -> x=k)+1)*(k+1)+1)
             /(k*(k+1)+1);
             t:=c*t;
          od;
          for p in Set(h) do
             d:=s*(q^2-1)*(p-1)+(q^2+q)*(t/2)+(q^2-q)*(r/2);
          od;
```

```
        Print("GL(2,",p,"^",Number(h, x -> x=p),") :",d,"\n");
    end;;
```

*. A program for $\psi(SL_2(q))$

```
sum:=function(q)
    local f, g, h, s, t, p, i, j, b , c, d;
      f:=Factors(q+1);;
      g:=Factors(q-1);;
      h:=Factors(q);
      s:=1;; t:=1;;
          for j in Set(g) do
              b:=(j^(2*Number(g, x -> x=j)+1)+1)/(j+1);
              s:=b*s;
          od;
          for i in Set(f) do
              c:=(i^(2*Number(f, x -> x=i)+1)+1)/(i+1);
              t:=c*t;
          od;
          for p in Set(h) do
              d:=3*(q^2-1)*(p-1)+(q^2+q)*(s/2)
              +(q^2-q)*(t/2);
          od;
      Print(d,"\n");
    end;;
```

*. A program for $\psi(L_2(2^n))$

```
sum:=function(n)
    local f, g, p, q, k, l, i, j;
      f:=Factors(2^n+1);;
      g:=Factors(2^n-1);;
      k:=1;; l:=1;;
          for i in Set(f) do
              p:=(i^(2*Number(f, x -> x=i)+1)+1)/(i+1);
              k:=p*k;
          od;
          for j in Set(g) do
              q:=(j^(2*Number(g, x -> x=j)+1)+1)/(j+1);
               l:=q*l;
          od;
      Print((2^(n-1))*((2^n+1)*l+(2^n-1)*k)+2^(2*n)-1,"\n");
    end;;
```

REFERENCES

[1] E. Adan-Bante – J.M. Harris: "On conjugacy classes of SL(2, q)", *Rev. Colombiana Mat.* 46, No. 2 (2012), 97–111.

[2] H. Amiri – S.M. Jafarian Amiri – I.M. Isaacs: "Sums of element orders in finite groups", *Comm. Algebra* 6 37 (2009), 2978–2980.

[3] H. Amiri – S.M. Jafarian Amiri: " Sum of element orders on finite groups of the same order", *J. Algebra Appl.* 10, No. 2 (2011), 187–190.

[4] M. Baniasad Azad – B. Khosravi: "On the sum of element orders of PSL(2, p) for some p", *Italian J. Pure App. Math.* 42 (2019), 18–24.

[5] W. Bosma – J.J. Cannon – C. Fieker – A. Steel: "Handbook of Magma functions", v2.23, *University of Sidney*, Sydney (2017).

[6] H.K. Dey – A. Mondal: "An exact upper bound for sums of power of element orders in non-cyclic finite groups", *J. Pure Appl. Algebra* 228, No. 6 (2024), 107580.

[7] The GAP Group: "GAP — Groups, Algorithms, and Programing", v4.6.5 (2013); url:`www.gap-system.org`

[8] M. Herzog – P. Longobardi – M. Maj: "Properties of finite and periodic groups determined by their element orders (a survey)", in Group Theory and Computation, *Springer*, Singapore (2018), 59–90.

[9] M. Herzog – P. Longobardi – M. Maj: " An exact upper bound for sums of element orders in non-cyclic finite groups", *J. Pure Appl. Algebra* 222 (2018), 1628–1642.

[10] S.M. Jafarian Amiri: "Characterization of A_5 and PSL(2, 7) by sum of element orders", *Int. J. Group Theory* 2 (2013), 35–39.

[11] M. Jahani – Y. Marefat – H. Refaghat – B. Vakili: "The minimum sum of element orders of finite groups", *Int. J. Group Theory* 10, No. 2 (2021), 55–60.

[12] G. James – M. Liebeck: "Representations and Characters of Groups", *Cambridge University Press*, Cambridge (2001).

[13] Y. Marefat – A. Iranmanesh – A. Tehranian: "On the sum of element orders of finite simple groups", *J. Algebra Appl.* 12 (2013), 1350026, 4pp.

[14] M. Tarnauceanu – D.G. Fedor: "On the sum of element orders of finite abelian group", *Sci. An. Univ. "A1.I. Cuza" Iasi, Ser. Math* 60 (2014), 1–7.

Seiyede Nazanin Banisadr
Department of Mathematics
Tabriz Branch, Islamic Azad University
Tabriz (Iran)
e-mail: nazaninbanisadr@yahoo.com

Hasan Refaghat (*corresponding author*)
Department of Mathematics
Tabriz Branch, Islamic Azad University
Tabriz (Iran)
e-mail: h.refaghat@iaut.ac.ir

Yadollah Marefat
Department of Mathematics
Shabestar Branch, Islamic Azad University
Shabestar (Iran)
e-mail: marefat@iau.ir; marefat@iaushab.ac.ir

Bahram Vakili
Department of Mathematics
Shabestar Branch, Islamic Azad University
Shabestar (Iran)
e-mail: b.vakil@iaushab.ac.ir