

Advances in Group Theory and Applications
© 2025 AGTA - www.advgrouptheory.com/journal
21 (2025), pp. 155–160 ISSN: 2499-1287
DOI: 10.32037/agta-2025-008

The Number of Supercharacter Theories of Cyclic Groups of Order pq

E. Armioun

(*Received Aug. 4, 2023; Accepted Feb. 21, 2024 — Communicated by N. Mazza*)

Abstract

In this paper, we determine the number of distinct supercharacter theories of cyclic group of order pq where p and q are distinct odd primes.

Mathematics Subject Classification (2020): 20C15, 20K27

Keywords: character; supercharacter; cyclic group

1 Introduction

The concept of supercharacter theory of a finite group was introduced by André [1] and Yang [9], as an approximation of the character table for special families of groups. Diaconis and Isaacs [2], formulated this for every finite group. A supercharacter theory of a finite group G is a pair $(\mathcal{K}, \mathcal{X})$ where $\mathcal{K}$ is a partition of G and $\mathcal{X}$ is a partition of irreducible characters of G such that: (a) $|\mathcal{K}| = |\mathcal{X}|$; (b) $\{1\} \in \mathcal{K}$; (c) for every $X \subseteq \mathrm{Irr}(G)$ the character associated with X which is denoted by χ_X is constant on every element of K. In [2], it is proved χ_X is a multiple of $\sigma_X = \sum_{\chi \in X} \chi(1)\chi$. Without loss of generality we can assume $\chi_X = \sigma_X$ (see [2], Lemma 2.1). The elements of $\mathcal{K}$ are called superclasses and functions σ_X are called supercharacters. The set of all supercharacter theories of G is denoted by $\mathrm{Sup}(G)$. Throughout the paper, the identity element of G is denoted as 1 and

the trivial irreducible character of G is represented by $\mathbb{1}$. Every finite group G with order more than two has multiple trivial supercharacter theories.

1. Minimal supercharacter theory. In this supercharacter theory, every conjugacy class of G is regarded as a superclass and every irreducible character of G is considered as a supercharacter. It is denoted by $\mathfrak{m}(G)$.

2. Maximal supercharacter theory. This supercharacter theory is the following: $M(G) = (\{\{1\}, G \setminus \{1\}\}, \{\{\mathbb{1}\}, \mathrm{Irr}(G) \setminus \{\mathbb{1}\}\})$.

Let N be a subgroup of a finite group G. We say $C = (\mathcal{K}, \mathcal{X}) \in \mathrm{Sup}(N)$ is a G-inavariant supercharacter theory of N, if the superclasses of C are unions of conjugacy classes of G. The set of all G-invariant supercharacter theories of N is denoted by $\mathrm{Sup}_G(N)$. Also we say N is C-normal subgroup of G, if it is a union of superclasses of C. There are several methods to construct supercharacter theories of G. We now consider only those which are used in this article.

Theorem 1 (see [6], Propositions 3.9 and 3.25) *Let G be a finite group, and A a group with compatible actions on the conjugacy classes of G and on the irreducible characters of G. Assume that the set of A-orbits on conjugacy classes of G is denoted by $\mathcal{K}$ and the set of A-orbits on $\mathrm{Irr}(G)$ is denoted by $\mathcal{X}$. Then $(\mathcal{K}, \mathcal{X})$ is a supercharacter theory which is known as the automorphic supercharacter theory of G generated by A. If G is a cyclic group of order p where p is a prime, then all its supercharacter theories are automorphic. The number of supercharacter theories of G is $\tau(p-1)$ in which $\tau(p-1)$ is equal to the number of divisors of $p-1$.*

Proposition 2 (see [5], Proposition 8.1) *Let*

$$G = H \times K, \quad S = (\mathcal{K}, \mathcal{X}) \in \mathrm{Sup}(H) \quad \text{and} \quad T = (\mathcal{L}, \mathcal{Y}) \in \mathrm{Sup}(K).$$

Suppose

$$\mathcal{M} = \mathcal{K} \times \mathcal{L} = \{K \times L : K \in \mathcal{K}, L \in \mathcal{K}\}$$

and

$$\mathcal{Z} = \mathcal{X} \times \mathcal{Y} = \{X \times Y : X \in \mathcal{X}, Y \in \mathcal{Y}\}.$$

Then the pair $(\mathcal{M}, \mathcal{Z})$ is a supercharacter theory of G, known as the direct product of S and T.

Theorem 3 (see [5], Theorem 4.3) *Suppose* G *is a finite group and* N *is a normal subgroup of* G. *Let* $\mathcal{C} = (\mathcal{K}, \mathcal{X})$ *is a* G*-invariant supercharacter theory of* N *and* $\mathcal{D} = (\mathcal{L}, \mathcal{Y})$ *be a supercharacter theory of* G/N. *Then*

$$(\mathcal{K} \cup \widetilde{\mathcal{L}} \setminus \{N\}, \mathcal{Y} \cup \mathcal{X}^G \setminus \mathrm{Irr}(G/N))$$

is a supercharacter theory of G *in which* $\widetilde{\mathcal{L}} = \{\widetilde{L} = \cup_{Ng \in L} Ng, L \in \mathcal{L}\}$ *and* $\mathcal{X}^G = \{X^G : X \in \mathcal{X}\}$ *where* $X^G = \{\chi \in \mathrm{Irr}(G) : \langle \chi_N, \psi \rangle > 0$ *for some* $\psi \in Z\}$ *for every part* Z *of* $\mathcal{X}$. *This supercharacter theory is denoted by* $\mathcal{C} \star \mathcal{D}$.

Suppose $S = (\mathcal{K}, \mathcal{X}) \in \mathrm{Sup}(G)$ and N be a S-normal subgroup of G. The restriction of the supercharacter theory S to N, S_N, and the deflated supercharacter theory S to G/N, $S^{G/N}$, are defined as follows:

- $S_N = \Big(\{K \in \mathcal{K} : K \subseteq N\}, \{X_N : X \in \mathcal{X}, X \nsubseteq \mathrm{Irr}(G/N)\} \cup \{\{\mathbb{1}\}\}\Big)$ where $X_N = \{f \in \mathrm{Irr}(N) : \exists \chi \in X : \langle \chi_N, f \rangle > 0\}$;
- $S^{G/N} = \Big(\{\{N\}\} \cup \{\{gN : g \in K\} : K \in \mathcal{K}, K \nsubseteq N\}, \{X \in \mathcal{X} : X \subseteq \mathrm{Irr}(G/N)\}\Big)$

S_N and $S^{G/N}$ are supercharacter theories of N and G/N, respectively (see [4], Proposition 6.4).

Proposition 4 (see [4], Theorem 7.2) *Let* G *be a finite group with normal subgroups* M *and* N *such that* $N \leqslant M$. *Let*

$$S = (\mathcal{K}, \mathcal{X}) \in \mathrm{Sup}_G(M) \quad \text{and} \quad T = (\mathcal{L}, \mathcal{Y}) \in \mathrm{Sup}(G/N).$$

Suppose N *is* S*-normal subgroup of* G, M/N *is* T*-normal subgroup of* G/N *and* $S^{M/N} = T_{M/N}$. *Then there is a unique supercharacter theory of* G *in which the superclasses are* $\mathcal{K} \cup \{\bigcup_{gN \in L} gN : L \in \mathcal{L}, L \nsubseteq M/N\}$, *and the supercharacters are* $\mathcal{Y} \cup \{X^G : X \in \mathcal{X}, X \nsubseteq \mathrm{Irr}(M/N)\}$. *This supercharacter theory of* G *is called* Δ*-product of* S *and* T, *and is denoted by* $S \Delta T$.

Lemma 5 (see [7]) *Let* G *be a finite group and let* M *and* N *be subgroups of* G *such that* $G \simeq M \times N$. *Assume that* S *is a supercharacter theory of* G *which comes from direct product of supercharacter theories of* S_M *and* S_N. *Then* S *comes from automorphism if and only if* S_M *and* S_N *are generated by automorphism.*

Lemma 6 (see [8], Theorem 1) *Let* $G \simeq \mathbb{Z}_n \times \mathbb{Z}_m$, *where* $n, m \geqslant 1$. *Then the number of subgroups of* G *is* $\sum_{d_1 | m-1, d_2 | n-1} \gcd(d_1, d_2)$.

Lemma 7 (see [5], Theorem 9.1) *Every non maximal supercharacter theory of a cyclic group, can be classified as either automorphic supercharacter theory,* Δ*-product or direct product of supercharacter theories.*

2 Main Results

In the following p and q are distinct odd prime numbers. The number of supercharacter theories of cyclic p-groups is obtained in [4] by Hendrickson. Inspired by Lemma 7, we will find a formula to compute the number of supercharacter theories of $\mathbb{Z}_{pq}$ where p and q are distinct primes.

Lemma 8 *For distinct odd primes p and q, assume $G = \mathbb{Z}_{pq}$. Let H_1 and H_2 be two subgroups of $\mathrm{Aut}(G)$. For each $i \in \{1,2\}$, we let $\mathcal{C}_i$ be the automorphic supercharacter theory of G generated by H_i. If $H_1 \neq H_2$, then $\mathcal{C}_1 \neq \mathcal{C}_2$.*

Proof — Let $G = \langle a \rangle$ and assume $\mathcal{C}_1 = \mathcal{C}_2$. We have ${\mathcal{C}_1}_{\langle a^p \rangle} = {\mathcal{C}_2}_{\langle a^p \rangle}$ and ${\mathcal{C}_1}_{\langle a^q \rangle} = {\mathcal{C}_2}_{\langle a^q \rangle}$. Since ${\mathcal{C}_i}_{\langle a^p \rangle}$ and ${\mathcal{C}_i}_{\langle a^q \rangle}$ $(i = 1,2)$ are automrphic supercharacter theories of $\langle a^p \rangle$ and $\langle a^q \rangle$, and for each divisor of $p-1$ and $q-1$ there is a unique supercharacter theory for $\langle a^q \rangle$ and $\langle a^p \rangle$. Therefore ${H_1}_{\langle a^p \rangle} = {H_2}_{\langle a^p \rangle}$ and also ${H_1}_{\langle a^q \rangle} = {H_2}_{\langle a^q \rangle}$. Clearly, the superclasses of ${\mathcal{C}_i}_{\langle a^p \rangle}$ and ${\mathcal{C}_i}_{\langle a^q \rangle}$ $(i = 1,2)$ determine all the other superclasses which lie in $G \setminus \langle a^p \rangle \cup \langle a^q \rangle$. It follows that

$$ {\mathcal{C}_1}_{G \setminus \langle a^p \rangle \cup \langle a^q \rangle} = {\mathcal{C}_2}_{G \setminus \langle a^p \rangle \cup \langle a^q \rangle}. $$

So $H_1 = H_2$ which is a contradiction. □

Theorem 9 *Let $G \simeq \mathbb{Z}_{pq}$. Then*

$$ |\mathrm{Sup}(G)| = 2\tau(p-1)\tau(q-1) + \sum_{d_1 | p-1, d_2 | q-1} \gcd(d_1, d_2) + 1. $$

Proof — Let $S \in \mathrm{Sup}(G)$. By [5], Theorem 9.1, S can be obtained by at least one of the folowing methods: automorphism, direct product or Δ-product. To count $\star$-product supercharacter theories of G, we have two choices for N, a normal subgroup of $\mathbb{Z}_{pq}$. If $N = \mathbb{Z}_p$, then $G/N \simeq \mathbb{Z}_q$. Therefore the number of $\star$-product supercharacter theories of N and G/N is $\tau(p-1)\tau(q-1)$. If $N = \mathbb{Z}_q$, then $G/N \simeq \mathbb{Z}_p$. So the number of $\star$-product supercharacter theories of N and G/N is $\tau(p-1)\tau(q-1)$. To enumerate the number of automorphic supercharacter theories, it is sufficient to count the number of subgroups of $\mathrm{Aut}(\mathbb{Z}_{pq})$. Since $\mathrm{Aut}(\mathbb{Z}_{pq}) \simeq \mathbb{Z}_{p-1} \times \mathbb{Z}_{q-1}$ (see [3], Theorem 6.5), using Lemma 6, the number of subgroups of $\mathrm{Aut}(\mathbb{Z}_{pq})$ is equal

to $\sum_{d_1|p-1,d_2|q-1} \gcd(d_1, d_2)$. Note that we have not counted the maximal supercharacter theory in any method of constructing supercharacter theories of G. Therefore

$$|\mathrm{Sup}(G)| = 2\tau(p-1)\tau(q-1) + \sum_{d_1|p-1,d_2|q-1} \gcd(d_1, d_2) + 1$$

In calculating the number of supercharacter theories of $\mathbb{Z}_{pq}$, we do not consider the supercharacter theories which are generated by direct product because, according to Lemma 5, they are regarded as automorphic supercharacter theories. □

Acknowledgment

I would like to express my gratitude to the anonymous referees for their valuable comments and recommendations.

REFERENCES

[1] C.A.M. André: "Basic characters of the unitriangular group", *J. Algebra* 175 (1995), 287–319.

[2] P. Diaconis – I. Isaacs: "Supercharacters and superclasses for algebra groups", *Trans. Amer. Math.Soc.* 360 (2008), 2359–2392.

[3] J. Gallian: "Contemporary Abstract Algebra", *Chapman & Hall*, London (2021).

[4] A. Hendrickson: "Supercharacter Theories of Cyclic p-Groups", Ph.D. Thesis, *University of Wisconsin*, Wisconsin, USA (2008).

[5] A. Hendrickson: "Supercharacter theory constructions corresponding to Schur ring products" , *Comm. Algebra* 40 (2012), 4420–4438.

[6] J.P. Lamar: "Lattices of Supercharacter Theories", Ph.D. Thesis, *University of Colorado at Boulder*, Colorado, USA (2018).

[7] A. Lang: "On the Classification of Supercharacter Theories", Ph.D. Thesis, *University of California*, California, USA (2016).

[8] L. Tóth: "Subgroups of finite abelian groups having rank two via goursat's lemma", *Tatra Mt. Math. Publ.* 59 (2014), 93–103.

[9] N. Yan: "Representation Theory of the Finite Unipotent Linear Groups", Ph.D. Thesis, *University of Pennsylvania*, Pennsylvania, USA (2001).

E. Armioun
Department of Mathematics and Computer Science
University of Kharazmi
Tehran (Iran)
e-mail: earmioun@gmail.com