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Burnside Groups and Groups Acting on Rooted Trees

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Abstract

Branch groups stem from one of the most influential problems in group theory: The famous *Burnside Problem*, which arose in 1902 and asks if a finitely generated torsion group, i.e. in which every element has finite order, can be itself infinite. Now branch groups are groups acting on rooted trees, that have a rich tree-like subgroup structure. There are many examples of branch groups with remarkable algebraic properties, and branch groups have many applications within group theory and also to other areas of mathematics, such as to dynamics, analysis, algebraic geometry and cryptography.

These lecture notes aim to introduce branch groups and some of their well-studied generalisations. An overview of the wide array of applications of branch groups will be given, including their use to answer several open questions. We will then focus on some new developments and some big open problems in the subject, such as those concerning maximal subgroups of branch groups.

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1 Introduction

In 1902, William Burnside asked if a finitely generated group can be infinite while every element of the group has finite order. This question, known as the *Burnside Problem*, is one of the oldest and most influential questions in group theory. It

was answered in 1964, when Golod and Shafarevich [36] produced an example, using group algebras. This was followed suit by Adjan and Novikov [4], Aleshin [5], Olshanskii [58], Grigorchuk [37], and Gupta and Sidki [43], to name a few. Fairly recently, Schlage-Puchta [66] gave a straightforward proof of the existence of such groups by introducing the notion of p-deficiency.

Of course, prior to 1902 there were many known examples of infinite torsion groups, such as the direct product of infinitely many copies of the cyclic group of order p, i.e. $C_p \times C_p \times \ldots$ or $\mathbb{Q}/\mathbb{Z}$ (or even its Sylow p-subgroups). However all these examples of infinite torsion groups are not finitely generated.

One also refers to infinite finitely generated torsion groups as *Burnside groups*. Note that the constructions by Golod and Shafarevich, Adjan and Novikov and Olshanskii are non-explicit and most of these constructions are complicated. On the other hand, the constructions by Grigorchuk, and Gupta and Sidki were explicit and gave more geometric examples of Burnside groups, which were also easy to describe. These examples are groups acting on rooted trees, and were the first explicit examples of Burnside groups.

Certainly, one of the challenges of the Burnside problem arose from the fact that the condition "finitely generated torsion" gives little insight to the structure of such a group. By imposing more conditions to better tackle the problem, this led to three other variants of the Burnside Problem: The *Bounded Burnside Problem*, the *Restricted Burnside Problem* and the *Related Burnside Problem*. Let us note that the original question posed by William Burnside is sometimes referred to as the *General Burnside Problem*.

The *Bounded Burnside Problem* is the following: If G is a finitely generated group with every element of order at most n, is G necessarily finite? This problem was immediately suggested by Burnside after he posed his first question. Clearly, if $n = 2$ then G is finite. Burnside [14] also showed in 1902 that the

same holds for $n = 3$. Sanov [67] in 1940 dealt with the case $n = 4$ and Hall [44] similarly proved the case $n = 6$ in 1958. However, for sufficiently large n, there are such infinite finitely generated groups of with every element of order at most n (Adjan, Novikov, Ivanov and Lysenok, 1968–2015; and, recently, Atkarskaya, Rips and Tent, 2024). Note that the Bounded Burnside Problem is still unsolved for small values of n, in particular for $n = 5$.

The *Restricted Burnside Problem* asks: Are there only finitely many finite groups with m generators, such that every element has order at most n? The link between this problem and the original one is as follows: Let $B(d, n)$ be the d-generated free Burnside group of exponent n, and let N be the intersection of all finite-index subgroups of $B(d, n)$. Besides, consider $B_0(d, n) = B(d, n)/N$. Clearly, the finiteness of $B(d, n)$ implies the finiteness of $B_0(d, n)$, but the converse does not hold, as we will comment on below. As every finite d-generated group of exponent n is a quotient of $B_0(d, n)$, an affirmative answer to the Restricted Burnside Problem is equivalent to an affirmative answer to $B_0(d, n)$ being finite. Using Lie-theoretic methods, Kostrikin [50] in 1959 proved there are only finitely many finite groups with every element having prime order. So, in other words $B_0(d, p)$ is finite, for a prime p; however the construction by Olshanskii [58] provides a 2-generated infinite group of exponent p, for a sufficiently large prime p. Hence, we see that the finiteness of $B_0(d, n)$ does not imply the finiteness of $B(d, n)$. In 1989, building on the work of Hall and Higman [45] and profiting from the classification of finite simple groups, the general case of the Restricted Burnside Problem was solved by Zelmanov (see [78] and [79]), who won a Fields Medal for his work.

The final variant of the Burnside Problem is the *Related Burnside Problem*, which asks if there exists a finitely presented infinite group, in which every element has finite order. This question is still completely open, with very little work been done

towards answering it (see [73]).

2 Groups acting on rooted trees

Groups acting on trees appear naturally in many contexts, and some of the early considerations of groups acting on rooted trees were by Rex Dark. There are many kinds of groups acting on rooted trees, from finite to infinite. It is amongst the infinite ones that are of most interest to us, and in particular, the class of branch groups. Roughly speaking, a branch group is a group acting spherically transitively on an infinite rooted tree T and admits a structure of subnormal subgroups similar to the corresponding structure in the full automorphism group $\mathrm{Aut}(T)$ of the tree T. This will be made precise below. All concepts defined here for the rest of this section also apply to more general rooted trees, but for conciseness we do not consider this more general setting. We refer the reader instead to [10].

There are many examples of branch groups with remarkable algebraic properties, and very often it was a branch group that gave the first example of a group with a given combination of exotic algebraic properties. This was already witnessed in the 1980s and these examples include the first explicit examples to the Burnside Problem, the first example of a finitely generated group with intermediate growth and the first example of an amenable but not elementary amenable group [38], which provided an answer to the Mahlon Day problem. Branch groups have also been used recently to answer key open questions, such as finding the first examples of infinite simple groups separated by the finiteness properties F_n [71], the first examples of groups with so-called jump rational cohomology of height 1 and infinite rational cohomological dimension [30], the first examples of infinite finitely generated residually finite non-torsion groups whose profinite completion is a pro-p group for a prime p [23], the first example of an uncountable family of finitely generated

residually finite non-amenable groups that are all profinitely isomorphic to a finitely generated residually finite amenable group [47], and the first example of a finitely generated non-amenable torsion branch group [48].

Branch groups are also important because the class of just infinite branch groups is one of the three classes that partition the class of all just infinite groups [77]. Here an infinite group is *just infinite* when all its non-trivial normal subgroups have finite index. In the theory of infinite groups, the role played by just infinite groups is akin to that of simple groups in the theory of finite groups. Note also that every finitely generated group has a just infinite quotient.

In addition, the recent works [20] and [42] has seen branch groups feature in algebraic geometry for the first time, where they have been used to deliver new instances of Beauville groups. In particular, in [20] many infinite families of Beauville 2-groups and 3-groups were constructed, which significantly enlarged the previously-known small collection of such families. Branch groups also feature in the proof of the recent result that big mapping class groups do not satisfy Tits alternative [7]. Apart from that, branch groups have been used to create new examples of simple totally disconnected locally compact groups [32], contributing notably to the search for more infinite simple groups. There are also links to dynamics (see [53]), C^*-algebras (see [54] and [55]), and cryptography (see [40]).

In this section, we define branch groups, and a natural generalisation of them called weakly branch groups, and then we will see some key examples from both families of groups.

In order to define these groups, we first need to consider properties of the rooted tree T and it automorphism group $\mathrm{Aut}(T)$, which we do next.

2.1 A rooted tree and its fractal-like structure

Let p be a prime and let T be the p-*adic tree*, i.e. the rooted tree having p descendants at every vertex. If we choose an

alphabet X with p letters, the tree T can be represented as the graph whose vertices are the elements of the free monoid X^*, the root is the empty word $\varnothing$, and w is a descendant of u provided that $w = ux$ with $x \in X$.

For a given vertex u, the set of vertices uv with $v \in X^*$ are said to *succeed* u. They form a tree T_u rooted at u, which is isomorphic to T. We denote by $|u|$ the length of u as a word. For every $n \in \mathbb{N} \cup \{0\}$, the set L_n of all words of length n is called the n*th layer* of the tree.

Graph automorphisms of T form a group $\mathrm{Aut}(T)$ under composition. Let u be a vertex of T and let $f \in \mathrm{Aut}(T)$. We use exponential notation for images of automorphisms and, more generally, permutations. Thus the image of u under $f \in \mathrm{Aut}(T)$ is u^f; this notation is in accordance with the fact that the action of elements in $\mathrm{Aut}(T)$ on T will be on the right. The *label* of f at u is the permutation $f(u)$ of the alphabet X defined by the rule

$$(ux)^f = u^f x^{f(u)},$$

for every $x \in X$. The *portrait* of f is the set of all labels of f, and there is a one-to-one correspondence between automorphisms of T and portraits. The *support* of f in T is the set of vertices with non-trivial label. We say that f is *rooted* if its support in T is contained in the root, and f is *directed* if its support in T is infinite and consists only of descendants of a given infinite path starting at the root.

In a similar way, the section f_u of f at u is the automorphism of T defined by

$$(uv)^f = u^f v^{f_u},$$

for every $v \in X^*$. Note that

$$(f_u)_v = f_{uv} \qquad \text{and} \qquad (fg)_u = f_u g_{u^f},$$

for all $f, g \in \mathrm{Aut}(T)$ and $u, v \in X^*$.

2.2 Key subgroups of $\mathrm{Aut}(T)$

For a vertex u of T, the *vertex stabiliser* $\mathrm{st}(u)$ is the subgroup consisting of all automorphisms of T fixing u. Moreover, the map $\varphi_u : f \mapsto f_u$ is a homomorphism from $\mathrm{st}(u)$ onto $\mathrm{Aut}(T)$. For every $n \in \mathbb{N}$, the n*th level stabiliser* is

$$\mathrm{St}(n) = \bigcap_{u \in L_n} \mathrm{st}(u).$$

Note that $\mathrm{St}(n)$ is a normal subgroup of $\mathrm{Aut}(T)$ and we observe that $\mathrm{Aut}(T)$ is isomorphic to the inverse limit of the finite groups $\mathrm{Aut}(T)/\mathrm{St}(n)$. In other words, the group $\mathrm{Aut}(T)$ is a profinite group with $\{\mathrm{St}(n)\}_{n\in\mathbb{N}}$ as a basis of neighbourhoods of the identity. For every $n \in \mathbb{N}$, we have an isomorphism

$$\begin{aligned} \psi_n : \mathrm{St}(n) &\longrightarrow \mathrm{Aut}(T) \times \ldots \times \mathrm{Aut}(T) \\ f &\longmapsto (f_u)_{u \in L_n} \end{aligned}$$

For simplicity, we write $\psi = \psi_1$.

Now, for $G \leqslant \mathrm{Aut}(T)$, write

$$\mathrm{st}_G(u) = \mathrm{st}(u) \cap G \qquad \text{and} \qquad \mathrm{St}_G(n) = \mathrm{St}(n) \cap G.$$

Thus, we say that G is:

- *spherically transitive* (or *level-transitive*) if it acts transitively on L_n, for every $n \in \mathbb{N}$;
- *self-similar* if all sections of elements of G at all vertices belong to G.

Note that if G is self-similar then $\psi_n(\mathrm{St}_G(n)) \subseteq G \times \ldots \times G$, for all n.

The *rigid vertex stabiliser* $\mathrm{rist}_G(u)$ of a vertex u in G is the subgroup consisting of all automorphisms in G that fix all vertices outside T_u. Then, for every $n \in \mathbb{N}$, we define the *rigid* n*th level*

stabiliser as:

$$\mathrm{Rist}_G(n) = \left\langle \mathrm{rist}_G(u) \mid u \in L_n \right\rangle = \prod_{u \in L_n} \mathrm{rist}_G(u) \trianglelefteq G.$$

Let G be spherically transitive. We say that G is a *branch group* if $|G : \mathrm{Rist}_G(n)| < \infty$, for all $n \in \mathbb{N}$, and that it is *weakly branch* if $\mathrm{Rist}_G(n) \neq 1$, for all n. Moreover, if G is spherically transitive and self-similar, and $K \times \ldots \times K \subseteq \psi(\mathrm{St}_K(1))$, for some $K \leqslant G$, we say that G is

- *regular branch over* K, if $|G : K| < \infty$
- *weakly regular branch over* K, if $K \neq 1$.

Of course, being (weakly) regular branch implies being (weakly) branch. In particular, weakly branch groups are infinite.

2.3 Topologies

Let $T_{[n]}$ be the finite subtree of T of vertices up to level n. Recall that the full automorphism group $\mathrm{Aut}(T)$ is a profinite group:

$$\mathrm{Aut}(T) = \varprojlim_{n \to \infty} \mathrm{Aut}(T)_{[n]}$$

The topology of $\mathrm{Aut}(T)$ is defined by the open subgroups $\mathrm{St}(n)$, for $n \in \mathbb{N}$. We say that $G \leqslant \mathrm{Aut}(T)$ has the *congruence subgroup property* if every finite-index subgroup of G contains a level stabiliser $\mathrm{St}_G(n)$, for some $n \in \mathbb{N}$. Equivalently, the group G has the congruence subgroup property if the profinite completion of G equals its closure in $\mathrm{Aut}(T)$. For other weaker versions of the congruence subgroup properties for groups acting on rooted trees, we refer the reader to [34] and [68].

2.4 General properties

As mentioned above, weakly branch groups are infinite groups. It is also known that a weakly branch group satisfies no laws [2]. Furthermore, we have that branch groups are virtually abelian

(see [39, Theorem 4]), and every finitely generated branch group which has the congruence subgroup property must be just infinite (see [25, Theorem 4.4.4]).

2.5 The boundary of the tree

Let ∂T denote the boundary of T, i.e. the set of all infinite simple paths in T starting at the root. Given a path $\xi \in \partial T$ and a vertex v of T, we write $v \leqslant \xi$ if ξ passes through v.

For a vertex v, we define the *cylinder set* at v to be

$$C_v = \{\xi \in \partial T \mid v \leqslant \xi\}.$$

The set of all cylinder sets form a base for a topology on ∂T.

Now any automorphism of T naturally extends to a homeomorphism of ∂T. We recall the *support* of $g \in \mathrm{Aut}(T)$ in ∂T is

$$\mathrm{Supp}(g) = \{\xi \in \partial T \mid g\xi \neq \xi\},$$

i.e. the complement of the set of fixed points of the action of g on the boundary ∂T of the tree. We note the following result, which will be of later use.

Lemma 1 (see [26], Lemma 2.10) *Let* $g \in \mathrm{Aut}(T)$. *Then* $\mathrm{Supp}(g)$ *is an open set of* ∂T.

Proof — Since g is a homeomorphism of a first-countable Hausdorff space, its set of fixed points is closed. Hence, $\mathrm{Supp}(g)$ is open. □

3 Examples

3.1 The (first) Grigorchuk group

Let T be the 2-adic tree. We denote by a the rooted automorphism that swaps the vertices at the first level. The (first) Grigorchuk group $\mathfrak{G}$ is generated by a, b, c, d, where b, c, d are directed automorphisms defined as follows:

$$\psi(b) = (a, c), \qquad \psi(c) = (a, d), \qquad \psi(d) = (1, b).$$

Theorem 2 *The Grigorchuk group $\mathfrak{G}$ is an infinite 2-group.*

Sketch of the proof — To show that $\mathfrak{G}$ is infinite, it suffices to observe that $\psi(\mathrm{St}_{\mathfrak{G}}(1))$ is a subdirect product in $\mathfrak{G} \times \mathfrak{G}$. In particular, the proper subgroup $\mathrm{St}_{\mathfrak{G}}(1)$ projects onto $\mathfrak{G}$ in each factor.

We give a sketch of the proof that every element has order a power of 2. Let $g \in \mathfrak{G}$ and let $k = |g|$, where $|g|$ is the word length of g with respect to the standard set of generators $\{a, b, c, d\}$. We proceed by induction on the length k. Clearly, if $k = 0$ then $g = 1$, and if $k = 1$ we have $g \in \{a, b, c, d\}$ and so $g^2 = 1$. A direct check shows that if $k = 2$ then g has order dividing 2^4.

For the induction step, we first suppose that k is odd and let w be a word of length k representing g. Then either the first and last letters of w are both a, in which case g is conjugate to the element aga of length at most $k-2$, or the first letter u and the last letter v of w are both in $\{b, c, d\}$. In this latter case, the element g is then conjugate to the element ugu, which is of length at most $k-1$. Thus the induction hypothesis applies.

For the remaining case when k is even, we proceed similarly, though there are more cases to consider. We omit this part and refer the reader instead to [37] (or [16]). □

Being a 2-group has special benefits, as certain tools only apply for such groups; e.g. in showing that the Grigorchuk group is not an Engel group [9].

Some other properties of the Grigorchuk group include being regular branch over $K = \langle (ab)^2 \rangle^{\mathfrak{G}}$, and having the congruence subgroup property.

The *second* Grigorchuk group is a rather different group, which was also defined in [37]; it is in fact similar to the groups of the next subsection. There is also the twisted twin of the Grigorchuk group, with all generators the same apart from b, which

instead is defined as $\psi(b) = (c, a)$. This is an infinite 2-group, but does not have the congruence subgroup property [11].

When the Grigorchuk group was defined, it was introduced alongside a family of Grigorchuk groups G_ω, where G_ω acts on the 2-adic tree and is determined by the parameter

$$\omega = (\omega_1, \omega_2, \dots),$$

such that $\omega_i \in \{0, 1, 2\}$, for any $i \in \mathbb{N}$. For an account on these groups see [37] or [38].

If $\omega = \{0, 1, 2, 0, 1, 2, \dots\}$, it turns out that G_ω is the first Grigorchuk group.

3.2 The GGS-groups

Let T be the p-adic tree, for an odd prime p. By a we denote the rooted automorphism, corresponding to the p-cycle

$$(1\,2\,\dots\,p) \in \mathrm{Sym}(p),$$

that cyclically permutes the vertices at the first level. Given a non-zero vector $\mathbf{e} = (e_1, e_2, \dots, e_{p-1}) \in (\mathbb{F}_p)^{p-1}$, we recursively define a directed automorphism $b \in \mathrm{St}(1)$ via

$$\psi(b) = \left(a^{e_1}, a^{e_2}, \dots, a^{e_{p-1}}, b\right).$$

We call the subgroup $G_{\mathbf{e}} = \langle a, b\rangle$ of $\mathrm{Aut}(T)$ the *Grigorchuk–Gupta–Sidki group* (or *GGS-group* for short) associated to the defining vector $\mathbf{e}$.

If $G_{\mathbf{e}} = \langle a, b\rangle$ is a GGS-group corresponding to the defining vector $\mathbf{e}$, then G is an infinite p-group if and only if

$$\sum_{j=1}^{p-1} e_j = 0 \quad \text{in } \mathbb{F}_p$$

(compare with [39]). Besides, it was shown in [24, Lemma 4.2] and [23, Theorem 3.7] that a GGS-group $G_{\mathbf{e}}$ is regular branch if

and only if **e** is not the constant vector. For further information on GGS-groups (we refer the reader to [24] and [23]).

Being 2-generated has advantages, such as enabling one to use results concerning identifying Nielsen equivalence classes (see [52]) and to construct Beauville groups (see [42]).

3.3 The multi-EGS groups

Here we consider a large family of groups, generalising the previous one. Before introducing this large family, it is helpful to first define the subclass of multi-GGS groups, which more closely resemble GGS-groups.

Let T be the p-adic tree, for an odd prime p. Let us observe that, as before, the rooted automorphism a corresponding to the p-cycle $(1\,2\,\ldots\,p) \in \mathrm{Sym}(p)$, that cyclically permutes the vertices at the first level. Given $1 \leqslant r \leqslant p-1$ and a finite r-tuple $\mathbf{E}$ of $\mathbb{F}_p$-linearly independent vectors

$$\mathbf{e}_i = (e_{i,1}, e_{i,2}, \ldots, e_{i,p-1}) \in (\mathbb{F}_p)^{p-1}, \qquad i \in \{1, \ldots, r\},$$

we recursively define directed automorphisms $b_1, \ldots, b_r \in \mathrm{St}(1)$ via

$$\psi(b_i) = (a^{e_{i,1}}, a^{e_{i,2}}, \ldots, a^{e_{i,p-1}}, b_i), \qquad i \in \{1, \ldots, r\}.$$

The subgroup $G = G_{\mathbf{E}} = \langle a, b_1, \ldots, b_r \rangle$ of $\mathrm{Aut}(T)$ is named the *multi-GGS group* associated to the defining vectors $\mathbf{E}$. We observe that $\langle a \rangle \simeq C_p$ and $\langle b_1, \ldots, b_r \rangle \simeq C_p^r$ are elementary abelian p-groups, and we note that a multi-GGS group G acts spherically transitively on the tree T.

These groups were introduced in [6], and were generalised in [49] to multi-EGS groups, which are defined below. Specifically, when $r = 1$, then $G = \langle a, b_1 \rangle$ is a GGS-group.

From [6, Proposition 4.3], we have that for $G = \langle a, b_1, \ldots, b_r \rangle$,

$$G/G' = \langle aG', b_1G', \ldots, b_rG' \rangle \simeq C_p^{r+1}.$$

If $G_{\mathbf{E}} = \langle a, b_1, \ldots, b_r \rangle$ is a multi-GGS group corresponding to an r-tuple of defining vectors $\mathbf{E} = (\mathbf{e}_i)_{i=1}^{r}$, then G is an infinite p-group if and only if, for every $\mathbf{e}_i = (e_{i,1}, \ldots, e_{i,p-1})$, we have $\sum_{j=1}^{p-1} e_{i,j} \equiv 0 \pmod{p}$ (compare with [39] and [76]).

Observe that a multi-GGS group G, apart from the GGS-group defined by the constant vector, is regular branch over $\gamma_3(G)$ (see [6, Proposition 3.5]). Furthermore, if G is not a GGS-group, namely, if $G = \langle a, b_1, \ldots, b_r \rangle$ with $r \geqslant 2$, then G is regular branch over G' (compare with [33, Lemma 2]). All regular branch multi-GGS groups have the congruence subgroup property [23].

We now define multi-EGS groups, where EGS stands for *Extended-Gupta–Sidki*. For $j \in \{1, \ldots, p\}$, let

$$r_j \in \{0, 1, \ldots, p-1\} \quad \text{with } r_j \neq 0,$$

for at least one index j. We fix the numerical datum

$$\mathbf{E} = (\mathbf{E}^{(1)}, \ldots, \mathbf{E}^{(p)}),$$

where each $\mathbf{E}^{(j)} = (\mathbf{e}_1^{(j)}, \ldots, \mathbf{e}_{r_j}^{(j)})$ is an r_j-tuple of $\mathbb{F}_p$-linearly independent vectors

$$\mathbf{e}_i^{(j)} = \big(e_{i,1}^{(j)}, \ldots, e_{i,p-1}^{(j)}\big) \in (\mathbb{F}_p)^{p-1}, \quad i \in \{1, \ldots, r_j\}.$$

The *multi-EGS group* associated to $\mathbf{E}$ is the group

$$G_{\mathbf{E}} = \langle a, \mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(p)} \rangle = \big\langle \{a\} \cup \{b_i^{(j)} \mid 1 \leqslant j \leqslant p,\ 1 \leqslant i \leqslant r_j\} \big\rangle$$

where, for each $j \in \{1, \ldots, p\}$, the generator family

$$\mathbf{b}^{(j)} = \{b_1^{(j)}, \ldots, b_{r_j}^{(j)}\}$$

consists of commuting directed automorphisms $b_i^{(j)} \in \mathrm{St}(1)$

along the directed path

$$\big(\varnothing,\ (p-j+1),\ (p-j+1)(p-j+1),\ \dots\big) \in \partial T$$

that satisfy the recursive relations

$$\psi\big(b_i^{(j)}\big) = \big(a^{e_{i,j}^{(j)}},\ \dots,\ a^{e_{i,p-1}^{(j)}}, b_i^{(j)}, a^{e_{i,1}^{(j)}},\ \dots,\ a^{e_{i,j-1}^{(j)}}\big).$$

The multi-EGS groups are infinite and act spherically transitively on T. A multi-EGS group $G_{\mathbf{E}}$ is a torsion group if and only if for every $\mathbf{e} = (e_1, \dots, e_{p-1}) \in \mathbf{E}$, we have $\sum_{j=1}^{p-1} e_j \equiv 0 \pmod{p}$ (compare with [74, Lemma 3.13]). From [74, Lemma 3.3], we have that if $\mathbf{E}$ is not the set of non-zero constant vectors, then the multi-EGS group $G_{\mathbf{E}}$ is regular branch.

Multi-EGS groups are significant since they provide the first extensive family of finitely generated branch groups without the congruence subgroup property [74].

3.4 The Šunić groups

Let T be the p-adic tree, for a prime p, and for $m \geqslant 2$ let

$$f(x) = x^m + a_{m-1}x^{m-1} + \dots + a_1 x + a_0$$

be an invertible polynomial over $\mathbb{F}_p$. The Šunić group $G_{p,f}$ is generated by the rooted automorphism a corresponding to the p-cycle $(1\,2\,\dots\,p) \in \mathrm{Sym}(p)$, and by the m directed generators $b_1, \dots, b_m$ defined as follows:

$$\begin{aligned}
\psi(b_1) &= (1, \dots, 1, b_2)\\
\psi(b_2) &= (1, \dots, 1, b_3)\\
&\vdots\\
\psi(b_{m-1}) &= (1, \dots, 1, b_m)\\
\psi(b_m) &= (1, \dots, 1, b_1^{-a_0} b_2^{-a_1} \dots b_m^{-a_{m-1}}).
\end{aligned}$$

From [72, Lemma 1 and 6], the groups $G_{p,f}$ are regular branch, and have the congruence subgroup property [27]. The Šunić groups were introduced in [72] as a further generalisation of the first Grigorchuk group, but one that preserves the self-similar (i.e. fractal-like) property. The Šunić groups were initially defined to answer questions concerning the Hausdorff dimension in profinite groups (see [72] for more details).

3.5 The Basilica group

Let T be the d-adic tree, for some $d \geqslant 2$ not necessarily prime. Let now σ be the automorphism of T that cyclically permutes the vertices at the root $\varnothing$, i.e., the element $\sigma = (1\,2\,\ldots\,d)$. Then, the d-*Basilica group*, is generated by a and b, which are recursively defined as follows:

$$a = \psi^{-1}(1, \ldots, 1, b) \qquad \text{and} \qquad b = \psi^{-1}(1, \ldots, 1, a)\sigma.$$

The 2-Basilica group is more commonly known as the *Basilica group*, which was the first such group in this family to be defined. It is significant because it is the first example of a finitely generated subexponentially amenable but not elementary amenable group. The Basilica group acting on the 2-adic tree also has link to dynamics, being the iterated monodromy group of the complex polynomial $z^2 - 1$; see [53, Section 6.12.1].

Properties of the d-Basilica group B include the fact that B is torsion-free, the abelianisation satisfies $B/B' \simeq \mathbb{Z} \times \mathbb{Z}$, the group B is a weakly regular branch group over B'. The d-Basilica group is not branch. We refer the reader to [18] and [64] for the proofs of these facts.

Another 2-generated weakly branch, but not branch, group similar to the Basilica group is the *generalised Brunner–Sidki–Vieira group* (see [22]). The generators are:

$$a = \psi^{-1}(1, \ldots, 1, a)\sigma \qquad \text{and} \qquad b = \psi^{-1}(1, \ldots, 1, b^{-1})\sigma$$

3.6 The odometer

The infinite cyclic group $\mathbb{Z}$ can act on a rooted tree:

$$a = \psi^{-1}(1, \ldots, 1, a)\sigma.$$

This group $\langle a \rangle \simeq \mathbb{Z}$ is also called the *adding machine* or the *odometer*. It is an example of a group acting on a rooted tree that is not weakly branch.

Petschick and Rajeev [64] introduced the idea of the *Basilica operation*, which produces many new examples of groups acting on d-adic trees, similarly to the Basilica group, and their construction starts from the odometer.

4 Maximal subgroups

Maximal subgroups, which are proper subgroups that are maximal with respect to containment, are one of the most fundamental aspects of a group, and the study of maximal subgroups dates back to the early days of group theory, e.g. to the time of Galois and Frattini in the 19th century. Especially in the study of finite groups, maximal subgroups play a key role. The situation is more subtle for infinite groups, as an infinite group may not even have a maximal subgroup, e.g. $\mathbb{Q}$ has none. However finitely generated infinite groups always have maximal subgroups.

4.1 Maximal subgroups of finitely generated branch groups

For branch groups, the key question about their maximal subgroups is when can their index be infinite, and a main goal in this subject is to classify finitely generated branch groups with maximal subgroups of infinite index. A first step in this direction, would be to identify more explicit families of branch groups with maximal subgroups of infinite index.

The first results here were provided by Pervova (see [60] and [61]), who proved that the torsion Grigorchuk groups and the torsion GGS-groups do not contain maximal subgroups of infinite index. Pervova's result was generalised to the torsion multi-EGS groups in [49]. Furthermore, it was shown in [29] that the non-torsion GGS-groups do not have maximal subgroups of infinite index.

In 2010, Bondarenko [12] gave the first example (nevertheless non-explicit) of a finitely generated branch group that does have maximal subgroups of infinite index. His construction is a subgroup of the iterated wreath product of finite perfect transitive groups, and his method does not apply to groups acting on the 2-adic and 3-adic trees. However, the influential paper by Francoeur and Garrido [27] provided the first explicit examples of finitely generated branch groups, acting on the 2-adic tree, with maximal subgroups of infinite index. Their examples are the non-torsion Šunić groups; the Šunić groups are also called siblings of the Grigorchuk group. Recently, further non-explicit examples of finitely generated branch groups with maximal subgroups of infinite index were produced by Kionke and Schesler in [47].

As mentioned above, one of the major aims in this subject is to classify finitely generated branch groups according to the existence of a maximal subgroup of infinite index. Such a characterisation has been achieved for other families of groups, as seen by the classical theorem of Margulis and Soifer [51], which states that a finitely generated linear group has maximal subgroups only of finite index if and only if it is virtually solvable. Recall that a group is *virtually solvable* if it has a solvable finite-index subgroup. Other such characterisations have been achieved in other geometric settings, for example, for subgroups of mapping class groups and word hyperbolic groups [35]. Each of these characterisations has very valuable applications within group theory and geometry, hence classifying branch groups in this manner is of high importance. What is more, the class

of branch groups with maximal subgroups of infinite index already shows promise of being rather different from the above classified geometric groups with maximal subgroups of infinite index. Indeed, we have that the non-torsion Šunić groups acting on the 2-adic tree have countably many maximal subgroups of infinite index, whereas the other geometric groups mentioned above have either none or uncountably many maximal subgroups of infinite index. Furthermore, for these geometric groups, all groups that have maximal subgroups of infinite index contain free subgroups and these groups are of exponential growth; this is not the case for the non-torsion Šunić groups. This also leads to the following interesting question.

Problem 3 (Grigorchuk) Does there exist finitely generated branch groups with uncountably many maximal subgroups?

Identifying finitely generated branch groups with maximal subgroups of infinite index naturally has motivations in permutation group theory, as for G a branch group and $M < G$ a maximal subgroup of infinite index, the action of G on the infinite coset space G/M is primitive. On the other hand, a motivation for identifying which finitely generated branch groups have maximal subgroups only of finite index, is the following notoriously difficult conjecture of Kaplansky on the group ring $K[G]$ of a finitely generated group G over a field K of characteristic p. The conjecture states that, if the Jacobson radical of $K[G]$ coincides with the augmentation ideal of $K[G]$, then G is a finite p-group. We recall that the *Jacobson radical* $\mathrm{Jac}(K[G])$ is the intersection of all maximal right ideals of $K[G]$, and the *augmentation ideal* $\mathrm{Aug}(K[G])$ is the kernel of the augmentation map from $K[G]$ to K defined by

$$\sum_{g\in G} k_g g \mapsto \sum_{g\in G} k_g.$$

Passman [59] showed that if $\mathrm{Jac}(K[G]) = \mathrm{Aug}(K[G])$ then G is a p-group (possibly infinite) and every maximal subgroup of G

is normal of index p. Hence among the branch groups with maximal subgroups only of finite index, those that are p-groups and have maximal subgroups being normal of index p, yield natural candidates for testing Kaplansky's conjecture. Such groups include the torsion GGS-groups, however only a small subclass of these groups are known to not satisfy

$$\mathrm{Jac}(K[G]) = \mathrm{Aug}(K[G])$$

(see [75]). In [49], a subfamily of multi-EGS groups was ruled out as counterexamples to the Kaplansky conjecture.

Another link to results concerning maximal subgroups of branch groups is in the subject of invariable generation, where we say that a subset S of a group G *invariably generates* G if the set $\{s^{g(s)} \mid s \in S\}$ is a generating set for G, for all choices of $g(s)$ in G. This concept was introduced by Dixon in [21] and there are several recent applications to number theory and topology [69]. Since finite groups are all invariably generated (IG), there is interest in finding the smallest invariable generating set for particular families of groups. If a group has a finite invariable generating set, then we call it FIG. Recent interest has arguably occurred due to the paper by Kantor, Lubotzky, and Shalev [46]. This work gave many tools for deciding whether a group is IG, with an emphasis on studying infinite linear groups. In particular the aforementioned classical result by Margulis and Soifer features in this paper, and the authors prove in [46] that a finitely generated linear group is FIG if and only if it is virtually solvable. It was noted in [46] that the first Grigorchuk group is FIG but not virtually solvable, meaning that the linearity assumption is necessary here. They also show that any finitely generated group with all maximal subgroups of finite index must be IG, but they note that this is not a necessary condition. Despite the recent work by various authors on invariable generation, the families of groups where it is understood is still relatively small. We refer the reader

to [15] for some results concerning IG for branch groups.

4.2 Maximal subgroups of weakly branch, but not branch, groups

The above study of maximal subgroups of finitely generated branch groups relies on Pervova's method from [60], which only works for branch groups. In 2020, Francoeur [26] extended Pervova's method to weakly branch groups, which paved the way for maximal subgroups of weakly branch, but not branch, groups to be studied. There are so far only a handful of results, as listed below, all of which only have maximal subgroups of finite index.

Theorem 4 (see [26]) *The Basilica group has maximal subgroups only of finite index.*

Theorem 5 (see [29]) *The weakly branch (not branch) GGS-groups have maximal subgroups only of finite index.*

Theorem 6 (see [18]) *The p-Basilica group has maximal subgroups only of finite index.*

Theorem 7 (see [65]) *The generalised Basilica groups have maximal subgroups only of finite index.*

4.3 The strategy of all proofs

Here we recall *Francoeur's method* from [26]. Note that the results presented below are stated just for the p-adic tree T, but they hold more generally. For convenience we denote by $\mathcal{MF}$ the class of finitely generated groups with all maximal subgroups of finite index.

Recall also that a subgroup H of G is k-*subnormal* if there exist subgroups

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{k-1} \trianglelefteq H_k = G$$

where $H_i \trianglelefteq H_{i+1}$.

Lemma 8 (see [26], Lemma 2.16) *Let* G *be a weakly branch group acting on* T*, and let* $H \leqslant G$ *be a* k*-subnormal subgroup of* G*, for some* $k \in \mathbb{N}$*. If* $v \in T$ *is a vertex not stabilised by* H*, then*

$$\mathrm{rist}_G^{(k)}(v) \leqslant H,$$

where $\mathrm{rist}_G^{(k)}(v)$ *is the* k*th derived subgroup of* $\mathrm{rist}_G(v)$*.*

Proof — We proceed by induction on k, and without loss of generality suppose that H is always a k-subnormal subgroup where k is chosen minimally. For $k = 0$, we have $H = G$ and the result is clear. So suppose the result is true for some $k \in \mathbb{N}$, and we will show it for $k+1$.

Let $K \leqslant G$ be a k-subnormal subgroup of G such that $H \trianglelefteq K$, and let v be a vertex not stabilised by H. Then v is not stabilised by K, so by induction $\mathrm{rist}_G^{(k)}(v) \leqslant K$.

Let $h \in H$ be such that $v^h \neq v$. For any $r_1, r_2 \in \mathrm{rist}_G^{(k)}(v)$, observe that $[h, r_1] \in H$ and $[[h, r_1], r_2] \in H$, since H is normal in K and $\mathrm{rist}_G^{(k)}(v) \leqslant K$. Also $h^{-1}r_1^{-1}h \in \mathrm{rist}_G^{(k)}(v^h)$. Since $v^h \neq v$, this means that $h^{-1}r_1^{-1}h$ commutes with r_1 and r_2. Therefore, we have that $[[h, r_1], r_2] = [r_1, r_2]$ and so $[r_1, r_2] \in H$. It follows that $\mathrm{rist}_G^{(k+1)}(v) \leqslant H$, as wanted. □

Lemma 9 (see [26], Lemma 2.17) *Let* G *be a weakly branch group acting on* T*. Then* $\mathrm{rist}_G(v)$ *is not solvable for any vertex* $v \in T$*.*

Proof — We need to show that $\mathrm{rist}_G^{(k)}(v)$ is non-trivial, for any vertex $v \in T$ and for any $k \in \mathbb{N}$. We proceed by induction on k. The case $k = 0$ follows directly from the definition of weakly branch. So we assume that there exists $k \in \mathbb{N}$ such that $\mathrm{rist}_G^{(k)}(v) \neq 1$, for all $v \in T$. Then, for a given $v \in T$, there exist $g \in \mathrm{rist}_G^{(k)}(v)$ and $w \in T_v$ such that $w^g \neq w$. Since $w \in T_v$ we have $\mathrm{rist}_G^{(k)}(w) \leqslant \mathrm{rist}_G^{(k)}(v)$, and by induction there exists a non-trivial element $r \in \mathrm{rist}_G^{(k)}(w)$. Then

$$\left[g^{-1}, r\right] = gr^{-1}g^{-1}r \in \mathrm{rist}_G^{(k+1)}(v).$$

since $w^g \neq w$, the subgroups $\mathrm{rist}_G^{(k)}(w)$ and $\mathrm{rist}_G^{(k)}(w^g)$ commute. Therefore, as $r \in \mathrm{rist}_G^{(k)}(w)$ and $gr^{-1}g^{-1} \in \mathrm{rist}_G^{(k)}(w^g)$, we obtain that $[g^{-1}, r]$ is non-trivial. Hence, the claim follows. □

The above lemmas imply the following result.

Proposition 10 *Let* G *be a weakly branch group acting on* T*, and let* $H_1, H_2 \leqslant G$ *be two subnormal subgroups of* G*. If* $[H_1, H_2] = 1$*, then* $\mathrm{Supp}(H_1) \cap \mathrm{Supp}(H_2) = \varnothing$*, where*

$$\mathrm{Supp}(H_i) = \left\{\xi \in \partial T \mid h\xi \neq \xi \text{ for some } h \in H_i\right\}.$$

Proof — Suppose by contradiction that

$$\mathrm{Supp}(H_1) \cap \mathrm{Supp}(H_2) \neq \varnothing.$$

Then there exist $\xi \in \partial T$, $h_1 \in H_1$, and $h_2 \in H_2$ such that $h_i\xi \neq \xi$, for $i \in \{1, 2\}$, and hence so there exists $v \in T$ such that $v^{h_i} \neq v$, for $i \in \{1, 2\}$. By Lemma 8, there is a number $k \in \mathbb{N}$ such that $\mathrm{rist}_G^{(k)}(v) \leqslant H_i$, for $i \in \{1, 2\}$. Therefore this gives that

$$\mathrm{rist}_G^{(k+1)}(v) \leqslant [H_1, H_2],$$

which contradicts Lemma 9. □

We obtain the next corollary, which will be useful in the sequel.

Corollary 11 *Let* G *be a weakly branch group acting on* T *and let* H_1, H_2 *be two subnormal subgroups of* G *such that* $[H_1, H_2] = 1$*. If* $H_2 \neq 1$*, then there exists* $v \in T$ *such that*

$$H_1 \subseteq \mathrm{st}_G(v) \qquad \textit{and} \qquad \varphi_v(H_1) = 1.$$

Proof — By Proposition 10, H_1 and H_2 have disjoint support. Since $H_2 \neq 1$, its support is non-empty. As the support is open by Lemma 1, there is $v \in T$ such that $C_v \subseteq \mathrm{Supp}(H_2)$. This implies that $C_v \cap \mathrm{Supp}(H_1) = \varnothing$; in other words H_1 acts trivially on C_v. Consequently, $v^{H_1} = v$ and $\varphi_v(H_1) = 1$. □

Next, we recall that we can view maximal subgroups of infinite index in a branch group from a topological point of view, namely, a maximal subgroup M of infinite index is dense with respect to the profinite topology. Indeed we have that $MN = G$, for all $1 \neq N \trianglelefteq_f G$. More generally, as considered in [26], recall that a subgroup $H \leqslant G$ is *prodense* if $HN = G$ for all non-trivial normal subgroups $N \trianglelefteq G$.

Proposition 12 (see [26], Proposition 2.22) *Let G be a finitely generated infinite group such that every proper quotient of G belongs to the class $\mathcal{MF}$. Then G has a proper prodense subgroup if and only if G has a maximal subgroup of infinite index.*

Proof — "$\Longrightarrow$" Let $H < G$ be a proper prodense subgroup of G. Since G is finitely generated, the subgroup H is contained in a maximal subgroup $M < G$, which must also be prodense. In particular, M cannot contain a non-trivial normal subgroup of G. This implies that M must be of infinite index in G. Indeed, if M was of finite index, it would contain a normal subgroup $N \trianglelefteq G$ of finite index, which would necessarily be non-trivial, since G is infinite.

"$\Longleftarrow$" Let $M < G$ be a maximal subgroup of infinite index, and let $N \trianglelefteq G$ be a non-trivial normal subgroup. If $MN \neq G$, then $N \leqslant M$ by the maximality of M. Therefore by the correspondence theorem M/N is a maximal subgroup of infinite index in G/N, which is a contradiction since $G/N \in \mathcal{MF}$. We conclude that $MN = G$ for all non-trivial normal subgroups of G, and so M is a proper prodense subgroup of G. □

So in theory, determining here the existence (or non-existence) of maximal subgroups of infinite index is equivalent to determining the existence (respectively non-existence) of proper prodense subgroups. In order to do so, we need some additional tools, as given below. First, however, we note a useful result.

Proposition 13 (see [26], Proposition 2.23) *Let G be a finitely generated branch group. Then every proper quotient of G is in $\mathcal{MF}$. In*

particular, G *has a maximal subgroup of infinite index if and only if it has a proper prodense subgroup.*

Proof — By Lemma 8, if N is a non-trivial normal subgroup of G, there exists $n \in \mathbb{N}$ such that $\mathrm{Rist}_G^{(1)}(n) \leqslant N$. Note that, since being in $\mathcal{MF}$ passes to quotients, it suffices to show that $G/\mathrm{Rist}_G^{(1)}(n) \in \mathcal{MF}$, for all $n \in \mathbb{N}$. Since G is branch, we have that $\mathrm{Rist}_G(n)$ is of finite index in G, which implies that $G/\mathrm{Rist}_G^{(1)}(n)$ is a finitely generated virtually abelian group. Thus $G/\mathrm{Rist}_G^{(1)}(n) \in \mathcal{MF}$. □

We now provide additional tools for studying maximal subgroups. Now for $G \leqslant \mathrm{Aut}(T)$, for convenience we write

$$G_u = \varphi_u(\mathrm{st}_G(u)).$$

Recall that $\varphi_u : \mathrm{st}_G(u) \to \mathrm{Aut}(T)$ is given by $f \mapsto f_u$.

Lemma 14 (see [26], Lemma 3.1) *Let* G *be a weakly branch group acting on* T*. If* $H \leqslant G$ *is prodense, then, for any* u *a vertex of* T*, the subgroup* H_u *is prodense in* G_u*.*

Proof — It suffices to show that

$$H_u \,\mathrm{Rist}_{G_u}^{(1)}(n) = G_u,$$

for all $n \in \mathbb{N}$. In fact, since G_u is weakly branch, for all non-trivial normal subgroup $N \trianglelefteq G_u$, by Lemma 8, there exists $n \in \mathbb{N}$ such that $\mathrm{Rist}_{G_u}^{(1)}(n) \leqslant N$. Furthermore, by Lemma 9 we have that $\mathrm{Rist}_{G_u}^{(1)}(n)$ is non-trivial, for every $n \in \mathbb{N}$.

Suppose that the level of u is m, and let $n \in \mathbb{N}$ be arbitrary. Note that

$$\mathrm{Rist}_G(m+n) \leqslant \mathrm{St}_G(m) \leqslant \mathrm{st}_G(u).$$

Then also

$$\varphi_u(\mathrm{Rist}_G(n+m)) \leqslant \mathrm{Rist}_{G_u}(n),$$

and therefore $\varphi_u(\mathrm{Rist}_G^{(1)}(n+m)) \leqslant \mathrm{Rist}_{G_u}^{(1)}(n)$.

As $\mathrm{Rist}_G^{(1)}(n+m)$ is a non-trivial normal subgroup of G, by assumption

$$H\,\mathrm{Rist}_G^{(1)}(n+m) = G. \tag{4.1}$$

By definition, there is $\widetilde{g} \in \mathrm{st}_G(u)$ such that $\varphi_u(\widetilde{g}) = g$, for every $g \in G_u$. By (4.1), there are $h \in H$ and $r \in \mathrm{Rist}_G^{(1)}(n+m)$ such that $hr = \widetilde{g}$. As $\widetilde{g}, r \in \mathrm{st}_G(u)$, it follows that $h \in \mathrm{st}_G(u)$. Thus

$$\varphi_u(h)\varphi_u(r) = g,$$

with $\varphi_u(h) \in H_u$ and $\varphi_u(r) \in \mathrm{Rist}_{G_u}^{(1)}(n)$. This shows that

$$H_u\,\mathrm{Rist}_{G_u}^{(1)}(n) = G_u,$$

and, since n is arbitrary, the result follows. □

Theorem 15 (see [26], Theorem 3.2) *Let G be a weakly branch group acting on T. Suppose H is a prodense subgroup of G and let $u \in T$. Then H is a proper subgroup of G if and only if H_u is a proper subgroup of G_u.*

Proof — If $H_u \neq G_u$, then certainly $H \neq G$. So now assuming that $H \neq G$, we shall show that $H_u \neq G_u$. Note that it suffices to prove this fact for a first-level vertex u, as we can then use induction to prove the result for an arbitrary vertex (cf. Lemma 14).

So, let $u \in L_1$ and suppose by contradiction that $H \neq G$ but $H_u = G_u$.

Observe that $\mathrm{rist}_H(u) = \mathrm{rist}_G(u) \cap H \trianglelefteq \mathrm{st}_H(u)$. Since $H_u = G_u$, we have that $\mathrm{rist}_H(u) \trianglelefteq \mathrm{st}_G(u)$. Indeed, for any $g \in \mathrm{st}_G(u)$, there exists $h \in \mathrm{st}_H(u)$ such that $\varphi_u(g) = \varphi_u(h)$. Hence, since any $r \in \mathrm{rist}_H(u)$ acts trivially outside of the subtree T_u rooted at u, we have

$$grg^{-1} = hrh^{-1} \in \mathrm{rist}_H(u).$$

We also have $\mathrm{rist}_H(u) \trianglelefteq St_G(1)$ since $St_G(1) \leqslant \mathrm{st}_G(u)$.

Now, note that, since G acts transitively at the first level and since $HSt_G(1) = G$, we obtain that H must also act transitively

on L_1. Therefore, for any $v \in L_1$, there exists $h \in H$ such that $st_H(v) = hst_H(u)h^{-1}$. Hence, for all $v \in L_1$,

$$H_v = \varphi_v(st_H(v)) = h_u G_u h_u^{-1} = G_v.$$

It follows that $rist_H(v) \trianglelefteq St_G(1)$, for all $v \in L_1$. Thus

$$Rist_H(1) = \prod_{v \in L_1} rist_H(v) \triangleleft St_G(1).$$

Since $Rist_H(1) \trianglelefteq H$ and $H\,St_G(1) = G$, we conclude that

$$Rist_H(1) \trianglelefteq G.$$

This implies that $Rist_H(1) = \{1\}$. Indeed, otherwise, by hypothesis, we would have $H\,Rist_H(1) = G$, which is impossible since $H\,Rist_H(1) = H$ and $H \neq G$.

For a subset $U \subseteq L_1$, we define

$$rist_H(U) = \Big\{ h \in H \mid v^h = v \quad \forall\, v \notin \bigcup_{u \in U} T_u \Big\}.$$

For $|U| = 1$, this coincides with the usual rigid stabiliser of a vertex.

Now, let $U \subseteq L_1$ such that $rist_H(U) \cap St_H(1) \neq \{1\}$ and whose cardinality is minimal for this property. Note that such a set must exist, since

$$rist_H(L_1) \cap St_H(1) = St_H(1),$$

which is clearly non-trivial. By transitivity, we can assume that $u \in U$. As we have seen above, $rist_H(u) = \{1\}$, which implies that $2 \leqslant |U| \leqslant |L_1|$.

Now we will show that there must exist a surjective homomorphism

$$\alpha : \varphi_u(St_H(|v|)) \longrightarrow \varphi_v(St_H(|v|)),$$

for some $v \in T \backslash T_u$.

Let us consider the (possibly trivial) subgroup

$$\mathrm{rist}_H(L_1\backslash\{u\}) \cap \mathrm{St}_H(1)$$

of elements of $\mathrm{St}_H(1)$ that are sent to the identity by φ_u. We have that

$$\big[\,\mathrm{rist}_H(L_1\backslash\{u\}) \cap \mathrm{St}_H(1),\, \mathrm{rist}_H(U) \cap \mathrm{St}_H(1)\big]$$

$$\leqslant \mathrm{rist}_H((L_1\backslash\{u\}) \cap U) \cap \mathrm{St}_H(1) = \mathrm{rist}_H(U\backslash\{u\}) \cap \mathrm{St}_H(1) = 1,$$

where the last equality is due to the minimality of U. We conclude that $\mathrm{rist}_H(L_1\backslash\{u\}) \cap \mathrm{St}_H(1)$ and $\mathrm{rist}_H(U) \cap \mathrm{St}_H(1)$ commute.

Now, recalling that $|U| \geqslant 2$, let us fix some $w \in U\backslash\{u\}$. Since the map $(\varphi_w)_{|_{\mathrm{st}_H(w)}}$ is a homomorphism, we have that

$$\varphi_w(\mathrm{rist}_H(L_1\backslash\{u\}) \cap \mathrm{St}_H(1)) \quad \text{and} \quad \varphi_w(\mathrm{rist}_H(U) \cap \mathrm{St}_H(1))$$

commute. Besides, notice that both these subgroups are normal in $\varphi_w(\mathrm{St}_H(1))$, since

$$\mathrm{rist}_H(L_1\backslash\{u\}) \cap \mathrm{St}_H(1) \quad \text{and} \quad \mathrm{rist}_H(U) \cap \mathrm{St}_H(1)$$

are normal subgroups of $\mathrm{St}_H(1)$. Now, since $\mathrm{St}_H(1)$ is normal in $\mathrm{st}_H(w)$, and since $\varphi_w(\mathrm{st}_H(w)) = G_w$, we get that $\varphi_w(\mathrm{St}_H(1))$ is a normal subgroup of G_w. Thus both

$$\varphi_w(\mathrm{rist}_H(L_1\backslash\{u\}) \cap \mathrm{St}_H(1)) \quad \text{and} \quad \varphi_w(\mathrm{rist}_H(U) \cap \mathrm{St}_H(1))$$

are 2-subnormal subgroups of G_w.

By the minimality of $|U|$, we know that $\varphi_w(\mathrm{rist}_H(U) \cap \mathrm{St}_H(1))$ is non-trivial. Therefore it follows from Corollary 11 that there exists some $v \in T_w$ (possibly equal to w) such that

$$\mathrm{rist}_H(L_1\backslash\{u\}) \cap \mathrm{St}_H(1) \leqslant \mathrm{st}_H(v)$$

and

$$\varphi_v(\mathrm{rist}_H(L_1\backslash\{u\}) \cap \mathrm{St}_H(1)) = 1.$$

This implies that the kernel of the map

$$(\varphi_u)_{|_{St_H(|v|)}} : St_H(|v|) \longrightarrow \varphi_u(St_H(|v|))$$

is contained in the kernel of the map

$$(\varphi_v)_{|_{St_H(|v|)}} : St_H(|v|) \longrightarrow \varphi_v(St_H(|v|)).$$

Therefore we have a well-defined homomorphism

$$\begin{aligned} \alpha : \varphi_u(St_H(|v|)) &\longrightarrow \varphi_v(St_H(|v|)) \\ g &\longmapsto \varphi_v(\widetilde{g}) \end{aligned}$$

where $\widetilde{g} \in St_H(|v|)$ is any element such that $\varphi_u(\widetilde{g}) = g$. Since for any $\widetilde{g} \in St_H(|v|)$ we have $\alpha(\varphi_u(\widetilde{g})) = \varphi_v(\widetilde{g})$, we see that α is surjective.

We will now derive a contradiction from the existence of this homomorphism and from the fact that H is prodense in G.

Let $1 \neq r \in rist_G(v)$ (recall that G is weakly branch). Notice that since we have shown that $rist_H(w) = 1$ for all $w \in L_1$, we must have $r \notin H$.

Consider $\varphi_v(r) \in \varphi_v(St_G(|v|))$. Since $1 \neq r \in rist_G(v)$, we obtain $\varphi_v(r) \neq 1$. Consequently, there must exist $n \in \mathbb{N}$ such that $\varphi_v(r) \notin St_{G_v}(n)$. Let us write

$$K = \alpha^{-1}\big(St_{\varphi_v(St_H(|v|))}(n)\big) \leqslant \varphi_u(St_H(|v|)).$$

Note that, as $St_{\varphi_v(St_H(|v|))}(n)$ is a normal subgroup of finite index in $\varphi_v(St_H(|v|))$, then K is a normal subgroup of finite index in $\varphi_u(St_H(|v|))$. Now, we know that $St_H(|v|)$ is a normal subgroup of finite index in $st_H(u)$. Therefore, $\varphi_u(St_H(|v|))$ is a normal subgroup of finite index in $\varphi_u(st_H(u)) = G_u$. This means that K is a subgroup of finite index in G_u. Let L be the normal core of K in G_u. Then L is a normal subgroup of G_u of finite index. It follows from Lemma 8 and from the fact that G_u is weakly branch, that there exists some $k \in \mathbb{N}$ such

that $\mathrm{Rist}^{(1)}_{G_u}(k) \leqslant L$.

Set $m = \max\{k, |v| + n - 1\}$, and consider $\mathrm{Rist}^{(1)}_G(m+1)$. We know that this is a normal subgroup of G. Thus, since H is prodense, we have that $H\,\mathrm{Rist}^{(1)}_G(m+1) = G$. In particular, there exist $h \in H$ and $g \in \mathrm{Rist}^{(1)}_G(m+1)$ such that $hg = r$, for r as defined above. Since $r \in \mathrm{St}_G(|v|)$ and

$$\mathrm{Rist}^{(1)}_G(m+1) \leqslant \mathrm{St}_G(m+1) \leqslant \mathrm{St}_G(|v|),$$

we must have $h \in \mathrm{St}_G(|v|)$, and thus $h \in \mathrm{St}_H(|v|)$.

Furthermore, since $\varphi_u(r) = 1$, we find that $\varphi_u(h) = \varphi_u(g)^{-1}$. As $g \in \mathrm{Rist}^{(1)}_G(m+1)$, it follows that $\varphi_u(g) \in \mathrm{Rist}^{(1)}_{G_u}(m) \leqslant L$. Therefore, $\varphi_u(h) \in L$.

Let us now consider $\varphi_v(r) = \varphi_v(h)\varphi_v(g)$. Hence, recalling that $h \in \mathrm{St}_H(|v|)$, we obtain

$$\varphi_v(h) = \alpha(\varphi_u(h)) \in \alpha(L) \leqslant \alpha(K) \leqslant \mathrm{St}_{G_v}(n).$$

From $g \in \mathrm{Rist}_G(m+1) \leqslant \mathrm{St}_G(m+1) \leqslant \mathrm{St}_G(|v|+n)$, we see that $\varphi_v(g) \leqslant \mathrm{St}_{G_v}(n)$. Therefore, we get $\varphi_v(r) \in \mathrm{St}_{G_v}(n)$, which contradicts our choice of r and n. Hence, $H_u \neq G_u$, as required. □

Here is a useful application of the above results.

Theorem 16 (see [26], Theorem 3.3) *Let G be a weakly branch group acting on T, and suppose every proper quotient of G is in $\mathcal{MF}$. If M is a maximal subgroup of G of infinite index, then M_u is a maximal subgroup of infinite index in G_u, for every $u \in T$.*

Proof — Since M is a maximal subgroup of infinite index and since every proper quotient of G is in $\mathcal{MF}$, we have that M is a proper prodense subgroup of G. Let $v \in T$. Then, by Theorem 15, we know that M_v is a proper prodense subgroup of G_v. Thus it only remains to show that it is maximal.

Suppose on the contrary that M_v is not maximal in G_v. Then

there exists $g \in G_v$ such that

$$M_v < \langle M_v, g \rangle < G_v.$$

Hence, by the definition of G_v, there exists $\widetilde{g} \in st_G(v)$ such that $\varphi_v(\widetilde{g}) = g$. Since M is prodense in G, we have that

$$MSt_G(|v|) = G.$$

Therefore, there exists $\widetilde{m} \in M$ and $\widetilde{s} \in St_G(|v|)$ such that $\widetilde{g} = \widetilde{m}\widetilde{s}$. Since both $\widetilde{g}$ and $\widetilde{s}$ belong to $st_G(v)$, we must have $\widetilde{m} \in st_G(v)$. Thus

$$\varphi_v(\widetilde{s}) = \varphi_v(\widetilde{m}^{-1}\widetilde{g}) = \varphi_v(\widetilde{m}^{-1})g.$$

Note that $\varphi_v(\widetilde{m}^{-1}) \in M_v$ and thus

$$\langle M_v, g \rangle = \langle M_v, \varphi_v(\widetilde{m}^{-1})g \rangle.$$

Therefore, replacing g by $\varphi_v(\widetilde{m}^{-1})g$ and $\widetilde{g}$ by $\widetilde{s}$ if necessary, we can assume that $\widetilde{g} \in St_G(|v|)$.

Now, let $w \in L_{|v|}$ be any vertex of level $|v|$ different from v. Observe that, since $rist_G(w)$ is a normal subgroup of $st_G(w)$, we have that $\varphi_w(rist_G(w))$ is a normal subgroup of G_w. By Theorem 15, we know that M_w is a proper prodense subgroup of G_w. Therefore,

$$M_w\varphi_w(rist_G(w)) = G_w$$

and, in particular, there exist $m_w \in M_w$ and $r_w \in rist_G(w)$ such that $m_w\varphi_w(r_w) = \varphi_w(\widetilde{g})$. Set

$$\widehat{g} = \widetilde{g} \prod_{w \in L_{|v|}\setminus\{v\}} r_w^{-1} \in St_G(|v|).$$

Then, for every $w \in L_{|v|}\setminus\{v\}$, we have

$$\begin{aligned}\varphi_w(\widehat{g}) &= \varphi_w(\widetilde{g}) \textstyle\prod_{w' \in L_{|v|}\setminus\{v\}} \varphi_w(r_{w'})^{-1} \\ &= \varphi_w(\widetilde{g})\varphi_w(r_w)^{-1} = m_w \in M_w\end{aligned}$$

where in the second equality we use the fact that $r_{w'} \in \mathrm{rist}_G(w')$, so $\varphi_w(r_{w'}) = 1$ if $w \neq w'$. Furthermore, by a similar computation, we have that $\varphi_v(\widehat{g}) = g$.

Since $g \notin M_v$ by construction, we must have $\widehat{g} \notin M$. Let

$$H = \langle M, \widehat{g} \rangle.$$

By the maximality of M, we obtain $H = G$. However, we now show that

$$H_v = \langle M_v, g \rangle < G_v,$$

which contradicts the fact that $H = G$.

Let $h \in \mathrm{st}_H(v)$ be arbitrary. Since $h \in H = \langle M, \widehat{g} \rangle$, there exist $n \in \mathbb{N}$, $i_1, \ldots, i_n \in \mathbb{Z}$ and $\mu_1, \ldots, \mu_{n+1} \in M$ such that

$$h = \mu_1 \widehat{g}^{i_1} \mu_1^{-1} \mu_2 \widehat{g}^{i_2} \mu_2^{-1} \ \ldots \ \mu_n \widehat{g}^{i_n} \mu_n^{-1} \mu_{n+1}.$$

As $\widehat{g} \in \mathrm{St}_G(|v|)$, we have $\mu_j \widehat{g}^{i_j} \mu_j^{-1} \in \mathrm{St}_G(|v|)$, for all $1 \leqslant j \leqslant n$. Since $h \in \mathrm{st}_G(v)$, we then get that $\mu_{n+1} \in \mathrm{st}_M(v)$.

We now claim that, for all $1 \leqslant j \leqslant n$, we have

$$\varphi_v(\mu_j \widehat{g}^{i_j} \mu_j^{-1}) \in \langle M_v, g \rangle.$$

Indeed, suppose first that $\mu_j \in \mathrm{st}_G(v)$. Then

$$\varphi_v(\mu_j \widehat{g}^{i_j} \mu_j^{-1}) = \varphi_v(\mu_j) g^{i_j} \varphi_v(\mu_j)^{-1} \in \langle M_v, g \rangle.$$

If $\mu_j \notin \mathrm{st}_G(v)$, then $v^{\mu_j^{-1}} \neq v$. Setting $w = v^{\mu_j^{-1}}$ yields

$$\varphi_v(\mu_j \widehat{g}^{i_j} \mu_j^{-1}) = \varphi_w(\mu_j) m_w^{i_j} \varphi_w(\mu_j)^{-1}.$$

Now, since $m_w^{i_j} \in M_w$, there exists some $\widehat{m} \in \mathrm{st}_M(w)$ such that $\varphi_w(\widehat{m}) = m_w^{i_j}$. Clearly,

$$\varphi_v(\mu_j \widehat{g}^{i_j} \mu_j^{-1}) = \varphi_v(\mu_j \widehat{m} \mu_j^{-1}).$$

Since $\widehat{m} \in st_M(w)$ and $\mu_j \in M$, we have

$$\mu_j \widehat{m} \mu_j^{-1} \in st_M(w^{\mu_j}) = st_M(v).$$

Therefore, we conclude that $\varphi_v(\mu_j \widehat{g}^{i_j} \mu_j^{-1}) \in M_v$.

Finally, since $\mu_{n+1} \in st_M(v)$, we have that $\varphi_v(\mu_{n+1}) \in M_v$. Thus, $\varphi_v(h) \in \langle M_v, g \rangle$. The proof is now complete. □

Hence in practice, to show that all maximal subgroups are of finite index, one considers a prodense subgroup H. In many weakly branch groups G, such as the GGS-groups, one has that G' is a proper subgroup. So due to prodensity, we have that $HG' = G$. Let $\{g_1, \ldots, g_n\}$ be a generating set for G. The previous equation yields "approximations"

$$h_1 = g_1 z_1, \ \ldots, \ h_n = g_n z_n$$

of the generators, where $h_1, \ldots, h_n \in H$ and $z_1, \ldots, z_n \in G'$. Since H_u is prodense in G_u, for all $u \in T$, the strategy is to show that there is a vertex u such that $H_u = G_u$; this is done by considering appropriate projections down the tree, with the aim of reducing the word length of the "error" terms $z_1, \ldots, z_n$. Then H is not proper. As H was arbitrary, the result follows.

Now we will see an interesting property of maximal subgroups of infinite index. First, a preliminary result.

Lemma 17 (see [26], Lemma 3.4) *Let G be a weakly branch group acting on T, and suppose every proper quotient of G is in $\mathcal{MF}$. Let M be a maximal subgroup of G of infinite index. Then*

$$\mathrm{Rist}_M(n) = \mathrm{Rist}_G(n) \cap M,$$

for all $n \in \mathbb{N}$.

Proof — Clearly, $\mathrm{Rist}_M(n) \leqslant \mathrm{Rist}_G(n) \cap M$, so it suffices to show the reverse inclusion. Let $g \in \mathrm{Rist}_G(n) \cap M$. Then from

the definition of $\mathrm{Rist}_G(n)$,

$$g = \prod_{v \in L_n} g_v,$$

for some $g_v \in \mathrm{rist}_G(v)$. Since $g \in M$, we have $\varphi_v(g) \in M_v$, for all $v \in L_n$. Note that $\varphi_v(g) = \varphi_v(g_v)$, since $\varphi_v(g_w) = 1$, for all $w \neq v$.

Fix $v \in L_n$ and set $H := \langle M, g_v \rangle$. From the remarks above, we observe that $H_v = M_v$. As M_v is a proper subgroup of G_v by Theorem 15, we conclude that $H \neq G$. Therefore, by the maximality of M, we obtain that $g_v \in \mathrm{Rist}_M(n)$. As $v \in L_n$ was arbitrary, we furthermore obtain that $g \in \mathrm{Rist}_M(n)$. □

Theorem 18 (see [26], Theorem 3.5) *Let* G *be a branch group acting on* T*, and let* M *be a maximal subgroup of* G *of infinite index. Then* M *is a branch group acting on* T*.*

Proof — By Lemma 17 we have $\mathrm{Rist}_M(n) = \mathrm{Rist}_G(n) \cap M$, for all $n \in \mathbb{N}$. Therefore,

$$|M : \mathrm{Rist}_M(n)| = |M : \mathrm{Rist}_G(n) \cap M| \leqslant |G : \mathrm{Rist}_G(n)| < \infty,$$

and so M is branch. Indeed, the subgroup M acts spherically transitively, as it is prodense, so in particular $M\mathrm{St}_G(n) = G$, for all $n \in \mathbb{N}$. □

Theorem 19 (see [26], Theorem 3.6) *Let* G *be a weakly branch group acting on* T*, and suppose every proper quotient of* G *is in* $\mathcal{MF}$*. Let* M *be a maximal subgroup of* G *of infinite index. Then* M *is a weakly branch group acting on* T*.*

Proof — Again by Lemma 17 we have

$$\mathrm{Rist}_M(n) = \mathrm{Rist}_G(n) \cap M,$$

for all $n \in \mathbb{N}$. We thus need to show that $\mathrm{Rist}_G(n) \cap M \neq 1$, for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be arbitrary, and let $1 \neq r_1 \in \mathrm{Rist}_G(n)$. Hence, if $r_1 \in M$, then $\mathrm{Rist}_G(n) \cap M \neq 1$. If not, since $r_1 \neq 1$, we have that $r_1 \notin \mathrm{St}_G(m)$, for some $m > n$. In particular, $r_1 \notin \mathrm{Rist}_G(m)$. As M is prodense, we have $h \in M$ and $r_2 \in \mathrm{Rist}_G(m)$ such that $hr_2 = r_1$. Therefore, $h = r_1 r_2^{-1}$. Since $r_1 \notin \mathrm{Rist}_G(m)$, it follows that $r_1 r_2^{-1} \neq 1$, and as $m > n$, we obtain that

$$\mathrm{Rist}_G(m) \leqslant \mathrm{Rist}_G(n).$$

Therefore, $r_1 r_2^{-1} \in \mathrm{Rist}_G(n)$. So $r_1 r_2^{-1} \in \mathrm{Rist}_G(n) \cap M$, and thus $\mathrm{Rist}_G(n) \cap M \neq 1$, as required. □

We end these notes with some open questions.

Problem 20 If G is weakly branch and has infinite-index maximal subgroups, must G be branch?

In the study of maximal subgroups, note that most examples of branch and weakly branch groups that have been considered are subgroups of the Sylow pro-p subgroup $\varprojlim C_p \wr \ldots \wr C_p$ of $\mathrm{Aut}(T)$.

Problem 21 Let T be the p-adic tree, for a prime p. Classify finitely generated branch groups within a Sylow pro-p subgroup of $\mathrm{Aut}(T)$ that have maximal subgroups of infinite index.

Finally, returning to our starting point on Burnside groups, we note that recent work by Petschick [63] has led to new explicit examples of groups acting on rooted trees that are Burnside groups. In relation to the Related Burnside Problem, we document the following question.

Problem 22 Does there exist a finitely presented torsion branch group?

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