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The Interplay of R-matrices and Quantum Groups

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Abstract

These are the expanded lecture notes of a mini-course given by the author in Milan in June 2024, during the conference *GABY: Groups and Algebras in Bicocca for Young algebraists*. We present a concise introduction to the theory of quantum groups, focusing on their ability to produce solutions of the quantum Yang–Baxter equation.

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1 Introduction

In classical mechanics, a physical system is characterized by a space of states X, typically a manifold, and an algebra of observables $A = \mathcal{O}(X)$, consisting of *regular* functions on X. Clearly, A is a commutative and associative algebra. For instance, in the case of a particle moving along a line, $X = \mathbb{R}^2$ and $A = \mathbb{C}[q, p]$, where q is the position and p is the momentum. Classical observables are given by quantities such as the energy of the particle

$$H(p, q) = \frac{p^2}{2} + U(q).$$

Quantization is the mathematical tool to understand the transition from classical to quantum mechanics. Intuitively, classical mechanics is the limit of quantum mechanics as the *quantum parameter* tends to zero and, conversely, quantization is

"an algorithm by which a quantum system corresponds to a classical dynamical one"

(F. Berezin)

The algebra of *quantum observables*, denoted $A_\hbar$, is a non-commutative, yet still associative, *deformation* of the algebra A, which depends on a quantization parameter $\hbar$, typically the Planck constant. For example, in the case of $X = \mathbb{R}^2$, the quantum observable algebra $A_\hbar$ is generated by q and $p = -i\hbar\frac{\partial}{\partial q}$, so that the quantum commutation relation holds:

$$[p, q] = -i\hbar.$$

When the space of states is a group $X = G$, the algebra of observables is naturally equipped with the structure of a commutative Hopf algebra. Thus, the transition from classical to quantum physics amounts to deforming a commutative Hopf algebra into a non-commutative one. This motivates the term *quantum group*, which is often used more broadly as a synonym for a non-commutative, non-cocommutative Hopf algebra.

Since their first appearance in the eighties, quantum groups quickly gained widespread interest for their spectacular connections with many areas of mathematics and theoretical physics, such as representation theory, low-dimensional topology, algebraic geometry, combinatorics, quantum field theory, gauge theory, and integrability.

In these notes, we focus on a phenomenon at the heart of the theory, providing its central connection to integrable systems: quantum groups are a natural source of R-matrices. Recall than an R-matrix is a function

$$\mathcal{R}(s)\colon \mathbb{C} \to \mathrm{End}(V \otimes V)\,,$$

valued in the algebra of linear endomorphisms of the second tensor power of a (typically finite-dimensional) vector space V, which satisfies the Yang–Baxter equation:

$$\mathcal{R}_{12}(s_1)\mathcal{R}_{13}(s_1+s_2)\mathcal{R}_{23}(s_2) = \mathcal{R}_{23}(s_2)\mathcal{R}_{13}(s_1+s_2)\mathcal{R}_{12}(s_1) \quad [*]$$

in $\mathrm{End}(V \otimes V \otimes V)$. The dependence of $\mathcal{R}(s)$ on the parameter s is a central feature. The R-matrix $\mathcal{R}(s)$ is referred to as constant (resp. rational, meromorphic, formal) if its matrix entries are constant (resp. rational, meromorphic, formal) functions in s.

A fundamental principle in the theory of quantum groups is that R-matrices often arise as intertwiners for the action of a Hopf algebra on $V \otimes V$. In these notes, we discuss the construction of R-matrices in two settings: Drinfeld–Jimbo quantum groups and Yangians. Drinfeld–Jimbo quantum groups are equipped with a *universal* R-matrix, yielding a *constant* R-matrix on any representation. The theory of Yangians provides instead a uniform method for the construction of *rational* and *meromorphic* R-matrices. The material covered in the first part is quite standard and can be found in Drinfeld's foundational paper [17], in many excellent textbooks, see, e.g., [14], [33], [37], [43], [46], [48], [54], or lecture notes, see, e.g., [24]. The second part, concerning the construction of the R-matrix for Yangians in both finite and affine types, is meant as a brief account of the recent papers [3] and [29].

2 Hopf algebras

In this section, we provide a concise overview of the concept of a Hopf algebra, outlining its basic properties and its connec-

[*] Here, the indices denote the tensor factors on which the operator is acting. For instance, if $\mathbf{R} = \sum_i a_i \otimes b_i \in \mathrm{End}(V \otimes V)$, then

$$\mathbf{R}_{13} = \sum_i a_i \otimes \mathrm{id} \otimes b_i \in \mathrm{End}(V \otimes V \otimes V).$$

tion to the framework of monoidal categories. Throughout, $\mathbb{K}$ denotes a base field over which all structures are defined.

2.1 Definition

A *bialgebra* is a tuple $(A, m, \eta, \Delta, \epsilon)$, where:

- the triple (A, m, η) is an associative algebra, with product $m : A \otimes A \to A$ and unit $\eta : \mathbb{K} \to A$, i.e., the diagrams

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m\otimes id} & A \otimes A \\ \downarrow{\scriptstyle id\otimes m} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array}$$

and

$$\begin{array}{ccccc} A & \xrightarrow{\rho^{-1}} & A \otimes \mathbb{K} & \xrightarrow{id\otimes\eta} & A \otimes A \\ \downarrow{\scriptstyle \lambda^{-1}} & & & & \downarrow{\scriptstyle m} \\ \mathbb{K} \otimes A & \xrightarrow[\eta\otimes id]{} & A \otimes A & \xrightarrow[m]{} & A \end{array}$$

where $\lambda\colon \mathbb{K} \otimes A \to A$ and $\rho\colon A \otimes \mathbb{K} \to A$ are the natural identifications, commute;

- the triple (A, Δ, ϵ) is a *coassociative coalgebra*, with coproduct $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to \mathbb{K}$, i.e., the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta\otimes id} \\ A \otimes A & \xrightarrow[id\otimes\Delta]{} & A \otimes A \otimes A \end{array}$$

and

$$\begin{array}{ccccc} A & \xrightarrow{\Delta} & A \otimes A & \xrightarrow{\epsilon\otimes id} & \mathbb{K} \otimes A \\ \downarrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle \lambda} \\ A \otimes A & & \xrightarrow[id\otimes\epsilon]{} A \otimes \mathbb{K} & \xrightarrow[\rho]{} & A \end{array}$$

commute;

- the coproduct Δ and the counit ϵ are morphisms of algebras, i.e., $\Delta \circ \eta = \eta \otimes \eta$, $\epsilon \circ \eta = id_{\mathbb{K}}$, and the diagrams

$$\begin{array}{ccccc} A \otimes A & & \xrightarrow{\quad m \quad} & & A \\ \downarrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle \Delta} \\ A \otimes A \otimes A \otimes A & \xrightarrow{(2\,3)} & A \otimes A \otimes A \otimes A & \xrightarrow{m\otimes m} & A \otimes A \end{array}$$

and

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ \downarrow{\scriptstyle \epsilon\otimes\epsilon} & & \downarrow{\scriptstyle \epsilon} \\ \mathbb{K} \otimes \mathbb{K} & \xrightarrow[\cdot]{} & \mathbb{K} \end{array}$$

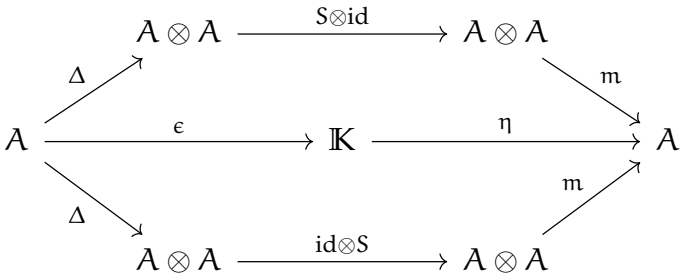
where $(2\,3)$ denotes the flip on the second and third components, commute.

By replacing the product (resp. the coproduct) with its opposite $m^{op} = m \circ (1\,2)$ (resp. $\Delta^{op} = (1\,2) \circ \Delta$), we obtain another bialgebra denoted A^{op} (resp. A^{cop}).

Observe also that the notion of a bialgebra is self-dual. Assume A is finite-dimensional, and let A^* be its dual. Then, we have that $(A^*, \Delta^t, \epsilon^t, m^t, \eta^t)$ is a bialgebra, where $(\cdot)^t$ denotes the dual of a linear map.

A bialgebra $(A, m, \eta, \Delta, \epsilon)$ is a *Hopf algebra* if there is a mor-

phism $S\colon A \to A$, called the *antipode*, such that the diagram



(2.1)

commutes.

Remark 1

1. The set $\mathrm{End}(A)$ is naturally an algebra with respect to the convolution product

$$f \star g = m \circ f \otimes g \circ \Delta$$

and with unit $\eta \circ \epsilon \in \mathrm{End}(A)$. Therefore, (2.1) reads

$$S \star \mathrm{id} = \eta \circ \epsilon = \mathrm{id} \star S\,,$$

i.e., the antipode is the convolution inverse of the identity in $\mathrm{End}(A)$. Thus, it is formally expressed through the geometric series

$$S = \frac{\eta \circ \epsilon}{\eta \circ \epsilon + (\mathrm{id} - \eta \circ \epsilon)} = \sum_{n \geqslant 0} (-1)^n (\mathrm{id} - \eta \circ \epsilon)^{\star n}$$

$$= \sum_{n \geqslant 0} (-1)^n m^{(n)} \circ (\mathrm{id} - \eta \circ \epsilon)^{\otimes n} \circ \Delta^{(n)}\,,$$

where $m^{(n)}$ and $\Delta^{(n)}$ denote the nth iterated product and coproduct, respectively. In particular, this shows that, if it exists, the antipode is unique (and it can therefore be regarded as a property of the bialgebra rather than an extra datum).

2. Let A, B be Hopf algebras and $f\colon A \to B$ a morphism of the underlying bialgebras, i.e., f is a morphism of both algebras and coalgebras. Then, by uniqueness,

$$f \circ S_A = S_B \circ f,$$

i.e., f is automatically a morphism of Hopf algebras.

3. The counit ϵ and the antipode S are uniquely determined by the coproduct Δ.

4. The antipode S is both an algebra and a coalgebra anti-automorphism.

5. If A is either commutative or cocommutative, then $S^2 = \mathrm{id}$.

6. A non-zero element $a \in A$ is *group-like* if $\Delta(a) = a \otimes a$. In this case, $\epsilon(a) = 1$ and a is invertible with $a^{-1} = S(a)$. An element $a \in A$ is *primitive* if $\Delta(a) = a \otimes 1 + 1 \otimes a$. In this case, $\epsilon(a) = 0$ and $S(a) = -a$.

Example 2

1. Let G be a finite group and $\mathcal{O}(G)$ the set of functions on G. Clearly, $\mathcal{O}(G)$ is a commutative unital associative algebra with respect to the standard pointwise operations on functions. However, $\mathcal{O}(G)$ has a natural Hopf algebra structure, obtained by dualizing the group operations on G.

   Namely, under the identification $\mathcal{O}(G \times G) \simeq \mathcal{O}(G) \otimes \mathcal{O}(G)$, the product on G induces a coassociative coproduct

$$\Delta\colon \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$$

given by

$$\Delta(\phi)(x, y) = \phi(xy),$$

for any $\phi \in \mathcal{O}(G)$ and $x, y \in G$. The unit $e \in G$ induces a

counit $\epsilon\colon \mathcal{O}(G) \to \mathbb{K}$ given by

$$\epsilon(\phi) = \phi(e),$$

for any $\phi \in \mathcal{O}(G)$. Finally, the inverse map induces the *antipode* $S\colon \mathcal{O}(G) \to \mathcal{O}(G)$ given by

$$S(\phi)(x) = \phi(x^{-1})$$

for any $\phi \in \mathcal{O}(G)$ and $x \in G$. Up to a suitable completion of the tensor product, the same construction works for the algebra of continuous functions over a Lie group or regular functions over an algebraic group.

2. Let G be a group. Then, the group-algebra $\mathbb{K}G$ is a cocommutative Hopf algebra with

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1, \quad \text{and} \quad S(g) = g^{-1},$$

for any $g \in G$.

3. Let $\mathfrak{g}$ be a Lie algebra. Then, the universal enveloping algebra $U(\mathfrak{g})$ is a cocommutative Hopf algebra with

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \epsilon(x) = 0, \quad \text{and} \quad S(x) = -x,$$

for any $x \in \mathfrak{g}$.

2.2 Quantum groups

The term *quantum group* is often used as a synonym for a non-commutative, non-cocommutative Hopf algebra (see [17]). The simplest example of a quantum group is given by the quantum deformation of $U(\mathfrak{sl}_2)$. Recall that $\mathfrak{sl}_2$ is the three-dimensional complex Lie algebra generated by e, f, h with bracket given by

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad \text{and} \qquad [e, f] = h\,.$$

Consider the ring of formal power series $\mathbb{C}[\![\hbar]\!]$ and set $q = e^{\hbar/2}$. The quantum group $U_\hbar(\mathfrak{sl}_2)$ is the algebra generated over $\mathbb{C}[\![\hbar]\!]$ by E, F, H with relations*

$$[H, E] = 2E, \quad [H, F] = -2F, \quad \text{and} \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}. \quad (2.2)$$

Note that, expanding in $\hbar$,

$$\frac{q^H - q^{-H}}{q - q^{-1}} = H + O(\hbar).$$

Thus, $U_\hbar(\mathfrak{sl}_2)$ is a deformation of $U(\mathfrak{sl}_2)$, since it reduces to $U(\mathfrak{sl}_2)$ at $\hbar = 0$. Moreover, $U_\hbar(\mathfrak{sl}_2)$ is a Hopf algebra over $\mathbb{C}[\![\hbar]\!]$ with coproduct

$$\Delta(E) = E \otimes q^H + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + q^{-H} \otimes F$$

and

$$\Delta(H) = H \otimes 1 + 1 \otimes H.$$

This determines the counit

$$\epsilon(E) = \epsilon(F) = \epsilon(H) = 0,$$

and the antipode

$$S(E) = -Eq^{-H}, \qquad S(F) = -q^H F, \qquad S(H) = -H.$$

Note that in this case the antipode is not involutive.

Remark 3

1. Set $K = q^H$. Then, the relations (2.2) reads

$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F,$$

* More precisely, we consider the algebra of formal power series with values in the free algebra generated by E, F, and H, and we mod out the $\hbar$-adic completion of the ideal generated by the relations (2.2).

and

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

As an algebra generated by $E, F, K^{\pm 1}$, this gives a presentation of quantum $\mathfrak{sl}_2$ which is defined over $\mathbb{C}$ for numerical values of $q \in \mathbb{C}$ ($q \neq 0, 1$) or over $\mathbb{C}(q)$ for q being an indeterminate. This is usually denoted by $U_q(\mathfrak{sl}_2)$.

2. $U_\hbar(\mathfrak{sl}_2)$ is a *quantized universal enveloping algebra* (QUE), i.e., a Hopf algebra deformation of a universal enveloping algebra, specifically of $U(\mathfrak{sl}_2)$. Since the coproduct is cocommutative modulo $\hbar$, it yields a map $\delta\colon \mathfrak{sl}_2 \to \mathfrak{sl}_2 \wedge \mathfrak{sl}_2$ by

$$\delta(x) = \frac{\Delta(\widetilde{x}) - \Delta^{21}(\widetilde{x})}{\hbar} \mod \hbar,$$

where $\widetilde{x} \in U_\hbar(\mathfrak{sl}_2)$ is any lift of $x \in \mathfrak{sl}_2$. For instance, we have $\delta(h) = 0$, while

$$\delta(e) = h \wedge e \qquad \text{and} \qquad \delta(f) = h \wedge f.$$

Then, δ is readily seen to be a *cobracket* on $\mathfrak{sl}_2$, i.e., the analogue for a Lie algebra of a coproduct map. As in the case of a bialgebra, the cobracket is essentially a *dual bracket* (as it gives rise to a bracket on the dual space $\mathfrak{g}^*$), satisfying anti-symmetry and a *co*-Jacobi identity. Moreover, it satisfies a suitable compatibility condition with the bracket, referred to as the *cocycle* condition, yielding on $\mathfrak{sl}_2$ a Lie bialgebra structure, see Section 4.1.

3. As an algebra, $U_\hbar(\mathfrak{sl}_2)$ is isomorphic to $U(\mathfrak{sl}_2)[\![\hbar]\!]$. An isomorphism can be constructed explicitly (see [14] and [19] for $\mathfrak{sl}_2$, and [2] for the case of $\mathfrak{sl}_n$, $n \geqslant 2$).

4. By definition, we have that a $U_\hbar(\mathfrak{sl}_2)$-module is a *topologically free* $\mathbb{C}[\![\hbar]\!]$-module equipped with an $\hbar$-linear action of $U_\hbar(\mathfrak{sl}_2)$. Recall that, roughly, a $\mathbb{C}[\![\hbar]\!]$-module $\mathcal{V}$ is topo-

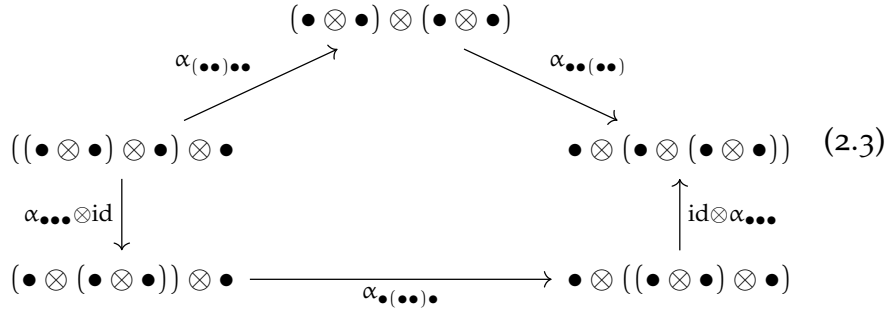
logically free if it is isomorphic to $V[\![\hbar]\!]$ for some complex vector space V. In this context, modules are never irreducible, as $\hbar V$ is always a submodule, nor are they finite-dimensional. Nevertheless, for simplicity, we will use the term *irreducible* to mean indecomposable and *finite-dimensional* to mean finite-rank (see also [14, Section 6.4E]).

2.3 Bialgebras and monoidal categories

Let A be a bialgebra. Then $\mathsf{Mod}(A)$ is a natural example of a *monoidal* category. Recall that a monoidal structure on a category $\mathcal{C}$ is the datum of

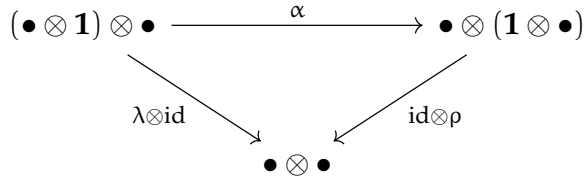
- a distinguished object $\mathbf{1} \in \mathcal{C}$, called *unit*,
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called *tensor product*,
- a natural isomorphism $\alpha\colon (\bullet \otimes \bullet) \otimes \bullet \Rightarrow \bullet \otimes (\bullet \otimes \bullet)$, called *associativity constraint*,
- two natural isomorphisms $\lambda\colon \mathbf{1} \otimes \bullet \Rightarrow \bullet$ and $\rho\colon \bullet \otimes \mathbf{1} \Rightarrow \bullet$, called, respectively, *left and right unit constraints*,

such that the diagrams



(2.3)

and



commute.

Remark 4 The identity (2.3) is called the *pentagon axiom*. Moreover, by MacLane's coherence theorem (see [47]), this implies that any two bracketings on a tensor product with any number of factors can be naturally identified in $\mathcal{C}$.

The category $\mathrm{Vect}_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$ is a monoidal category. The tensor product is given by the usual tensor product of vector spaces, the unit is the base field, the associativity constraint is given by the natural isomorphism

$$(U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

sending $(u \otimes v) \otimes w$ to $u \otimes (v \otimes w)$, and the unit constraints are given by the natural isomorphisms $V \otimes \mathbb{K} \simeq V \simeq \mathbb{K} \otimes V$.

Let A be a bialgebra. Then $\mathsf{Mod}(A)$ is a monoidal category with respect to the following structure.

- The tensor product in $\mathsf{Mod}(A)$ is induced by the coproduct Δ. For any $V, W \in \mathsf{Mod}(A)$, their tensor product $V \otimes W$ is naturally a module over $A \otimes A$. By pullback through Δ, one gets an action of A, i.e., for $v \in V$ and $w \in W$, the action of an element $a \in A$ on $V \otimes W$ is defined by

  $$a \cdot (v \otimes w) = \sum (a_{(1)} \cdot v) \otimes (a_{(2)} \cdot w),$$

  where

  $$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$$

  in Sweedler's notation.

- The unit object is the base field $\mathbb{K}$, treated as an A-module via the counit ϵ of A.

Finally, the associativity constraint and the unit constraints are inherited from the category of vector spaces.

Remark 5 For $\mathsf{Mod}(A)$ to be a monoidal category, A does not need to be a Hopf algebra. For example, if M is a monoid, then $A = \Bbbk M$ is still a bialgebra and $\mathsf{Mod}(A)$ is a monoidal category. However, if A is a Hopf algebra, then $\mathsf{Mod}(A)$ has *dual modules*.

3 Quasitriangular Hopf algebras

Quasitriangular Hopf algebras form a special class of Hopf algebras that play a key role in the theory of quantum groups and integrable systems. Introduced by Drinfeld [17], a quasitriangular Hopf algebra is a Hopf algebra equipped with a distinguished operator, called a *universal R-matrix*, which yields a solution to the quantum Yang–Baxter equation on any module.

3.1 R-matrices

Let A be a bialgebra, with coproduct $\Delta\colon A \to A \otimes A$ and opposite coproduct $\Delta^{\mathrm{op}} = (1\,2) \circ \Delta$. Then A is *quasitriangular* if there exists an invertible element $\mathbf{R} \in (A \otimes A)^{\times}$, called an *R-matrix*, which satisfies the following properties.

- The *intertwining identity*

$$\mathbf{R} \cdot \Delta(a) = \Delta^{\mathrm{op}}(a) \cdot \mathbf{R}, \tag{3.1}$$

  for any $a \in A$.

- The *coproduct identities*

$$(\Delta \otimes \mathrm{id})(\mathbf{R}) = \mathbf{R}_{13} \cdot \mathbf{R}_{23} \tag{3.2}$$

$$(\mathrm{id} \otimes \Delta)(\mathbf{R}) = \mathbf{R}_{13} \cdot \mathbf{R}_{12} \tag{3.3}$$

  where the subscripts indicate on which factor the operator $\mathbf{R}$ acts.

Note that, by combining (3.1) and either (3.2) or (3.3), it follows that $\mathbf{R}$ is a solution of the quantum Yang–Baxter equation

$$\mathbf{R}_{12} \cdot \mathbf{R}_{13} \cdot \mathbf{R}_{23} = \mathbf{R}_{23} \cdot \mathbf{R}_{13} \cdot \mathbf{R}_{12}$$

in $A \otimes A \otimes A$.

Remark 6 From (3.2) and (3.3), it follows that

$$(\epsilon \otimes \mathrm{id})(\mathbf{R}) = 1 = (\mathrm{id} \otimes \epsilon)(R).$$

Assume now that A is a Hopf algebra with invertible antipode $S\colon A \to A$. Then, one also has

$$(S \otimes \mathrm{id})(\mathbf{R}) = \mathbf{R}^{-1} \qquad \text{and} \qquad (\mathrm{id} \otimes S)(\mathbf{R}^{-1}) = \mathbf{R}. \tag{3.4}$$

3.2 Drinfeld element, balances, and sovereign elements

The *Drinfeld element* is the invertible element

$$u = m((S^{-1} \otimes \mathrm{id})(\mathbf{R})),$$

where $m\colon A \otimes A \to A$ denotes the multiplication map. It satisfies the identities[*]

$$\mathrm{Ad}(u) = S^2 \qquad \text{and} \qquad \Delta(u) = (\mathbf{R}_{21} \cdot \mathbf{R})^{-1} \cdot (u \otimes u)\,.$$

The Hopf algebra A is called

- *balanced* if there exists a central element $b \in A^\times$, called *balance*, such that $\Delta(b) = (b \otimes b) \cdot \mathbf{R}_{21} \cdot \mathbf{R}$;
- *sovereign* if there exists a group-like element $D \in A^\times$ such that
$$\mathrm{Ad}(D) = S^2.$$

It is clear that A is balanced if and only if it is sovereign, since (up to group-like central elements) the balance and the sovereign

[*] The element u is u_2^{-1} in [19].

element satisfy

$$D = b \cdot u.$$

Remark 7 The two definitions are essentially implicit in [19]. A balance, as presented above, is a weaker version of a ribbon as considered in [52]. For the terminology of sovereign Hopf algebras, see [12].

3.3 R-matrices and dual modules

Assume that A is balanced, with balance b and sovereign element D. Let V be an A-module. The algebraic dual V^* is naturally equipped with two A-module structures. Namely, the *right dual* module V^* and the *left dual* module ${}^*V$ are defined by the assignments

$$a_{V^*} = (S(a)_V)^t, \qquad a_{{}^*V} = (S^{-1}(a)_V)^t,$$

respectively, where $a \in A$ and $(\cdot)^t$ denotes the dual of a linear map. For all A-modules V, W one has

$$\mathbf{R}_{V^*,W} = (\mathbf{R}^{-1}_{V,W})^{t_1} \qquad \text{and} \qquad \mathbf{R}_{V,{}^*W} = (\mathbf{R}^{-1}_{V,W})^{t_2}$$

by (3.4), and, similarly,

$$\mathbf{R}_{V^{**},W} = (((\mathbf{R}^{-1}_{V,W})^{t_1})^{-1})^{t_1} \quad \text{and} \quad \mathbf{R}_{V,{}^{**}W} = (((\mathbf{R}^{-1}_{V,W})^{t_2})^{-1})^{t_2}.$$

When V is finite-dimensional, the action of the sovereign element provides a canonical isomorphism of modules $V \to V^{**}$ and $V \to {}^{**}V$. Thus, we have

$$\mathrm{Ad}(D \otimes 1)(\mathbf{R})_{V,W} = (((\mathbf{R}^{-1}_{V,W})^{t_1})^{-1})^{t_1} \tag{3.5}$$

when V is finite-dimensional, and

$$\mathrm{Ad}(1 \otimes D^{-1})(\mathbf{R})_{V,W} = (((\mathbf{R}^{-1}_{V,W})^{t_2})^{-1})^{t_2} \tag{3.6}$$

when W is finite-dimensional. The relations (3.3) and (3.3) are commonly referred to as *crossing symmetries*.

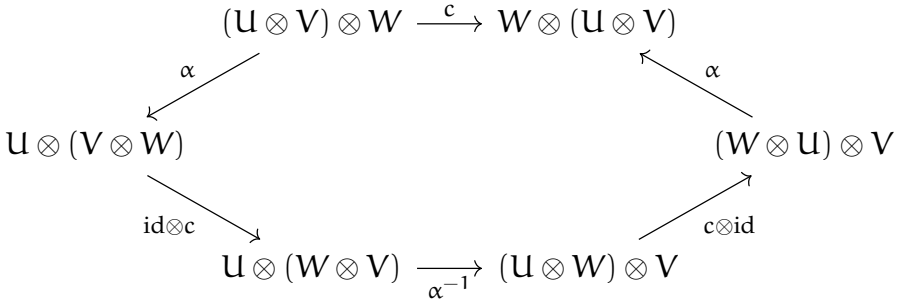
3.4 Braided monoidal categories

Quasitriangular Hopf algebras are a natural source of *braided* monoidal categories, as the R-matrix induces a braiding on the category of modules.

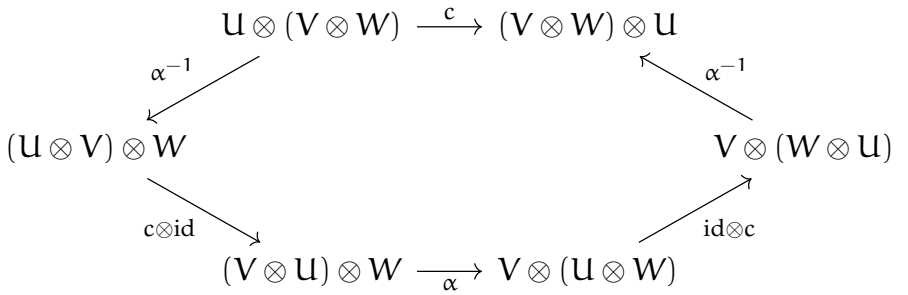
Recall that a *braided monoidal category* $(\mathcal{C}, \otimes, \mathbf{1}, c)$ is a monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ equipped with a natural isomorphism

$$c \colon \otimes \Rightarrow \otimes^{\mathrm{op}},$$

called the *braiding* or the *commutativity constraint*, such that the diagrams



and



commutes. For every object $V \in \mathcal{C}$ and $n > 0$, the tensor power $V^{\otimes n}$ is naturally acted upon by the *braid group* on n strands $\mathcal{B}_n$, which is generated by the elements $S_1, \ldots, S_{n-1}$, with relations

$$S_i \cdot S_{i+1} \cdot S_i = S_{i+1} \cdot S_i \cdot S_{i+1}$$

and $S_i S_j = S_j S_i$ if $|i-j| > 1$. The action $\rho\colon \mathcal{B}_n \to \mathrm{End}_{\mathcal{C}}(V^{\otimes n})$ is essentially* given by the assignment $\rho(S_i) = (c_{VV})_{i\,i+1}$.

When A is a quasi-triangular Hopf algebra with R-matrix $\mathbf{R}$, the category of A-modules becomes a braided monoidal category. The R-matrix defines a braiding by setting

$$c_{VW} = (1\,2) \circ \mathbf{R}_{VW}\,,$$

where $\mathbf{R}_{VW} \in \mathrm{End}(V \otimes W)$ denotes the action of $\mathbf{R}$ on $V \otimes W$. By the intertwining identity (3.1), $c_{VW} \in \mathrm{End}_A(V \otimes W)$ is an intertwiner, while the hexagon axioms above follows from the coproduct identities (3.2) and (3.3).

3.5 The R-matrix of quantum $\mathfrak{sl}_2$

Recall the definition of $U_\hbar(\mathfrak{sl}_2)$ as a Hopf algebra over $\mathbb{C}[\![\hbar]\!]$ from Section 2.2. In this case, one may attempt to compute the R-matrix directly by imposing a certain form on $\mathbf{R}$ and then determining its coefficients relying on one of the coproduct identities. This yields the following explicit expression of the R-matrix of $U_\hbar(\mathfrak{sl}_2)$

$$\mathbf{R} = q^{\frac{H\otimes H}{2}} \sum_{n\geqslant 0} q^{\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]!} E^n \otimes F^n\,,$$

where $q = e^{\hbar/2}$, $[n] = \frac{q^n - q^{-n}}{q-q^{-1}}$, and $[n]! = [n]\dots[1]$.

Since $(q-q^{-1})^n = \hbar^n + O(\hbar^{n+1})$, the R-matrix $\mathbf{R}$ is an *algebraic* element, i.e., $\mathbf{R} \in U_\hbar(\mathfrak{sl}_2) \otimes U_\hbar(\mathfrak{sl}_2)$, where $\otimes$ denotes the $\hbar$-adically complete tensor product. In particular, $\mathbf{R}$ acts on the tensor product of *any* two representations of $U_\hbar(\mathfrak{sl}_2)$ defined over $\mathbb{C}[\![\hbar]\!]$.

* More precisely, $V^{\otimes n}$ is equipped with a family of actions of $\mathcal{B}_n$ indexed by complete bracketings on n points. However, all actions are equivalent through conjugation by an associativity constraint.

In contrast, when $q \in \mathbb{C}$ or q is indeterminate, $\mathbf{R}$ is not an algebraic element, and it may not be well-defined when evaluated on representations. On the other hand, in analogy with the classical case of $\mathfrak{sl}_2$, the category $\mathsf{Mod}^{\mathrm{fd}}(U_q(\mathfrak{sl}_2))$ of finite-dimensional modules is semisimple, and irreducible representations are *highest weight*, i.e., they are generated by an eigenvector of H which is annihilated by E. These conditions guarantee that E and F act *locally nilpotently*, and therefore the action of $\mathbf{R}$ on $V \otimes W$ is well-defined for any finite-dimensional V.

Remark 8

1. It was noted in [38] and [39] that the R-matrix $\mathbf{R}$ is characterized by its action on finite-dimensional modules. Namely, $(1\,2) \circ \mathbf{R}_{VW}$ is the unique intertwiner on $V \otimes W$ such that

$$\mathbf{R}_{VW}(v \otimes w) = q^{\lambda \cdot \mu/2}(v \otimes w)\,,$$

   where $v \in V$ and $w \in W$ are the highest weight vector of weight λ and any weight vector of weight μ, respectively.

2. One readily demonstrates that $U_\hbar(\mathfrak{sl}_2)$ is also a sovereign Hopf algebra with $D = q^H$. In particular, it admits a balance $b = q^H u^{-1}$, where u is the Drinfeld element.

3.6 The Faddeev–Reshetikhin–Takhtajan formalism

One of the most interesting and fascinating phenomena in the theory of quantum groups is that they can be recovered solely from the action of the universal R-matrix on a representation. This approach is typically referred to as the Faddeev–Reshetikhin–Takhtajan (FRT) formalism (see [25]).

Let V be the two-dimensional representation of $U_\hbar(\mathfrak{g})$. Then, V has a basis $\{v_+, v_-\}$ with action of $U_\hbar(\mathfrak{g})$ given by

$$Hv_+ = +1\,, \qquad Ev_+ = 0\,, \qquad Fv_+ = v_-\,,$$
$$Hv_- = -1\,, \qquad Ev_- = v_+\,, \qquad Fv_- = 0\,.$$

The action of $\mathbf{R}$ on $V \otimes V$ reads

$$\mathbf{R} = \begin{bmatrix} q^{1/2} & & & \\ & q^{-1/2} & q^{-1/2}(q - q^{-1}) & \\ & & q^{-1/2} & \\ & & & q^{1/2} \end{bmatrix}$$

in the basis $\{v_+ \otimes v_+,\, v_+ \otimes v_-,\, v_- \otimes v_+,\, v_- \otimes v_-\}$.

Following [25], we introduce the FRT algebra $U_{\mathbf{R},V}$. Its generators are grouped into two matrices

$$\mathbf{L}^+ = \begin{bmatrix} \ell_{11} & \ell_{12} \\ & \ell_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{L}^- = \begin{bmatrix} \ell_{11}^{-1} & \\ \ell_{21} & \ell_{22}^{-1} \end{bmatrix},$$

which are to be thought of as elements in $\mathrm{End}(V) \otimes U_{\mathbf{R},V}$, and subject to the relations

$$\mathbf{R}_{VV}\mathbf{L}_1^{\pm}\mathbf{L}_2^{\pm} = \mathbf{L}_2^{\pm}\mathbf{L}_1^{\pm}\mathbf{R}_{VV} \qquad \text{and} \qquad \mathbf{R}_{12}\mathbf{L}_{13}^{+}\mathbf{L}_{23}^{-} = \mathbf{L}_{23}^{+}\mathbf{L}_{13}^{-}\mathbf{R}_{12}$$

in $\mathrm{End}(V) \otimes \mathrm{End}(V) \otimes U_{\mathbf{R},V}$. Then, $U_{\mathbf{R},V}$ has a natural Hopf algebra structure with coproduct given by

$$\mathrm{id} \otimes \Delta(\mathbf{L}^{\pm}) = \mathbf{L}_{13}^{\pm}\mathbf{L}_{12}^{\pm}$$

in $\mathrm{End}(V) \otimes U_{\mathbf{R},V} \otimes U_{\mathbf{R},V}$.

The relation between $U_{\mathbf{R},V}$ and $U_\hbar(\mathfrak{sl}_2)$ is readily obtained. On the one hand, we have that the universal R-matrix of $U_\hbar(\mathfrak{sl}_2)$ satisfies the Yang–Baxter equations

$$\mathbf{R}_{12}\mathbf{R}_{13}\mathbf{R}_{23} = \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12}\,, \qquad \mathbf{R}_{12}\mathbf{R}_{31}^{-1}\mathbf{R}_{32}^{-1} = \mathbf{R}_{32}^{-1}\mathbf{R}_{31}^{-1}\mathbf{R}_{12}\,,$$

and

$$\mathbf{R}_{12}\mathbf{R}_{13}\mathbf{R}_{32}^{-1} = \mathbf{R}_{32}^{-1}\mathbf{R}_{13}\mathbf{R}_{12}$$

in $U_\hbar(\mathfrak{sl}_2)^{\otimes 3}$. Moreover, by the coproduct identities, one has

$$\mathrm{id} \otimes \Delta(\mathbf{R}) = \mathbf{R}_{13}\mathbf{R}_{12}.$$

Hence, there is a natural Hopf algebra morphism $U_{\mathbf{R},V} \to U_\hbar(\mathfrak{sl}_2)$ given by the assignments

$$\mathbf{L}^+ \mapsto \mathbf{R}_{V,U_\hbar(\mathfrak{sl}_2)} \qquad \text{and} \qquad \mathbf{L}^- \mapsto (\mathbf{R}_{21})^{-1}_{V,U_\hbar(\mathfrak{sl}_2)}$$

in $\mathrm{End}(V) \otimes U_\hbar(\mathfrak{sl}_2)$.

On the other hand, let $U \subset U_\hbar(\mathfrak{sl}_2)$ be the Hopf subalgebra generated by E, F, and $K = q^H$. Then, there is a morphism of Hopf algebras $U \to U_{\mathbf{R},V}$ given by the assignments defined by

$$K \mapsto \ell_{11}\ell_{22}^{-1}, \qquad E \mapsto -q^{1/4}(q-q^{-1})^{-1}\ell_{11}\ell_{21},$$

and

$$F \mapsto q^{1/4}(q-q^{-1})^{-1}\ell_{12}\ell_{11}^{-1}\,.$$

Therefore, up to extending $U_{\mathbf{R},V}$ with the *logarithm* of $\ell_{11}\ell_{22}^{-1}$, we obtain an isomorphism of Hopf algebras between $U_\hbar(\mathfrak{sl}_2)$ and the FRT algebra $U_{\mathbf{R},V}$.

4 Quantum doubles and Drinfeld–Yetter modules

In this section, we review the quantum double construction [17] and its representation theory in terms of Drinfeld–Yetter modules contained in [21] and [61]. This framework underpins the construction of the R-matrix for $U_\hbar(\mathfrak{sl}_2)$ and, more generally, for the Drinfeld–Jimbo quantum groups associated to symmetrizable Kac–Moody algebras. The exposition of the material in this section is based on [6].

4.1 Lie bialgebras

A Lie bialgebra is a triple $(\mathfrak{b}, [\,,\,]_\mathfrak{b}, \delta_\mathfrak{b})$ where $(\mathfrak{b}, [\,,\,]_\mathfrak{b})$ is a Lie algebra, $(\mathfrak{b}, \delta_\mathfrak{b})$ a Lie coalgebra, and the cobracket $\delta_\mathfrak{b} : \mathfrak{b} \to \mathfrak{b} \otimes \mathfrak{b}$

satisfies the cocycle condition

$$\delta_{\mathfrak{b}}([X, Y]_{\mathfrak{b}}) = \mathrm{ad}(X)\,\delta_{\mathfrak{b}}(Y) - \mathrm{ad}(Y)\,\delta_{\mathfrak{b}}(X)\,.$$

4.2 The finite Drinfeld double

Following [17], the Drinfeld double of a finite-dimensional Lie bialgebra $(\mathfrak{b}, [\,,\,]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ is the Lie algebra $\mathfrak{g}_{\mathfrak{b}}$ defined as follows. As a vector space,

$$\mathfrak{g}_{\mathfrak{b}} = \mathfrak{b} \oplus \mathfrak{b}^*.$$

The duality pairing $\mathfrak{b}^* \otimes \mathfrak{b} \to \mathbb{K}$ extends uniquely to a symmetric, non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_{\mathfrak{b}}$, with respect to which both $\mathfrak{b}$ and $\mathfrak{b}^*$ are isotropic subspaces. The Lie bracket on $\mathfrak{g}_{\mathfrak{b}}$ is defined as the unique bracket which coincides with $[\,,\,]_{\mathfrak{b}}$ on $\mathfrak{b}$, with $\delta_{\mathfrak{b}}^t$ on $\mathfrak{b}^*$, and is compatible with $\langle \cdot, \cdot \rangle$, i.e., satisfies

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle,$$

for all $x, y, z \in \mathfrak{g}_{\mathfrak{b}}$. The mixed bracket of $x \in \mathfrak{b}$ and $\phi \in \mathfrak{b}^*$ is then given by

$$[x, \phi] = \mathrm{ad}^*(x)(\phi) + \phi \otimes \mathrm{id}_{\mathfrak{b}} \circ \delta(x)\,,$$

where ad^* is the coadjoint action of $\mathfrak{b}$ on $\mathfrak{b}^*$. Note that $(\mathfrak{g}_{\mathfrak{b}}, \mathfrak{b}, \mathfrak{b}^*)$ is a Manin triple [17], and any such triple arises this way.

4.3 Infinite-dimensional Lie bialgebras

Following [20], the Drinfeld double construction extends to the infinite-dimensional setting, where the dual space is replaced by the *continuous* dual. However, in presence of a finite grading, the construction is simpler. Namely, let $\mathfrak{b}$ be an infinite-dimensional Lie bialgebra which is $\mathbb{N}$-graded with finite-dimensional components, and such that the bracket and cobracket are homogeneous of degrees 0 and $d \in \mathbb{Z}$ respectively. In the sequel, we shall abusively refer to such a $\mathfrak{b}$ as

an $\mathbb{N}$-graded Lie bialgebra. The *restricted* double of $\mathfrak{b}$ is defined as

$$\mathfrak{g}_{\mathfrak{b}}^{\mathrm{res}} = \mathfrak{b} \oplus \mathfrak{b}^{\star}[d],$$

where $\mathfrak{b}^{\star}[d]_n = (\mathfrak{b}_{-n+d})^*$, and is a *restricted* Manin triple.

The restricted Drinfeld double $\mathfrak{g}_{\mathfrak{b}}^{\mathrm{res}}$ (and, in particular, the double of a finite-dimensional Lie bialgebra) is additionally endowed with a Lie bialgebra structure, with cobracket

$$\delta_{\mathfrak{g}_{\mathfrak{b}}^{\mathrm{res}}}(X) = [X \otimes 1 + 1 \otimes X, r]\,,$$

where r is the canonical element in $\mathfrak{b}\widehat{\otimes}\mathfrak{b}^{\star}$, with $\widehat{\otimes}$ the completion of the tensor product with respect to the grading, and

$$\delta_{\mathfrak{g}_{\mathfrak{b}}^{\mathrm{res}}} = \delta_{\mathfrak{b}} - \delta_{\mathfrak{b}^{\star}}.$$

Remark 9 The construction of the restricted Drinfeld double relies only on the mutual actions of $\mathfrak{b}$ on $\mathfrak{b}^{\star}$ and of $\mathfrak{b}^{\star}$ on $\mathfrak{b}$. Thus, it readily extends to any *matched pair* of Lie bialgebras, i.e., a pair of Lie bialgebras suitably acting on each other. The resulting quasitriangular Lie bialgebra is referred to as a *double crossed product* (see [48]).

4.4 Drinfeld–Yetter modules

According to [17] and [21], a Drinfeld–Yetter module over a Lie bialgebra $\mathfrak{b}$ is a triple (V, π_V, π_V^*), where (V, π_V) is a left $\mathfrak{b}$-module, (V, π_V^*) a right $\mathfrak{b}$-comodule, and the maps $\pi_V : \mathfrak{b} \otimes V \to V$ and $\pi_V^* : V \to \mathfrak{b} \otimes V$ satisfy the following compatibility condition in $\mathrm{End}(\mathfrak{b} \otimes V)$:

$$\begin{aligned}&\pi_V^* \circ \pi_V - \mathrm{id}_{\mathfrak{b}} \otimes \pi_V \circ (12) \circ \mathrm{id}_{\mathfrak{b}} \otimes \pi_V^* \\ &= [\cdot,\cdot]_{\mathfrak{b}} \otimes \mathrm{id}_V \circ \mathrm{id}_{\mathfrak{b}} \otimes \pi_V^* - \mathrm{id}_{\mathfrak{b}} \otimes \pi_V \circ \delta_{\mathfrak{b}} \otimes \mathrm{id}_V\,,\end{aligned}$$

The category $\mathsf{DY}_{\mathfrak{b}}$ of such modules is a symmetric tensor category. For any $V, W \in \mathsf{DY}_{\mathfrak{b}}$, the action and coaction on the tensor

product $V \otimes W$ are defined, respectively, by

$$\pi_{V\otimes W} = \pi_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \pi_W \circ (1\,2) \otimes \mathrm{id}_W\,,$$

$$\pi^*_{V\otimes W} = \pi^*_V \otimes \mathrm{id}_W + (1\,2) \otimes \mathrm{id}_W \circ \mathrm{id}_V \otimes \pi^*_W\,.$$

The associativity constraints are trivial, and the braiding is defined by $\beta_{VW} = (1\,2)$.

4.5 Representations of the Drinfeld double

The category $\mathsf{DY}_\mathfrak{b}$ is canonically isomorphic to the category $\mathcal{E}_{\mathfrak{g}_\mathfrak{b}}$ of *equicontinuous* $\mathfrak{g}_\mathfrak{b}$-modules [20], i.e., those endowed with a locally finite $\mathfrak{b}^*$-action. This condition yields a functor

$$\mathsf{E} : \mathcal{E}_{\mathfrak{g}_\mathfrak{b}} \to \mathsf{DY}_\mathfrak{b},$$

which assigns to any $V \in \mathcal{E}_{\mathfrak{g}_\mathfrak{b}}$ the Drinfeld–Yetter $\mathfrak{b}$-module

$$(V,\, \pi,\, \pi^*),$$

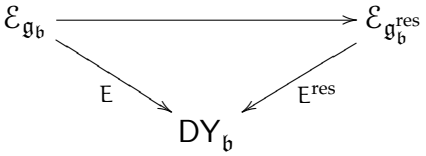
where π is the restriction of the action of $\mathfrak{g}_\mathfrak{b}$ to $\mathfrak{b}$, and the coaction π^* is given by

$$\pi^*(v) = \sum_i b_i \otimes b^i v \in \mathfrak{b} \otimes V\,,$$

where $\{b_i\}, \{b^i\}$ are dual bases of $\mathfrak{b}$ and $\mathfrak{b}^*$. The inverse functor is obtained by letting $\phi \in \mathfrak{b}^* \subset \mathfrak{g}_\mathfrak{b}$ act on $V \in \mathsf{DY}_\mathfrak{b}$ by $\phi \otimes \mathrm{id}_V \circ \pi^*$.

If $\mathfrak{b}$ is $\mathbb{N}$-graded with finite-dimensional homogeneous components, the formulae defining E similarly give rise to an isomorphism $\mathsf{E}\,\mathrm{res}$ between the category $\mathcal{E}_{\mathfrak{g}_\mathfrak{b}^{\mathrm{res}}}$ of equicontinuous modules over the restricted double of $\mathfrak{b}$ and $\mathsf{DY}_\mathfrak{b}$. Moreover, the categories $\mathcal{E}_{\mathfrak{g}_\mathfrak{b}}$ and $\mathcal{E}_{\mathfrak{g}_\mathfrak{b}^{\mathrm{res}}}$ are isomorphic, since any locally finite

action of $\mathfrak{b}^\star$ extends uniquely to one of $\mathfrak{b}^*$, and the diagram



is commutative.

4.6 Drinfeld–Yetter modules over a Hopf algebra

Following [61], a Drinfeld–Yetter module over a Hopf algebra $\mathfrak{B}$ is a triple $(\mathcal{V}, \pi_\mathcal{V}, \pi^*_\mathcal{V})$, where $(\mathcal{V}, \pi_\mathcal{V})$ is a left $\mathfrak{B}$-module, $(\mathcal{V}, \pi^*_\mathcal{V})$ a right $\mathfrak{B}$-comodule, and the maps

$$\pi_\mathcal{V} : \mathfrak{B} \otimes \mathcal{V} \to \mathcal{V} \qquad \text{and} \qquad \pi^*_\mathcal{V} : \mathcal{V} \to \mathfrak{B} \otimes \mathcal{V}$$

satisfy the following compatibility condition:

$$\pi^*_\mathcal{V} \circ \pi_\mathcal{V} = m^{(3)} \otimes \pi_\mathcal{V} \circ (1\,3)(2\,4) \circ S^{-1} \otimes \mathrm{id}^{\otimes 4} \circ \Delta^{(3)} \otimes \pi^*_\mathcal{V}\,,$$

where $m^{(3)} : \mathfrak{B}^{\otimes 3} \to \mathfrak{B}$ and $\Delta^{(3)} : \mathfrak{B} \to \mathfrak{B}^{\otimes 3}$ are the iterated multiplication and comultiplication respectively, and $S : \mathfrak{B} \to \mathfrak{B}$ is the antipode.

The category $\mathsf{DY}_\mathfrak{B}$ of such modules is a braided monoidal category. For any $\mathcal{V}, \mathcal{W} \in \mathsf{DY}_\mathfrak{B}$, the action and coaction on the tensor product $\mathcal{V} \otimes \mathcal{W}$ are defined by

$$\pi_{\mathcal{V}\otimes\mathcal{W}} = \pi_\mathcal{V} \otimes \pi_\mathcal{W} \circ (2\,3) \circ \Delta \otimes \mathrm{id}_{\mathcal{V}\otimes\mathcal{W}}\,,$$

$$\pi^*_{\mathcal{V}\otimes\mathcal{W}} = m^{21} \otimes \mathrm{id}_{\mathcal{V}\otimes\mathcal{W}} \circ (2\,3) \circ \pi^*_\mathcal{V} \otimes \pi^*_\mathcal{W}\,.$$

The associativity constraints are trivial, and the braiding is

$$\beta_{\mathcal{V}\mathcal{W}} = (1\,2) \circ \mathbf{R}_{\mathcal{V}\mathcal{W}},$$

where the R-matrix $\mathbf{R}_{VW} \in \mathrm{End}(\mathcal{V} \otimes \mathcal{W})$ is defined by

$$\mathbf{R}_{VW} = \pi_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{W}} \circ (1\,2) \circ \mathrm{id}_{\mathcal{V}} \otimes \pi^*_{\mathcal{W}} .$$

The linear map R_{VW} is invertible, with inverse

$$\mathbf{R}^{-1}_{VW} = \pi_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{W}} \circ S \otimes \mathrm{id}_{\mathcal{V}\otimes\mathcal{W}} \circ (1\,2) \circ \mathrm{id}_{\mathcal{V}} \otimes \pi^*_{\mathcal{W}} .$$

4.7 The finite quantum double

Following [17] and [48], let $\mathfrak{B}$ be a finite-dimensional Hopf algebra, and $\mathfrak{B}^\circ$ the dual Hopf algebra $\mathfrak{B}^*$ with opposite coproduct. The quantum double (or double cross product) of $\mathfrak{B}$ is the unique quasitriangular Hopf algebra $(D\mathfrak{B}, \mathbf{R})$ such that:

- $D\mathfrak{B} = \mathfrak{B} \otimes \mathfrak{B}^\circ$ as vector spaces;
- $\mathfrak{B}$ and $\mathfrak{B}^\circ$ are Hopf subalgebras of $D\mathfrak{B}$;
- $\mathbf{R}$ is the canonical element in $\mathfrak{B} \otimes \mathfrak{B}^\circ \subset D\mathfrak{B} \otimes D\mathfrak{B}$.

Proceeding as in Section 4.5, the category $\mathsf{Mod}(D\mathfrak{B})$ is readily seen to be canonically isomorphic to $DY_{\mathfrak{B}}$ as a braided monoidal category (see, e.g., [5, Appendix A]).

Remark 10 As in the case of Lie bialgebras, the construction of the quantum double extends to any *matched pair* of Hopf algebras $(\mathfrak{A}, \mathfrak{B})$, yielding a quasitriangular Hopf algebra $\mathfrak{A} \bowtie \mathfrak{B}$ referred to as a *double crossed product* (see [48]).

Interestingly, this can be defined for any matched pairs of bialgebras. In this case, $\mathfrak{A} \bowtie \mathfrak{B}$ is generated by $\mathfrak{A}$ and $\mathfrak{B}$, and it is not assumed to be isomorphic to $\mathfrak{A} \otimes \mathfrak{B}$ as vector spaces. The two constructions coincide whenever $\mathfrak{A}$ and $\mathfrak{B}$ are Hopf algebras.

4.8 Quantum double for QUEs

The construction of the quantum double can be adapted to quantized universal enveloping algebras (QUE). Recall that a QUE is a topological Hopf algebra $\mathfrak{B}$ over $\mathbb{K}[\![\hbar]\!]$ which reduces modulo $\hbar$ to an enveloping algebra $U\mathfrak{b}$ for some Lie bialgebra $\mathfrak{b}$, and is such that, for any $x \in \mathfrak{b}$,

$$\delta(x) = \frac{\Delta(\widetilde{x}) - \Delta^{21}(\widetilde{x})}{\hbar} \mod \hbar,$$

where $\widetilde{x} \in \mathfrak{B}$ is any lift of x. A QUE is of finite type if the underlying Lie bialgebr $\mathfrak{b}$ is finite-dimensional. In this case, the dual $\mathfrak{B}^*$ is a quantized formal series Hopf algebra (QFSH), i.e., a topological Hopf algebra over $\mathbb{K}[\![\hbar]\!]$ which reduces modulo $\hbar$ to $\widehat{S\mathfrak{b}} = \prod_n S^n\mathfrak{b}$. Conversely, the dual of a QFSH of finite type is a QUE (see [17] and [30]).

If $\mathfrak{B}$ is a QUE, set

$$\mathfrak{B}' = \left\{ b \in \mathfrak{B} \mid (\mathrm{id} - \iota \circ \varepsilon)^{\otimes n} \circ \Delta^{(n)}(b) \in \hbar^n \mathfrak{B}^{\otimes n} \text{ for any } n \geqslant 0 \right\},$$

where $\Delta^{(n)} : \mathfrak{B} \to \mathfrak{B}^{\otimes n}$ is the iterated coproduct. Then, $\mathfrak{B}'$ is a Hopf subalgebra of $\mathfrak{B}$, and a QFSH.

In particular, if $\mathfrak{B}$ is of finite type, $\mathfrak{B}^\vee = (\mathfrak{B}')^*$ is a QUE. As in Section 4.7, $(\mathfrak{B}, \mathfrak{B}^\vee)$ is a matched pair of Hopf algebras (see [5, Appendix A.5] or [48]). The double cross product

$$D\mathfrak{B} = \mathfrak{B} \bowtie \mathfrak{B}^\vee$$

is a quasitriangular QUE, whose R-matrix is the canonical element $R \in \mathfrak{B}' \otimes \mathfrak{B}^\vee$. The underlying Lie bialgebra the Drinfeld double $\mathfrak{g}_\mathfrak{b} = \mathfrak{b} \oplus \mathfrak{b}^*$.

This construction extends to the case of *finitely* $\mathbb{N}$*-graded* QUEs, i.e., $\mathbb{N}$-graded Hopf algebras

$$\mathfrak{B} = \bigoplus_{n \geqslant 0} \mathfrak{B}_n$$

such that $\mathfrak{B}_0$ is a QUE of finite type, and each $\mathfrak{B}_n$ is a finitely generated $\mathfrak{B}_0$-module. Note that such a QUE is a quantization of an $\mathbb{N}$-graded Lie bialgebra with finite-dimensional components and cobracket of degree $d = 0$, see Section 4.2. Moreover,

$$\mathfrak{B}' = \bigoplus_{n \geqslant 0} (\mathfrak{B}' \cap \mathfrak{B}_n)$$

is also graded, and its *restricted dual*

$$\mathfrak{B}^\star = \bigoplus_{n \geqslant 0} (\mathfrak{B}' \cap \mathfrak{B}_n)^*$$

is a finitely $\mathbb{N}$-graded QUE, which quantizes the restricted dual Lie bialgebra $\mathfrak{b}^\star$.

The double cross product $(D\mathfrak{B})^{res} = \mathfrak{B} \bowtie \mathfrak{B}^\star$ is called the *restricted* quantum double of $\mathfrak{B}$. Besides, $(D\mathfrak{B})^{res}$ is a quasitriangular, finitely $\mathbb{Z}$-graded QUE, whose R-matrix is the canonical element in the graded completion of $\mathfrak{B}' \otimes \mathfrak{B}^\star$. The underlying Lie bialgebra is the restricted Drinfeld double $\mathfrak{g}_{\mathfrak{b}}^{res} = \mathfrak{b} \oplus \mathfrak{b}^\star$.

Example 11 The Hopf algebra $U_\hbar(\mathfrak{sl}_2)$ is easily seen to be isomorphic to a quotient of the restricted quantum double of the Hopf subalgebra $U_\hbar^{\leqslant 0}(\mathfrak{sl}_2)$ generated by H and F. Moreover, the R-matrix $\mathbf{R}$ appears as the image of the canonical R-matrix of a quantum double under this isomorphism.

4.9 Admissible Drinfeld–Yetter modules over a QUE

If $\mathfrak{B}$ is a QUE, the categories of Drinfeld–Yetter $\mathfrak{B}$-modules and modules over $(D\mathfrak{B})^{res}$ are not equivalent, even when $\mathfrak{B}$ is of finite type. This motivates the following definition, due to P. Etingof (see [5]).

A Drinfeld–Yetter module $(\mathcal{V}, \pi_\mathcal{V}, \pi_\mathcal{V}^*)$ over $\mathfrak{B}$ is *admissible* if the coaction $\pi_\mathcal{V}^* : \mathcal{V} \to \mathfrak{B} \otimes \mathcal{V}$ factors through $\mathfrak{B}' \otimes \mathcal{V}$, where $\otimes$

is the $\hbar$-adic tensor product, and $\mathfrak{B}'$ is endowed with topology induced by the $\hbar$-adic topology on $\mathfrak{B}$, so that

$$\mathfrak{B}' \otimes \mathcal{V} \subset \mathfrak{B} \otimes \mathcal{V}.$$

Note that the induced topology on $\mathfrak{B}'$ coincides with the QFSH topology. We denote the category of such modules by $\mathsf{DY}^{\mathrm{adm}}_{\mathfrak{B}}$. If $\mathcal{B}$ is a quantization of $\mathfrak{b}$, the category $\mathsf{DY}^{\mathrm{adm}}_{\mathfrak{B}}$ reduces modulo $\hbar$ to $\mathsf{DY}_{\mathfrak{b}}$, i.e., every admissible Drinfeld–Yetter $\mathfrak{B}$-module is a quantization of a Drinfeld–Yetter $\mathfrak{b}$-module.

When $\mathfrak{B}$ is a finitely $\mathbb{N}$-graded QUE, there is a canonical isomorphism between $\mathsf{DY}^{\mathrm{adm}}_{\mathfrak{B}}$ and the category of $(\mathrm{D}\mathfrak{B})^{\mathrm{res}}$-modules with a locally finite action of $\mathfrak{B}^\star$ (in the $\hbar$-adic topology, see [7]). Note that, if $\mathcal{V}$ is a $(\mathrm{D}\mathfrak{B})^{\mathrm{res}}$-module, then the $\mathfrak{B}$-coaction naturally induced by the universal R-matrix of $(\mathrm{D}\mathfrak{B})^{\mathrm{res}}$ is admissible, since $\mathbf{R} \in \mathfrak{B}' \otimes \mathfrak{B}^\star$.

Example 12 We have that the category of finite-dimensional modules over $\mathrm{U}_\hbar(\mathfrak{sl}_2)$, with the braiding induced by the universal R-matrix $\mathbf{R}$, naturally identifies with a full tensor subcategory of $\mathsf{DY}^{\mathrm{adm}}_{\mathrm{U}^{\leqslant 0}_\hbar(\mathfrak{sl}_2)}$.

5 Drinfeld–Jimbo quantum groups

In this section, we consider the Drinfeld–Jimbo quantum groups associated to arbitrary symmetrizable complex Kac–Moody algebras. We review their (integrable) highest weight theory in relation with the category of Drinfeld–Yetter modules over the quantum negative Borel subalgebra.

5.1 Symmetrizable Kac–Moody algebras

Let $\mathbf{I}$ be a finite set and $\mathsf{A} = (a_{ij})_{i,j\in\mathbf{I}}$ a generalized Cartan matrix, i.e., $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\leqslant 0}$, $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A, i.e.,

- $\mathfrak{h}$ is a finite-dimensional complex vector space (note that, unlike [35], we do not require $\mathfrak{h}$ to have minimal dimension $2|\mathbf{I}| - \operatorname{rank}(A)$);
- $\Pi = \{\alpha_i\}_{i\in\mathbf{I}}$ is a linearly independent subset of $\mathfrak{h}^*$;
- $\Pi^\vee = \{h_i\}_{i\in\mathbf{I}}$ is a linearly independent subset of $\mathfrak{h}$;
- $\alpha_i(h_j) = a_{ji}$, for any $i, j \in \mathbf{I}$.

According to [35], the Kac–Moody algebra corresponding to A and the realisation $(\mathfrak{h}, \Pi, \Pi^\vee)$ is the complex Lie algebra $\mathfrak{g}$ generated by $\mathfrak{h}$ and elements $\{e_i, f_i\}_{i\in\mathbf{I}}$, with relations $[\mathfrak{h}, \mathfrak{h}] = 0$ and

$$[h, e_i] = \alpha_i(h)e_i\,, \qquad [h, f_i] = -\alpha_i(h)f_i\,, \qquad [e_i, f_j] = \delta_{ij}h_i\,,$$

and

$$\mathsf{ad}(e_i)^{1-a_{ij}}(e_j) = 0 = \mathsf{ad}(f_i)^{1-a_{ij}}(f_j),$$

for any $i \neq j$. It is finitely $\mathbb{Z}$-graded with

$$\deg(\mathfrak{h}) = 0 \qquad \text{and} \qquad \deg(e_i) = 1 = -\deg(f_i).$$

Assume that A is symmetrizable, and fix an invertible diagonal matrix $D = \operatorname{diag}(d_i)_{i\in\mathbf{I}}$ with coprime entries $d_i \in \mathbb{Z}_{>0}$ such that DA is symmetric. Then, there is a symmetric, non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$ such that $\langle h_i, -\rangle = d_i^{-1}\alpha_i$. The corresponding identification $\nu : \mathfrak{h} \to \mathfrak{h}^*$ intertwines the actions of W, satisfies $\nu(h_i) = d_i^{-1}\alpha_i$ and therefore restricts to an isomorphism $\mathfrak{h}' \simeq Q \otimes_{\mathbb{Z}} \mathbb{C}$, where $\mathfrak{h}'$ is the span of $\{h_i\}_{i\in\mathbf{I}}$ and $Q = \bigoplus_{i\in\mathbf{I}} \mathbb{Z}\alpha_i \subseteq \mathfrak{h}^*$ is the root lattice. Note that

$$\langle h_i, h_i\rangle = 2d_i^{-1},$$

while the induced form on $\mathfrak{h}^*$ satisfies $\langle\alpha_i, \alpha_i\rangle = 2d_i \in 2\mathbb{Z}_{>0}$.

By [35, Theorem 2.2], $\langle\cdot,\cdot\rangle$ uniquely extends to a non-degenerate, invariant symmetric bilinear form on $\mathfrak{g}$, which satisfies

$$\langle e_i, f_j\rangle = \delta_{ij}d_i^{-1} \quad \text{and} \quad [x, y] = \langle x, y\rangle \cdot t_\alpha,$$

for any $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, where $t_\alpha = \nu^{-1}(\alpha)$.

Remark 13 The Kac–Moody algebras are a natural generalization of finite-dimensional simple complex Lie algebras. For instance, $\mathfrak{sl}_2$ is the Kac–Moody algebra that corresponds to the 1×1 Cartan matrix (see [2]). This framework includes remarkable infinite-dimensional examples, such as affine Lie algebras (e.g., the central extension of the loop algebra $\mathfrak{sl}_2[t, t^{-1}]$).

5.2 The Lie bialgebra structure

Let $\mathfrak{n}^\pm \subset \mathfrak{g}$ be the Lie subalgebras generated by $\{e_i\}_{i\in\mathbf{I}}$ and $\{f_i\}_{i\in\mathbf{I}}$, respectively, and set $\mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm$. The bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ induces identifications $(\mathfrak{b}^\mp)^\star \simeq \mathfrak{b}^\pm$. This give rise to a Lie bialgebra structure on $\mathfrak{b}^\pm$ and therefore on $\mathfrak{g}$, which is compatible with the grading. Specifically, consider the Lie algebra $\mathfrak{g}^{(2)} = \mathfrak{g} \oplus \mathfrak{h}_\mathfrak{z}$, with $\mathfrak{h}_\mathfrak{z} = \mathfrak{h}$, and endow it with the inner product

$$\langle\cdot,\cdot\rangle^{(2)} = \langle\cdot,\cdot\rangle \oplus -\langle\cdot,\cdot\rangle|_{\mathfrak{h}_\mathfrak{z}\times\mathfrak{h}_\mathfrak{z}} \ .$$

Let $\pi_0 : \mathfrak{g} \to \mathfrak{h}$ be the projection arising from the root space decomposition, and $\mathfrak{b}^{(2)}_\pm \subset \mathfrak{g}^{(2)}$ the subalgebra

$$\mathfrak{b}^{(2)}_\pm = \{(X, h) \in \mathfrak{b}^\pm \oplus \mathfrak{h}_\mathfrak{z} \mid \pi_0(X) = \pm h\}.$$

Note that the projection $\mathfrak{g}^{(2)} \to \mathfrak{g}$ onto the first component restricts to an isomorphism $\mathfrak{b}^{(2)}_\pm \to \mathfrak{b}^\pm$ with inverse

$$\mathfrak{b}^\pm \ni X \to (X, \pm\pi_0(X)) \in \mathfrak{b}^{(2)}_\pm.$$

Then, $(\mathfrak{g}^{(2)}, \mathfrak{b}^{(2)}_-, \mathfrak{b}^{(2)}_+)$ is a restricted Manin triple. In particular, $\mathfrak{b}^{(2)}_\mp$ and $\mathfrak{g}^{(2)}$ are Lie bialgebras, with cobracket

$$\delta_{\mathfrak{b}^{(2)}_\mp} = [\cdot,\cdot]^t_{\mathfrak{b}^{(2)}_\pm} \qquad \text{and} \qquad \delta_{\mathfrak{g}^{(2)}} = \delta_{\mathfrak{b}^{(2)}_-} - \delta_{\mathfrak{b}^{(2)}_+}.$$

The central subalgebra $0 \oplus \mathfrak{h}_\mathfrak{z} \subset \mathfrak{g}^{(2)}$ is a coideal, so that the projection $\mathfrak{g}^{(2)} \to \mathfrak{g}$ induces a Lie bialgebra structure on $\mathfrak{g}$ and $\mathfrak{b}_\mp$

given by

$$\delta_{|_{\mathfrak{h}}} = 0\,, \qquad \delta(e_i) = d_i h_i \wedge e_i\,, \qquad \delta(f_i) = d_i h_i \wedge f_i\,.$$

The projection $\mathfrak{g}^{(2)} \to \mathfrak{g}$ maps the canonical elements

$$r_{\mathfrak{g}^{(2)}} \in \mathfrak{b}_-^{(2)} \widehat{\otimes} \mathfrak{b}_+^{(2)} \qquad \text{and} \qquad \Omega_{\mathfrak{g}^{(2)}} = r_{\mathfrak{g}^{(2)}} + r^{21}_{\mathfrak{g}^{(2)}} \in \mathfrak{g}^{(2)} \widehat{\otimes} \mathfrak{g}^{(2)}$$

corresponding to the inner product $\langle \cdot, \cdot \rangle^{(2)}$ to

$$r_{\mathfrak{g}} = \sum_i x_i \otimes x^i + \frac{1}{2} \sum_a t_a \otimes t^a\,,$$

and the canonical element $\Omega_{\mathfrak{g}} \in \mathfrak{g} \widehat{\otimes} \mathfrak{g}$ corresponding to $\langle \cdot, \cdot \rangle$, where $\{x_i\}$, $\{x^i\}$ are homogeneous dual bases of $\mathfrak{n}_-$ and $\mathfrak{n}_+$, respectively, and $\{t_a\}$, $\{t^a\}$ are dual bases of $\mathfrak{h}$. The $\frac{1}{2}$ factor in $r_{\mathfrak{g}}$ arises because if $t, t' \in \mathfrak{h}$ and

$$t_+ = (t, t) \quad \text{and} \quad t'_- = (t', -t')$$

are the corresponding elements of $\mathfrak{b}_\pm^{(2)}$, then $\langle t_+, t'_- \rangle^{(2)} = 2\langle t, t' \rangle$.

5.3 Category $\mathcal{O}_\infty$ representations

If V is an $\mathfrak{h}$-module and $\lambda \in \mathfrak{h}^*$, we denote the corresponding weight space of V by

$$V[\lambda] = \{v \in V \mid h\, v = \lambda(h) v,\ h \in \mathfrak{h}\}$$

and set $P(V) = \{\lambda \in \mathfrak{h}^* \mid V[\lambda] \neq 0\}$. A $\mathfrak{g}$-module V is

(C1) a *weight module* if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$;

(C2) *integrable* if it is a weight module, and the elements $\{e_i, f_i\}_{i \in \mathbf{I}}$ act locally nilpotently — this implies that $\lambda(h_i) \in \mathbb{Z}$, for any $\lambda \in P(V)$ and $i \in \mathbf{I}$, and also that V is completely re-

ducible as a (possibly infinite) direct sum of simple finite-dimensional modules over $\mathfrak{sl}_2^{\alpha_i} = \langle e_i, h_i, f_i \rangle \subset \mathfrak{g}$;

(C3) in the *category* $\mathcal{O}_{\infty,\mathfrak{g}}$ if the action of $\mathfrak{b}^+$ is locally finite, i.e., any $v \in V$ is contained in a finite-dimensional $\mathfrak{b}^+$-submodule of V (this implies in particular that V is the direct sum of its generalized weight spaces and that, for any $v \in V$, and $(U\mathfrak{n}^+)_\beta v = 0$ for all but finitely many $\beta \in Q_+$).

(C4) in the *category* $\mathcal{O}_\mathfrak{g}$ if it is a weight module with finite-dimensional weight spaces, such that

$$P(V) \subseteq D(\lambda_1) \cup \ldots \cup D(\lambda_m) \tag{5.1}$$

for some $\lambda_1, \ldots, \lambda_m \in \mathfrak{h}^*$, where $D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leqslant \lambda\}$ and $\mu \leqslant \lambda$ if and only if $\lambda - \mu \in Q_+ = \bigoplus_{i \in \mathbf{I}} \mathbb{N}\alpha_i$.

The categories $\mathcal{O}_\mathfrak{g} \subset \mathcal{O}_{\infty,\mathfrak{g}}$ are symmetric tensor categories. Let $\mathcal{O}_\mathfrak{g}^{\mathrm{int}} \subset \mathcal{O}_\mathfrak{g}$ and $\mathcal{O}_{\infty,\mathfrak{g}}^{\mathrm{int}} \subset \mathcal{O}_{\infty,\mathfrak{g}}$ be the full tensor subcategories of integrable representations. By [46, Theorem 6.2.2], both categories are semisimple with irreducible modules indexed by dominant weights, i.e., $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) \in \mathbb{Z}_{\geqslant 0}$, for all $i \in \mathbf{I}$.

Remark 14 When $\dim(\mathfrak{g}) < \infty$, $\mathcal{O}_\mathfrak{g}^{\mathrm{int}}$ is the category of finite-dimensional $\mathfrak{g}$-modules. Instead, when $\dim(\mathfrak{g}) = \infty$, $\mathcal{O}_\mathfrak{g}^{\mathrm{int}}$ contains no non-trivial finite-dimensional module.

5.4 From category $\mathcal{O}$ to Drinfeld–Yetter modules

The category $\mathcal{O}_{\infty,\mathfrak{g}}$ is isomorphic to the full tensor subcategory of $\mathcal{E}_{\mathfrak{g}^{(2)}}$ consisting of those modules carrying a trivial action of $\mathfrak{h}_3$. By Sections 5.2 and 4.5, the category $\mathsf{DY}_{\mathfrak{b}^-}$ of Drinfeld–Yetter $\mathfrak{b}^-$-modules is equivalent to the category $\mathcal{E}_{\mathfrak{g}^{(2)}}$ of $\mathfrak{g}^{(2)}$-modules which carry a locally finite action of $\mathfrak{b}_+^{(2)} \subset \mathfrak{g}^{(2)}$. Under the equivalence $\mathcal{E}_{\mathfrak{g}^{(2)}} \simeq \mathsf{DY}_{\mathfrak{b}^-}$, $\mathcal{O}_{\infty,\mathfrak{g}}$ is isomorphic to the full

tensor subcategory of $\mathsf{DY}_{\mathfrak{b}^-}$ consisting of those modules V such that

$$\rho_V = \langle \cdot, \cdot \rangle_{\mathfrak{h}} \otimes \mathrm{id}_V \circ \mathrm{id}_{\mathfrak{h}} \otimes \rho_V^* \,,$$

where ρ_V and ρ_V^* denote, respectively, the action and the coaction of $\mathfrak{h}$ on V.

5.5 Deformation category $\mathcal{O}_\infty$ representations

Analogous notions can be defined for $\mathfrak{g}$-modules in the category $\mathrm{Vect}_\hbar$ of topologically free $\mathbb{C}[\![\hbar]\!]$-modules. Specifically, a $\mathfrak{g}$-module $\mathcal{V} \in \mathrm{Vect}_\hbar$ is called

(D1) a *weight module* if $\mathcal{V} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{V}[\lambda]$, where $\bigoplus$ is the direct sum in $\mathrm{Vect}_\hbar$, i.e., the completion of the algebraic direct sum in the $\hbar$-adic topology (note that the eigenvalues of the action of $\mathfrak{h}$ on $\mathcal{V}$ are required to lie in $\mathfrak{h}^* \subsetneq \mathfrak{h}^*[\![\hbar]\!]$);

(D2) *integrable* if it is a weight module and, for any $i \in \mathbf{I}, v \in \mathcal{V}$,

$$\lim_{n\to\infty} e_i^n v = 0 = \lim_{n\to\infty} f_i^n v \,;$$

(D3) in the *category* $\mathcal{O}^\hbar_{\infty,\mathfrak{g}}$ if the action of $\mathfrak{b}^+$ on $\mathcal{V}/\hbar^n\mathcal{V}$ is locally finite, for any $n \geqslant 0$;

(D4) in the *category* $\mathcal{O}^\hbar_{\mathfrak{g}}$ if it is a weight representation with finite-rank weight spaces, and such that $P(\mathcal{V})$ satisfies (5.1).

It is easy to see that $\mathcal{V}$ is a weight (resp. integrable) module in $\mathrm{Vect}_\hbar$ if and only if $\mathcal{V}/\hbar^n\mathcal{V}$ is a weight (resp. integrable) module in Vect, for any $n \geqslant 0$. We denote by $\mathcal{O}^{\hbar,\mathrm{int}}_{\mathfrak{g}} \subset \mathcal{O}^\hbar_{\mathfrak{g}}$ and $\mathcal{O}^{\hbar,\mathrm{int}}_{\infty,\mathfrak{g}} \subset \mathcal{O}^\hbar_{\infty,\mathfrak{g}}$ the full tensor subcategories of integrable representations.

5.6 Braid group action

Integrable $\mathfrak{g}$-modules are equipped with an extra symmetry. Let W be the Weyl group of $\mathfrak{g}$, and $\{s_i\}_{i\in\mathbf{I}}$ its set of simple reflections. The braid group $\mathcal{B}_W$ is the group generated by the

elements $\{S_i\}_{i\in\mathbf{I}}$, with relations

$$\underbrace{S_i \cdot S_j \cdot S_i \ \dots}_{m_{ij}} = \underbrace{S_j \cdot S_i \cdot S_j \ \dots}_{m_{ij}}, \tag{5.2}$$

for any $i \neq j$, where m_{ij} is the order of $s_i s_j$ in W. If V is an integrable $\mathfrak{g}$-module in Vect or $\mathrm{Vect}_\hbar$, the operators

$$\widetilde{s}_i = \exp(e_i) \cdot \exp(-f_i) \cdot \exp(e_i) \in \mathrm{GL}(V)$$

are well-defined, and satisfy the braid relations (5.2) (see [59]). The corresponding action of $\mathcal{B}_W$ on V factors through the *Tits extension* $\widetilde{W}$, an extension of W by the sign group $\mathbb{Z}_2^{\mathbf{I}}$.

5.7 The Drinfeld–Jimbo quantum group

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra. Set

$$q = \exp(\hbar/2) \qquad \text{and} \qquad q_i = q^{d_i},$$

with $i \in \mathbf{I}$. Following [17] and [34], the Drinfeld–Jimbo quantum group of $\mathfrak{g}$ is the algebra $U_\hbar(\mathfrak{g})$ topologically generated over $\mathbb{C}[\![\hbar]\!]$ by $\mathfrak{h}$ and the elements $\{E_i, F_i\}_{i\in\mathbf{I}}$, subject to the relations $[h, h'] = 0$,

$$[h, E_i] = \alpha_i(h)E_i \quad [h, F_i] = -\alpha_i(h)F_i \quad [E_i, F_j] = \delta_{ij}\frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}},$$

for any $h, h' \in \mathfrak{h}$, $i, j \in \mathbf{I}$, and the q-Serre relations

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_i X_i^{1-a_{ij}-m} X_j X_i^m = 0$$

for $X = E, F$, $i \neq j \in \mathbf{I}$, where $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and, for any $k \leqslant n$,

$$[n]_i! = [n]_i \cdot [n-1]_i \dots [1]_i \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i! \cdot [n-k]_i}$$

We consider the Hopf algebra structure on $U_\hbar(\mathfrak{g})$ with coproduct $\Delta(h) = h \otimes 1 + 1 \otimes h$, and

$$\Delta(E_i) = E_i \otimes q_i^{h_i} + 1 \otimes E_i \qquad \Delta(F_i) = F_i \otimes 1 + q_i^{-h_i} \otimes F_i\,.$$

The counit and the antipode are given by

$$\varepsilon(h) = \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

and

$$S(h) = -h, \quad S(E_i) = -E_i q_i^{-h_i} \quad \text{and} \quad S(F_i) = -q_i^{h_i} F_i,$$

for any $h \in \mathfrak{h}$ and $i \in \mathbf{I}$.

We define weight, integrable, category $\mathcal{O}_\infty$ and $\mathcal{O}$ modules for $U_\hbar(\mathfrak{g})$ in $\mathrm{Vect}_\hbar$ analogously to Section 5.5. In particular, a representation V of $U_\hbar(\mathfrak{g})$ is in category $\mathcal{O}_\infty$ if the action of $U_\hbar(\mathfrak{b}^+)$ on $V/\hbar^n V$ is locally finite, for any $n \geqslant 0$. We denote by

$$\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar(\mathfrak{g})} \subset \mathcal{O}_{\infty,U_\hbar(\mathfrak{g})} \qquad \text{and} \qquad \mathcal{O}^{\mathrm{int}}_{U_\hbar(\mathfrak{g})} \subset \mathcal{O}_{U_\hbar(\mathfrak{g})}$$

the subcategories of integrable modules. As in the classical case, $\mathcal{O}^{\mathrm{int}}_{U_\hbar(\mathfrak{g})}$ and $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar(\mathfrak{g})}$ are both *semisimple* in that they are additively generated by indecomposable modules. Besides, whenever $\dim(\mathfrak{g}) < \infty$, the category $\mathcal{O}^{\mathrm{int}}_{U_\hbar(\mathfrak{g})}$ consists of finite-rank topologically free $U_\hbar(\mathfrak{g})$-modules.

5.8 The universal R-matrix

The Hopf algebra $U_\hbar(\mathfrak{g})$ is quasitriangular. Namely, let $U_\hbar(\mathfrak{n}^+)$ (resp. $U_\hbar(\mathfrak{n}^-)$) be the subalgebra generated by E_i, with $i \in \mathbf{I}$ (resp. $F_i, i \in \mathbf{I}$), and set $U_\hbar(\mathfrak{b}^\pm) = U_\hbar(\mathfrak{n}^\pm)U\mathfrak{h}[\![\hbar]\!]$. There is a unique non-degenerate Hopf pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{D}} : U_\hbar(\mathfrak{b}^-) \otimes U_\hbar(\mathfrak{b}^+) \to \mathbb{C}(\!(\hbar)\!)$$

such that $\langle 1, 1\rangle_{\mathcal{D}} = 1$,

$$\langle h, h'\rangle_{\mathcal{D}} = \frac{1}{\hbar}\langle h, h'\rangle \qquad \langle F_i, E_j\rangle_{\mathcal{D}} = \frac{\delta_{ij}}{q - q^{-1}}$$

and zero otherwise. It follows that $U_\hbar(\mathfrak{b}^+)$ identifies with the restricted dual of $U_\hbar(\mathfrak{b}^-)$, and $(U_\hbar(\mathfrak{b}^-), U_\hbar(\mathfrak{b}^+))$ is a matched pair of Hopf algebras with respect to the mutual adjoint actions. Then, $U_\hbar(\mathfrak{g})$ is readily isomorphic to a quotient of the restricted quantum double of $U_\hbar(\mathfrak{b}^-)$, and therefore it is equipped with a universal R-matrix, which is computed as follows (see [17] and [46]). Note that, as in the classical case, the quotient identifies the two copies of the Cartan sitting in $U_\hbar(\mathfrak{b}^\pm)$.

Let $\{x_a\}, \{x^a\} \subset \mathfrak{h}$ be dual bases. Note that the pairing respects the weight decomposition in $U_\hbar(\mathfrak{b}^\pm)$. For any $\mu \in Q_+$, let $\{X^\pm_{\mu,p}\}_p \subset U_\hbar(\mathfrak{n}^\pm)[\pm\mu]$ be dual bases with respect to $\langle\cdot,\cdot\rangle_{\mathcal{D}}$ and set $\Theta_\mu = \sum_p X^-_{\mu,p} \otimes X^+_{\mu,p}$. Then, $U_\hbar(\mathfrak{g})$ is a quasitriangular Hopf algebra with R-matrix

$$\mathbf{R} = q^{\Omega^0} \cdot \Theta \in U_\hbar(\mathfrak{b}^-)\widehat{\otimes}U_\hbar(\mathfrak{b}^+)\,,$$

where $\Omega^0 = \sum_a x_a \otimes x^a$ and $\Theta = \sum_{\mu>0}\Theta_\mu$. In particular, $\mathbf{R}$ satisfies the intertwining property $\Delta^{\mathrm{op}}(x) = \mathbf{R}\cdot\Delta(x)\cdot\mathbf{R}^{-1}$, as well as the cabling identities

$$\Delta\otimes 1(\mathbf{R}) = \mathbf{R}_{13}\cdot\mathbf{R}_{23} \qquad \text{and} \qquad 1\otimes\Delta(\mathbf{R}) = \mathbf{R}_{13}\cdot\mathbf{R}_{12}\,.$$

Moreover, the Yang–Baxter equation

$$\mathbf{R}_{12}\cdot\mathbf{R}_{13}\cdot\mathbf{R}_{23} = \mathbf{R}_{23}\cdot\mathbf{R}_{13}\cdot\mathbf{R}_{12}$$

follows. Thanks to the condition (D4), we have that the action of the R-matrix on a tensor product of representations in $\mathcal{O}_{\infty,U_\hbar(\mathfrak{g})}$ is well-defined and induces a braiding. By Section 4.5, $\mathcal{O}_{\infty,U_\hbar(\mathfrak{g})}$ identifies with a full subcategory of admissible Drinfeld–Yetter modules over $U_\hbar(\mathfrak{b}^-)$.

Remark 15 A *continuum* generalization of $U_\hbar(\mathfrak{g})$ has been introduced in [4]. Note that they arise as *topological* quantum doubles, and are therefore naturally equipped with a topological universal R-matrix. It is still unclear whether they possess an interesting representation theory.

5.9 Quantum Weyl group operators

Let $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar(\mathfrak{g})}$ be the category of integrable $U_\hbar(\mathfrak{g})$-modules in category $\mathcal{O}_\infty$, i.e., the action of the elements E_i, F_i, $i \in \mathbf{I}$, is locally nilpotent mod $\hbar^n$, for any $n \geqslant 0$. On these modules, the action of the R-matrix can be computed in terms of the quantum Weyl group operators, see [42], [46], and [57].

The operators $\mathbf{S}_i$ ($i \in \mathbf{I}$) naturally act on integrable modules, and consequently they can be thought of as elements of a *completion* of $U_\hbar(\mathfrak{g})$ with respect to $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar(\mathfrak{g})}$. Let $V \in \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar(\mathfrak{g})}$ and $i \in \mathbf{I}$. Then, the action of $\mathbf{S}_i$ on V is defined as follows. For any $v_\mu \in V[\mu]$,

$$\mathbf{S}_i(m) \;=\; \sum_{\substack{a,b,c\in\mathbb{Z}_{\geqslant 0}\\ a-b+c=-\mu(h_i)}} (-1)^b q_i^{b-ac} E_i^{(a)} F_i^{(b)} E_i^{(c)} \cdot v_\mu\,,$$

where $X_i^{(a)} = X^a/[a]_i!$, $X = E, F$. Clearly, we have $\mathbf{S}_i(V[\mu]) \subseteq V[s_i(\mu)]$. Note that the operators $\mathbf{S}_i$ are well-defined on any integrable $U_\hbar(\mathfrak{g})$-module. In the notation of [46, Section 5.2], $\mathbf{S}_i$ coincides with the operator $T''_{i,+1}$.

By [46, Section 39.4], the operators $\mathbf{S}_i$ ($i \in \mathbf{I}$), induce an action of the generalized braid group $\mathcal{B}_W$ on $V \in \mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar(\mathfrak{g})}$, which reduces at $\hbar = 0$ to the action by the triple exponential operators described in Section 5.6. Moreover, it recovers the *internal* braid group action on $U_\hbar(\mathfrak{g})$ defined in [45]. Namely, there is an algebra automorphism $\mathbf{T}_i$ of $U_\hbar(\mathfrak{g})$, such that, for any $u \in U_\hbar(\mathfrak{g})$ and $v \in V \in \mathcal{O}^{\mathrm{int}}_\infty$, one has $\mathbf{S}_i(u \cdot m) = \mathbf{T}_i(u) \cdot \mathbf{S}_i(m)$. In particular, for any $h \in \mathfrak{h}$, $\mathbf{T}_i(h) = s_i(h)$.

The quantum Weyl group operators recover also the action of the universal R-matrix as a multiplicative coboundary. Indeed, by [46, Section 5.3] (see also [8, Section 4.10] which follows our conventions), the operator $\mathbf{S}_i$ satisfies the coproduct identity

$$\Delta(\mathbf{S}_i) = (\mathbf{S}_i \otimes \mathbf{S}_i) \cdot \Theta_i \,,$$

where $\mathbf{R}_i \in U_\hbar(\mathfrak{b}_i^-)\widehat{\otimes}U_\hbar(\mathfrak{b}_i^+)$ is the universal R-matrix of the subalgebra $U_\hbar(\mathfrak{g}_i) = \langle E_i, F_i, h_i\rangle \subset U_\hbar(\mathfrak{g})$ corresponding to the simple root α_i and $\mathbf{R}_i = q_i^{h_i\otimes h_i/2}\cdot\Theta_i$. Note that $U_\hbar(\mathfrak{g}_i) \simeq U_{d_i\hbar}(\mathfrak{sl}_2)$.

Set $\mathbf{S}_{c,i} = q_i^{h_i^2/4}\cdot\mathbf{S}_i = \mathbf{S}_i\cdot q_i^{h_i^2/4}$. Then,

$$(\mathbf{S}_{c,i}\otimes\mathbf{S}_{c,i})^{-1}\cdot\Delta(\mathbf{S}_{c,i}) = \mathbf{R}_i\,.$$

From the identity $\Delta(\mathbf{S}_{c,i}) = \mathbf{R}_{i,21}\cdot\Delta^{op}(\mathbf{S}_{c,i})\cdot\mathbf{R}_{i,21}^{-1}$, we get the coproduct identity

$$\Delta(\mathbf{S}_{c,i}) = (\mathbf{S}_{c,i}\otimes\mathbf{S}_{c,i})\cdot\mathbf{R}_i = \mathbf{R}_i^{21}\cdot(\mathbf{S}_{c,i}\otimes\mathbf{S}_{c,i})\,.$$

This implies that $\mathbf{S}_{c,i}$ is a *half*-balance for $U_\hbar(\mathfrak{g}_i)$, i.e., it is a square root of the balance.

Remark 16 By [46, Section 5.2], we have that the squares of the operators $\mathbf{S}_i$ and $\mathbf{S}_{c,i}$ are particularly simple and related to the quantum Casimir element $C_{\hbar,i}$ of $U_\hbar(\mathfrak{g}_i)$. It acts on the indecomposable representation of rank $d+1$ as $d_i\cdot d\cdot(d+2)/2$. Set $\mathcal{K}_{\hbar,i} = C_{\hbar,i} - d_i\cdot h_i^2/2$. Then, we have

$$\mathbf{S}_i^2 = \exp(\pi\imath h_i)\cdot q^{\mathcal{K}_{\hbar,i}} \qquad \text{and} \qquad \mathbf{S}_{c,i}^2 = \exp(\pi\imath h_i)\cdot q^{C_{\hbar,i}}$$

Note that $\exp(\pi\imath h_i)$ is central, i.e., it commutes with the action of $U_\hbar(\mathfrak{g}_i)$ and therefore so is $\mathbf{S}_{c,i}^2$.

This result extends to $U_\hbar(\mathfrak{g})$. By [36], [42], [56], there is an operator $\mathbf{X}$ on $\mathcal{O}^{int}_{\infty,U_\hbar(\mathfrak{g})}$ such that

$$\Delta(\mathbf{X}) = (\mathbf{X}\otimes\mathbf{X})\cdot\mathbf{R} = \mathbf{R}^{21}\cdot(\mathbf{X}\otimes\mathbf{X})$$

and $\mathbf{X}^2$ is central. Moreover, in finite types, $\mathbf{X}$ is expressed, up to a Cartan correction, by an ordered product of quantum Weyl group operators.

5.10 The Faddeev–Reshetikhin–Takhtajan formalism

As we anticipated in Section 3.6, Faddeev, Reshetikhin, and Takhtajan provided an alternative approach to $U_\hbar(\mathfrak{g})$, which is commonly referred to as the FRT (or the R-matrix) formalism (see [25]). This yields a new presentation of $U_\hbar(\mathfrak{g})$ given in terms of a fixed matrix solution of the Yang–Baxter equation. This formalism was extended to quantum affine algebras of classical Lie type in [51].

More recently, Gautam, Rupert, and Wendlandt provided in [27] and [53] a uniform approach to the FRT formalism for $U_\hbar(\mathfrak{g})$ when $\mathfrak{g}$ is finite-dimensional, similar in spirit to the main result of [51].

Specifically, let V be an irreducible $U_\hbar(\mathfrak{g})$-module in $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar(\mathfrak{g})}$. As in Section 3.6, there is an FRT algebra $U_{\mathbf{R},V}$ arising from the evaluation of the universal R-matrix of $U_\hbar(\mathfrak{g})$ on $V \otimes V$. Then, we have that $U_\hbar(\mathfrak{g})$ is recovered as a Hopf quotient of $U_{\mathbf{R},V}$.

5.11 Trigonometric R-matrices

When $\dim(\mathfrak{g}) = \infty$, the category $\mathcal{O}^{\mathrm{int}}_{\infty,U_\hbar(\mathfrak{g})}$ contains no non-trivial finite-dimensional modules. However, when $\mathfrak{g}$ is an affine Lie algebra, i.e., the Cartan matrix of $\mathfrak{g}$ is positive semi-definite and $\mathfrak{g}$ is an extension of the loop algebra

$$\mathscr{L}\mathfrak{a} = \mathfrak{a}[t, t^{-1}]$$

for a finite-dimensional simple Lie algebra $\mathfrak{a}$, then $U_\hbar(\mathfrak{g})$ admits a subquotient $U_\hbar(\mathscr{L}\mathfrak{a})$, which quantizes $\mathscr{L}\mathfrak{a}$ and possesses a rich and interesting finite-dimensional representation theory, see, e.g., [14, Chapter 12].

The universal R-matrix of $U_\hbar(\mathfrak{g})$ does not necessarily act on finite-dimensional $U_\hbar(\mathscr{L}\mathfrak{a})$-modules, since the latter are not in

category $\mathcal{O}$. To remedy this, let z be a formal variable and consider the grading shift $\Sigma : U_\hbar(\mathfrak{g}) \to U_\hbar(\mathfrak{g})[z, z^{-1}]$ defined by

$$\Sigma(E_i) = zE_i, \qquad \Sigma(F_i) = z^{-1}F_i \qquad \text{and} \qquad \Sigma(h) = h,$$

for any $h \in \mathfrak{h}$ and $i \in \mathbf{I}$. Let $\mathbf{R}$ be the universal R-matrix of $U_\hbar(\mathfrak{g})$ and set

$$\mathbf{R}(z) = \Sigma \otimes \mathrm{id}(R).$$

Then, $R(z)$ is a formal power series in z whose coefficients have a well-defined and natural action on any tensor product of finite-dimensional $U_\hbar(\mathscr{L}\mathfrak{a})$-modules. Therefore, for any finite-dimensional $U_\hbar(\mathscr{L}\mathfrak{a})$-module V, $R(z)$ determines a matrix-valued formal series satisfying the *spectral* Yang–Baxter equation

$$\mathbf{R}_{12}(z)\mathbf{R}_{13}(zw)\mathbf{R}_{23}(w) = \mathbf{R}_{23}(w)\mathbf{R}_{13}(zw)\mathbf{R}_{12}(z)$$

in $\mathrm{End}(V \otimes V \otimes V)[\![w, z]\!]$. Note that this is a *multiplicative* version of the Yang–Baxter equation used in the introduction. By [23], the crossing symmetry (3.3) implies that, when $q \in \mathbb{C}^\times$, $\mathbf{R}_{VV}(z)$ is the expansion at $z = 0$ of a meromorphic function. Moreover, by a *generic irreducibility* argument (see, e.g., [13]), it follows that, when V is irreducible, then $\mathbf{R}_{VV}(z)$ can be normalized to a rational function in z which satisfies the Yang–Baxter equation and is commonly referred to as a *trigonometric* R-matrix (see [17], [26], [40]). A detailed account of the latest advancements in the theory of trigonometric R-matrices can be found in [32].

5.12 Quantum symmetric pairs and K-matrices

The reflection equation appeared in the eighties in the works of Cherednik [15] and Sklyanin [55] on integrable systems with a boundary condition. It arises as a factorization condition of particles moving on a half-line or a segment. As such, it is a boundary counterpart of the Yang–Baxter equation.

The solutions of the reflection equation are called *K-matrices*. In analogy with the theory of R-matrices, they naturally arise in

the representation theory of Drinfeld–Jimbo quantum groups (see [8], [10], and [11]). Their construction depends upon the choice of a distinguished coideal subalgebra, typically referred to as *quantum symmetric pair* or ıquantum group (see [44]). The construction of trigonometric K-matrices on the representation theory of quantum affine algebras is carried out in [9].

6 Rational R-matrices and Yangians

In this final section, we discuss the construction of *meromorphic* and *rational* R-matrices in the representation theory of Yangians, providing an outline of the recent works [3] and [29].

6.1 The spectral Yang–Baxter equation

Drinfeld's theory of Yangians provides a uniform method for constructing *rational* solutions of the Yang–Baxter equation with a spectral parameter. Recall that an R-matrix is a function

$$\mathcal{R}(s)\colon \mathbb{C} \to \mathrm{End}(V \otimes V)$$

(or a formal power series) satisfying the equation:

$$\mathcal{R}_{12}(s_1)\mathcal{R}_{13}(s_1+s_2)\mathcal{R}_{23}(s_2) = \mathcal{R}_{23}(s_2)\mathcal{R}_{13}(s_1+s_2)\mathcal{R}_{12}(s_1)\,. \quad (6.1)$$

The Yangian $Y_\hbar(\mathfrak{g})$ of a finite-dimensional simple complex Lie algebra $\mathfrak{g}$ was introduced in [16]. It is the canonical Hopf algebra deformation of the current algebra $U(\mathfrak{g}[z])$, and gives rise to rational solutions of (6.1) on its irreducible, finite-dimensional representations.

6.2 The Yangian $Y_\hbar(\mathfrak{g})$

Let us consider a finite-dimensional simple complex Lie algebra $\mathfrak{g}$ with Cartan matrix $A = (a_{ij})_{i,j\in\mathbf{I}}$. Similarly to Section 5.1, we fix an invertible diagonal matrix $D = \mathrm{diag}(d_i)_{i\in\mathbf{I}}$ with coprime entries $d_i \in \mathbb{Z}_{>0}$ such that DA is symmetric.

Following [16] and [18], the Yangian $Y_\hbar(\mathfrak{g})$ is the unital, associative algebra generated by elements

$$\{x^\pm_{i,r},\ \xi_{i,r}\}_{i\in\mathbf{I},\, r\in\mathbb{N}},$$

subject to the following relations:

(Y1) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{N}$: $[\xi_{i,r},\xi_{j,s}]=0$.

(Y2) For $i,j\in\mathbf{I}$ and $s\in\mathbb{N}$: $\left[\xi_{i,0},\,x^\pm_{j,s}\right]=\pm d_i a_{ij}x^\pm_{j,s}$.

(Y3) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{N}$:

$$\left[\xi_{i,r+1},\,x^\pm_{j,s}\right]-\left[\xi_{i,r},\,x^\pm_{j,s+1}\right]=\pm\hbar\frac{d_i a_{ij}}{2}\left(\xi_{i,r}x^\pm_{j,s}+x^\pm_{j,s}\xi_{i,r}\right).$$

(Y4) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{N}$:

$$\left[x^\pm_{i,r+1},x^\pm_{j,s}\right]-\left[x^\pm_{i,r},x^\pm_{j,s+1}\right]=\pm\hbar\frac{d_i a_{ij}}{2}\left(x^\pm_{i,r}x^\pm_{j,s}+x^\pm_{j,s}x^\pm_{i,r}\right).$$

(Y5) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{N}$: $[x^+_{i,r},x^-_{j,s}]=\delta_{ij}\xi_{i,r+s}$.

(Y6) Let $i\neq j\in\mathbf{I}$ and set $m=1-a_{ij}$. For any $r_1,\dots,r_m,s\in\mathbb{N}$:

$$\sum_{\pi\in\mathfrak{S}_m}\left[x^\pm_{i,r_{\pi(1)}},\left[x^\pm_{i,r_{\pi(2)}},\left[\dots,\left[x^\pm_{i,r_{\pi(m)}},x^\pm_{j,s}\right]\dots\right]\right]\right]=0,$$

where $\mathfrak{S}_m$ is the symmetric group on m elements.

It follows from the definition that the assignment

$$\sqrt{d_i}e_i\mapsto x^+_{i,0},\qquad \sqrt{d_i}f_i\mapsto x^-_{i,0},\qquad d_ih_i\mapsto\xi_{i,0},$$

for all $i\in\mathbf{I}$, extends to an algebra homomorphism $U(\mathfrak{g})\to Y_\hbar(\mathfrak{g})$.

We denote by $Y^0_\hbar(\mathfrak{g})$ and $Y^\pm_\hbar(\mathfrak{g})$ the unital subalgebras of $Y_\hbar(\mathfrak{g})$ generated by

$$\{\xi_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{\geqslant 0}}\qquad\text{and}\qquad\{x^\pm_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{\geqslant 0}},$$

respectively. Let $Y_\hbar^{\geqslant}(\mathfrak{g})$ (resp. $Y_\hbar^{\leqslant}(\mathfrak{g})$) denote the subalgebras of $Y_\hbar(\mathfrak{g})$ generated by $Y_\hbar^0(\mathfrak{g})$ and $Y_\hbar^+(\mathfrak{g})$ (resp. $Y_\hbar^0(\mathfrak{g})$ and $Y_\hbar^-(\mathfrak{g})$).

The Yangian $Y_\hbar(\mathfrak{g})$ is a Hopf algebra. Its coproduct

$$\Delta : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g})$$

is uniquely defined by the assignments

$$\Delta(\xi_{i,0}) = \xi_{i,0} \otimes 1 + 1 \otimes \xi_{i,0} \qquad \Delta(x_{i,0}^{\pm}) = x_{i,0}^{\pm} \otimes 1 + 1 \otimes x_{i,0}^{\pm}$$

and

$$\Delta(t_{i,1}) = t_{i,1} \otimes 1 + 1 \otimes t_{i,1} + \hbar\,[\xi_{i,0} \otimes 1, \Omega]\ ,$$

where

$$t_i(u) = \sum_{r\geqslant 0} t_{i,r} u^{-r-1} = \sum_{n\geqslant 0} \frac{(-1)^n}{n} \Big(\hbar \sum_{\ell\geqslant 0} \xi_{i,\ell} u^{-\ell-1}\Big)^n$$

and $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir tensor. Henceforth, we shall refer to Δ as the *Kac–Moody coproduct* of $Y_\hbar(\mathfrak{g})$.

Furthermore, $Y_\hbar(\mathfrak{g})$ is equipped with a one-parameter family of Hopf algebra automorphisms $\tau_s \colon Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})$ ($s \in \mathbb{C}$) given by

$$\tau_s(y_r) = \sum_{i=1}^{r} \binom{r}{i} s^{r-i} y_i\,,$$

where $y = x^{\pm}, \xi$. This can be regarded as a quantization of the shift automorphisms $z \mapsto z + s$ of $U(\mathfrak{g}[z])$. We set

$$\Delta_s = \tau_s \otimes \mathrm{id} \circ \Delta \qquad \text{and} \qquad \Delta_s^{\mathrm{op}} = \tau_s \otimes \mathrm{id} \circ \Delta^{\mathrm{op}}\,.$$

Remark 17 The definition of the Yangian naturally extends to any symmetrizable Kac–Moody algebra $\mathfrak{g}$, including the Cartan subalgebra $\mathfrak{h}$ among the generators to retain the embedding $\mathfrak{g} \to Y_\hbar(\mathfrak{g})$. However, it remains uncertain whether such an extension admits a coproduct in all cases. According to [31],

a coproduct exists for a specific class of Kac–Moody algebras, including all affine Lie algebras. The cases of affine rank one, in both untwisted and twisted type, are not covered in [31], and they need to be addressed separately. Note also that Ω is a topological tensor whenever $\dim(\mathfrak{g}) = \infty$. This is managed by incorporating a degree shift.

6.3 Drinfeld's R-matrix

By [16, Theorem 3], there exists a unique formal series

$$\mathcal{R}^{D}(s) \in 1^{\otimes 2} + s^{-1} Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})[\![s^{-1}]\!]$$

satisfying the intertwining equation

$$\Delta_s^{op}(x) = \mathcal{R}^{D}(s)\Delta_s(x)\mathcal{R}^{D}(s)^{-1},$$

for all $x \in Y_{\hbar}(\mathfrak{g})$, in addition to the coproduct identities

$$\Delta \otimes \mathrm{id}(\mathcal{R}^{D}(s)) = \mathcal{R}^{D}_{13}(s)\mathcal{R}^{D}_{23}(s)$$

and

$$\mathrm{id} \otimes \Delta(\mathcal{R}^{D}(s)) = \mathcal{R}^{D}_{13}(s)\mathcal{R}^{D}_{12}(s).$$

These readily imply that $\mathcal{R}^{D}(s)$ is a solution to (6.1) on $Y_{\hbar}(\mathfrak{g})^{\otimes 3}$.

6.4 Meromorphic and rational R-matrices

The algebra $Y_{\hbar}(\mathfrak{g})$ is known to have a rich finite-dimensional representation theory, with a continuous family of pairwise non-isomorphic irreducible modules classified by tuples of polynomials with complex coefficients, see, e.g., [14, Chapter 12].

Now, let us consider V_1, V_2 be two finite-dimensional irreducible $Y_{\hbar}(\mathfrak{g})$-modules, and $\mathcal{R}^{D}_{V_1,V_2}(s)$ the evaluation of $\mathcal{R}^{D}(s)$ on $V_1 \otimes V_2$. Upon normalizing by its eigenvalue on a tensor product of highest-weight vectors, the dependence of $\mathcal{R}^{D}_{V_1,V_2}(s)$

on s becomes rational [16, Theorem 4]. In particular, one obtains rational solutions to (6.1) valued in any finite-dimensional irreducible $Y_\hbar(\mathfrak{g})$-module.

For arbitrary finite-dimensional $Y_\hbar(\mathfrak{g})$-modules V_1, V_2, the analytic properties of $\mathcal{R}^D_{V_1,V_2}(s)$ were studied in [28] and [29]. This is proved to be a divergent series in s which admits two canonical *resummations* with the following representation theoretic interpretation. The category $\mathrm{Rep}_{fd}(Y_\hbar(\mathfrak{g}))$ admits no *rational* braiding, i.e., the rational normalization described above cannot be carried out uniformly for all modules, [29, Theorem 7.3]. In contrast, the two resummations of $\mathcal{R}^D_{V_1,V_2}(s)$ can be interpreted as two *meromorphic* braidings on $\mathrm{Rep}_{fd}(Y_\hbar(\mathfrak{g}))$.

These results stem from a novel and *constructive* proof of the existence of

$$\mathcal{R}^D(s) \in Y_\hbar(\mathfrak{g})^{\otimes 2}[\![s^{-1}]\!]\,.$$

Drinfeld's proof was based on cohomological methods and has not yet appeared in the literature.

6.5 The R-matrix of affine Yangians

The construction of the meromorphic R-matrices of the Yangians has been extended in [3] to the case of affine Yangians, i.e., Yangians corresponding to affine Lie algebras (with the exception of $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$). In this case, we have that finite-dimensional $Y_\hbar(\mathfrak{g})$-modules are replaced by category $\mathcal{O}$ representations, i.e., $Y_\hbar(\mathfrak{g})$-modules whose restriction to $\mathfrak{g}$ is in category $\mathcal{O}$; in fact, it is unclear whether affine Yangians admit non-trivial finite-dimensional represetations.

In contrast to the case of finite type, let us note that $Y_\hbar(\mathfrak{g})$ is not known to have a *universal* R-matrix. Nonetheless, every category $\mathcal{O}$ $Y_\hbar(\mathfrak{g})$-module V does give rise to a solution $\mathcal{R}_{VV}(s)$ of (6.1), which can be normalized to a rational solution in the case of highest weight representations.

Such rational R-matrices are completely new. A notable exception arises when $\mathfrak{g}$ is simply-laced, and V is the equivariant co-

homology of a quiver variety for the underlying affine Dynkin diagram. In this setting, a rational solution of the (6.1) corresponding to V has been constructed by Maulik and Okounkov in [49] using stable envelopes. However, it is expected, but still unclear, whether rational R-matrices constructed in [3] and [49] coincide.

In the remainder of this section, we provide an outline of the construction process for meromorphic R-matrices obtained in [3] and [29], highlighting the key steps and underlying principles. Henceforth, we fix $\hbar \in \mathbb{C}$, $\hbar \neq 0$.

6.6 Meromorphic R-matrices

Let $\mathcal{O}(Y_\hbar(\mathfrak{g}))$ be the category of $Y_\hbar(\mathfrak{g})$-modules whose restriction to $\mathfrak{g}$ is in category $\mathcal{O}$. We denote by $\underset{\mathrm{KM},s}{\otimes}$ the tensor product with respect to the shifted Kac–Moody coproduct.

The main result contained in [3] and [29] is the following. Let $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$. There are two meromorphic functions

$$\mathcal{R}^{\eta}_{V_1,V_2} \colon \mathcal{C} \to \mathrm{End}(V_1 \otimes V_2) \qquad\qquad \eta \in \{\uparrow, \downarrow\}$$

with the following properties:

(1) For $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$,

$$(1\,2) \circ \mathcal{R}^{\eta}_{V_1,V_2}(s) : V_1 \underset{\mathrm{KM},s}{\otimes} V_2 \to (V_2 \underset{\mathrm{KM},-s}{\otimes} V_1)(s)$$

is a $Y_\hbar(\mathfrak{g})$-intertwiner which is natural in V_1 and V_2.

(2) For $V_1, V_2, V_3 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, the cabling identities

$$\mathcal{R}^{\eta}_{V_1 \underset{\mathrm{KM},s_1}{\otimes} V_2,V_3}(s_2) = \mathcal{R}^{\eta}_{V_1,V_3}(s_1+s_2) \cdot \mathcal{R}^{\eta}_{V_2,V_3}(s_2)$$

and

$$\mathcal{R}_{V_1,V_2 \underset{\mathrm{KM},s_2}{\otimes} V_3}(s_1+s_2) = \mathcal{R}^{\eta}_{V_1,V_3}(s_1+s_2) \cdot \mathcal{R}^{\eta}_{V_1,V_2}(s_1)$$

is satisfied. In particular, $\mathcal{R}^{\eta}(s)$ is a solution of the quantum Yang–Baxter equation

$$\begin{aligned}&\mathcal{R}^{\eta}_{V_1,V_2}(s_1)\cdot\mathcal{R}^{\eta}_{V_1,V_3}(s_1+s_2)\cdot\mathcal{R}^{\eta}_{V_2,V_3}(s_2)\\&=\mathcal{R}^{\eta}_{V_2,V_3}(s_2)\cdot\mathcal{R}^{\eta}_{V_1,V_3}(s_1+s_2)\cdot\mathcal{R}^{\eta}_{V_1,V_2}(s_1)\end{aligned}$$

in $V_1\otimes V_2\otimes V_3$.

The properties (1) and (2) are equivalent to the existence of two *meromorphic* braidings on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$ with respect to the meromorphic (in fact, polynomial) tensor product $\underset{KM,s}{\otimes}$. Moreover, these are *translation invariant*, i.e., for $a,b\in\mathcal{C}$ and $V_1,V_2\in\mathcal{O}(Y_\hbar(\mathfrak{g}))$,

$$\mathcal{R}^{\eta}_{V_1(a),V_2(b)}(s)=\mathcal{R}^{\eta}_{V_1,V_2}(s+a-b)\,,$$

and related to each other through the *unitarity relation*

$$\mathcal{R}^{\uparrow}_{V_1,V_2}(s)^{-1}=(1\,2)\circ\mathcal{R}^{\downarrow}_{V_2,V_1}(-s)\circ(1\,2)\,,$$

for any $V_1,V_2\in\mathcal{O}(Y_\hbar(\mathfrak{g}))$.

Remark 18 In [58], Soibelman introduced the notion of a meromorphic braided tensor category. Nevertheless, that framework does not incorporate parameter-dependent tensor structures, such as those given by the shifted Kac–Moody coproduct on $Y_\hbar(\mathfrak{g})$. Thus, it does not apply to $\mathcal{O}(Y_\hbar(\mathfrak{g}))$.

In this section, instead, a meromorphic braiding on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$, with respect to $\underset{KM,s}{\otimes}$, is precisely an operator which depends meromorphically on s, it satisfies the properties (1) and (2) and it is translation invariant.

6.7 Relation to Drinfeld's R-matrix

The meromorphic R-matrices $\mathcal{R}^{\uparrow}_{V_1,V_2}(s)$ and $\mathcal{R}^{\downarrow}_{V_1,V_2}(s)$ possess further analytic properties. Set $\varepsilon(\uparrow)=+1$ and $\varepsilon(\downarrow)=-1$. Then, we have that $\mathcal{R}^{\eta}_{V_1,V_2}(s)$, $\eta\in\{\uparrow,\downarrow\}$, is holomorphic on

a half-plane $\varepsilon(\eta)\,\mathrm{Re}(s/\hbar) \gg 0$ and approaches the identity map as $\varepsilon(\eta)\,\mathrm{Re}(s/\hbar) \to \infty$. Moreover, $\mathcal{R}^{\uparrow}_{V_1,V_2}(s)$ and $\mathcal{R}^{\downarrow}_{V_1,V_2}(s)$ have the same asymptotic expansion as

$$\mathrm{Re}(s/\hbar) \to \infty \quad \text{and} \quad \mathrm{Re}(s/\hbar) \to -\infty,$$

respectively.

In finite type, by Drinfeld's uniqueness theorem, this expansion coincides with the action of $\mathcal{R}^{D}(s)$ on $V_1 \otimes V_2$.

6.8 Rational R-matrices

The meromorphic R-matrices $\mathcal{R}^{\uparrow}_{V_1,V_2}(s)$ and $\mathcal{R}^{\downarrow}_{V_1,V_2}(s)$ can be normalized to a rational operator on highest weight representations (the notion of a highest weight representation refers here to the triangular decomposition of the Yangian).

Let $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$ be representations which are generated by highest-weight vectors v_1, v_2, respectively. After normalization, $\mathcal{R}^{\uparrow}_{V_1,V_2}(s)$ and $\mathcal{R}^{\downarrow}_{V_1,V_2}(s)$ yield the *same* operator $R_{V_1,V_2}(s)$, whose matrix entries are rational in s, uniquely determined by the condition

$$R_{V_1,V_2}(s)v_1 \otimes v_2 = v_1 \otimes v_2\,.$$

Moreover, $R_{V_1,V_2}(\infty) = \mathrm{id}_{V_1\otimes V_2}$. As mentioned in Section 6.4, we have that $R_{V_1,V_2}(s)$ is not natural in V_1 and V_2, since no rational braiding exists on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$.

Remark 19 Even in finite types, the rationality property of the R-matrix presented in [3] is somewhat new. Namely, in Theorem 4 of [16] it is assumed that the representations are finite-dimensional and irreducible. Moreover, its proof is likely based on a *generic irreducibility* argument (as in the case of finite-dimensional representations of quantum affine algebras, see, e.g., [14]). Let us mention that Drinfeld's original proof has not appeared in print.

In [3], the result relies instead on the explicit calculation of the adjoint action of $\mathcal{R}^{0}(s)$ on operators from $Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g})$. It is

valid for (not necessarily irriducible) highest-weight representations, and assumes neither integrability nor generic cyclicity.

6.9 The abelianization method

The approach used in [3] and [29], revolves around a rather general principle, which is referred to as the *abelianization method*. Roughly, this amounts to building $\mathcal{R}(s)$ in the form

$$\mathcal{R}(s) = \mathcal{R}^{+}(s) \cdot \mathcal{R}^{0}(s) \cdot \mathcal{R}^{-}(s) \ ,$$

where $\mathcal{R}^{0}(s)$ is a meromorphic braiding with respect to the *Drinfeld tensor product*, $\mathcal{R}^{-}(s)$ is a rational twist relating the standard Kac–Moody tensor product and the *Drinfeld tensor product*, and $\mathcal{R}^{+}(s) = \mathcal{R}^{-}_{21}(-s)^{-1}$. Such factorizations, in the context of *Yangian doubles*, first appeared as a conjecture in the work of Khoroshkin and Tolstoy (see [41]), later proved in [60]. The same approach is expected to work in more general settings, e.g., Maulik–Okounkov Yangians, and their trigonometric and elliptic counterparts (see [1], [49], and [50]).

More precisely, the Drinfeld tensor product a meromorphic tensor structure on $\mathcal{O}(Y_{\hbar}(\mathfrak{g}))$ which depends *rationally* on s, i.e., it is a family of actions of $Y_{\hbar}(\mathfrak{g})$ on $V_1 \otimes V_2$, which is a rational function in s, denoted by $V_1 \underset{D,s}{\otimes} V_2$. This is induced by the *Drinfeld coproduct*

$$\underset{D,s}{\Delta} : Y_{\hbar}(\mathfrak{g}) \to Y_{\hbar}(\mathfrak{g}) \otimes Y_{\hbar}(\mathfrak{g})[s;\, s^{-1}]\!]$$

which is defined on the generators by

$$\underset{D,s}{\Delta}(t_{i,r}) = \tau_s(t_{i,r}) \otimes 1 + 1 \otimes t_{i,r}$$

and

$$\underset{D,s}{\Delta}(x^{+}_{i,k}) = \tau_s(x^{+}_{i,k}) \otimes 1 \ + \ 1 \otimes x^{+}_{i,k} +$$

$$+\,\hbar\sum_{N\geqslant 0} s^{-N-1}\Big(\sum_{n=0}^{N}(-1)^{n+1}\binom{N}{n}\,\xi_{i,n}\otimes x^{+}_{i,r+N-n}\Big)$$

Then, the abelianization method decouples the construction of the R-matrix for $\underset{KM,s}{\otimes}$ into two separate and independent problems. Assume the following datum is given:

(M1) two meromorphic braidings with respect to the Drinfeld tensor product on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$, i.e., a natural system of invertible intertwiners

$$(1\,2)\circ\mathcal{R}^{0,\uparrow/\downarrow}_{V_1,V_2}(s) : V_1 \underset{D,s}{\otimes} V_2 \to (V_2 \underset{D,-s}{\otimes} V_1)(s)\,,$$

for any $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, satisfying the analogue of Sections 6.6 and 6.8 with respect to the Drinfeld tensor product;

(M2) a meromorphic twist intertwining the standard tensor product and the Drinfeld tensor product on $\mathcal{O}(Y_\hbar(\mathfrak{g}))$, i.e., a natural system of invertible intertwiners

$$\mathcal{R}^{-}_{V_1,V_2}(s) : V_1 \underset{KM,s}{\otimes} V_2 \to V_1 \underset{D,s}{\otimes} V_2\,,$$

for any $V_1, V_2 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$, such that the cocycle equation

$$\mathcal{R}^{-}_{V_1 \underset{D,s_1}{\otimes} V_2,V_3}(s_2)\cdot\mathcal{R}^{-}_{V_1,V_2}(s_1) = \mathcal{R}^{-}_{V_1,V_2 \underset{D,s_2}{\otimes} V_3}(s_1+s_2)\cdot\mathcal{R}^{-}_{V_2,V_3}(s_2)$$

holds, for any $V_1, V_2, V_3 \in \mathcal{O}(Y_\hbar(\mathfrak{g}))$.

If the matrix entries of $\mathcal{R}^{-}(s)$ depend rationally on s, and

$$\mathcal{R}^{-}(\infty) = \mathrm{id},$$

then the functional nature of $\mathcal{R}^{0}(s)$ does not change upon multiplication by $\mathcal{R}^{-}(s)$ on the right and by $\mathcal{R}^{-}_{21}(-s)^{-1}$ on the left.

Therefore, the isomorphism of $Y_\hbar(\mathfrak{g})$-modules

$$(1\,2)\circ\mathcal{R}^{\uparrow/\downarrow}_{V_1,V_2}(s) : V_1 \underset{KM,s}{\otimes} V_2 \to (V_2 \underset{KM,-s}{\otimes} V_1)(s)\,,$$

defined by the commutativity of the diagram

$$\begin{array}{ccc} V_1 \underset{KM,s}{\otimes} V_2 & \xrightarrow{(1\,2)\circ\mathcal{R}^{\uparrow/\downarrow}_{V_1,V_2}(s)} & (V_2 \underset{KM,-s}{\otimes} V_1)(s) \\ \Big\downarrow{\scriptstyle \mathcal{R}^{-}_{V_1,V_2}(s)} & & \Big\downarrow{\scriptstyle \mathcal{R}^{-}_{V_2,V_1}(-s)}\,, \\ V_1 \underset{D,s}{\otimes} V_2 & \xrightarrow[(1\,2)\circ\mathcal{R}^{0,\uparrow/\downarrow}_{V_1,V_2}(s)]{} & (V_2 \underset{D,-s}{\otimes} V_1)(s) \end{array}$$

is readily seen to be a meromorphic R-matrix. In particular, one has

$$\mathcal{R}^{\uparrow/\downarrow}_{V_1,V_2}(s) = \mathcal{R}^{+}_{V_1,V_2}(s)\circ\mathcal{R}^{0,\uparrow/\downarrow}_{V_1,V_2}(s)\circ\mathcal{R}^{-}_{V_1,V_2}(s)\,,$$

where

$$\mathcal{R}^{+}_{V_1,V_2}(s) = (1\,2)\circ\mathcal{R}^{-}_{V_2,V_1}(-s)^{-1}\circ(1\,2)$$

is automatically a meromorphic twist $V_1 \underset{D,s}{\otimes} V_2 \to V_1 \underset{KM,s}{\otimes} V_2$.

The solutions to Problems (M1) and (M2) are described in greater detail in Sections 6.10 and 6.11.

6.10 The abelian R-matrix

In this subsection, we outline the construction of $\mathcal{R}^0(s)$ carried out in [3]. Let $\{t_{i,r}\}_{i\in I, r\in\mathbb{N}}$ be the generators of the commutative subalgebra $Y^0_\hbar(\mathfrak{g}) \subset Y_\hbar(\mathfrak{g})$. Since they are primitive with respect to the Drinfeld coproduct, we denote by $\mathrm{Prim}^D(Y_\hbar(\mathfrak{g}))$ the linear subspace spanned by them. In fact, it is expected that every primitive element belongs to $\mathrm{Prim}^D(Y_\hbar(\mathfrak{g}))$.

The operator $\mathcal{R}^0(s)$ is required to satisfy the analogues of the intertwining equation and the coproduct identites from Section 6.6 (1)–(2) with respect to the Drinfeld coproduct. By the

coproduct identities, it follows that $\mathcal{R}^0(s)$ is bound to have the form

$$\mathcal{R}^0(s) = \exp(\Lambda(s))$$

where $\Lambda(s) \in \mathrm{Prim}^D(Y_\hbar(\mathfrak{g}))^{\otimes 2}[\![s^{-1}]\!]$.

Clearly, every such operator would also satisfy the intertwining equation restricted to $Y^0_\hbar(\mathfrak{g})$. Thus, the only remaining constraint on $\mathcal{R}^0(s)$ is given by the intertwining equation with the operators $x^\pm_{i,r} \in Y_\hbar(\mathfrak{g})$. This *forces* $\Lambda(s)$ to be a solution of the difference equation

$$(p - p^{-1}) \det(\mathbf{B}(p)) \cdot \Lambda(s) = \mathcal{G}(s), \tag{6.2}$$

where:

- p is the shift operator defined by $p \cdot f(s) = f(s - \hbar/2)$,
- $\mathbf{B}(p)$ is the symmetrized p-Cartan matrix (this amount to replacing every entry of the symmetrized, finite or affine, Cartan matrix with the corresponding quantum number in p),
- $\mathcal{G}(s) = \sum_{ij} \mathbf{B}(p)^*_{ji} \cdot \mathcal{T}_{ij}(s)$, where $\mathbf{B}(p)^*$ is the adjoint matrix and $\mathcal{T}_{ij}(s)$ is the element

$$\mathcal{T}_{ij}(s) = \hbar^2 \sum_{m \geqslant 1} m! s^{-m-1} \sum_{\substack{a,b \geqslant 0 \\ a+b=m-1}} (-1)^a \frac{t_{i,a}}{a!} \otimes \frac{t_{j,b}}{b!}\,.$$

In both finite and affine types, one shows that (6.2) admits a unique formal solution $\Lambda(s) \in \mathrm{Prim}^D(Y_\hbar(\mathfrak{g}))^{\otimes 2}[\![s^{-1}]\!]$ (in affine types, (6.2) is an irregular difference equation, and needs to be replaced by a suitable regularization). Then, *an* abelian R-matrix for $Y_\hbar(\mathfrak{g})$ is given by $\mathcal{R}^0(s) = \exp(\Lambda(s))$.

The functional nature of this formal object is determined by evaluating (6.2) on a tensor product of two representations from $\mathcal{O}(Y_\hbar(\mathfrak{g}))$. Focusing on one matrix entry at a time, the problem becomes scalar valued. Relying on certain techniques

from complex analysis, one proves the existence and uniqueness of two fundamental solutions to the scalar analogue of (6.2). Hence, (6.2) admits two fundamental solutions $\Lambda^{\uparrow/\downarrow}_{V_1,V_2}(s)$ whose exponentials yield the meromorphic braidings $\mathcal{R}^{0,\uparrow/\downarrow}_{V_1V_2}(s)$. Moreover, their asymptotic expansions as $\mathrm{Re}(s/\hbar) \to \pm\infty$ coincide and recover the action of $\Lambda(s)$.

6.11 The rational twist

We now outline the solution to the problem (M2). The rational twist $\mathcal{R}^-(s)$ is obtained from the action of a canonical element in $(Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g}))[\![s^{-1}]\!]$, which is uniquely, and explicitly, determined by an intertwining equation.

Let $Y^\pm_\hbar(\mathfrak{g})$ (resp. $Y^0_\hbar(\mathfrak{g})$) be the unital subalgebras of $Y_\hbar(\mathfrak{g})$ generated by the Drinfeld generators $\{x^\pm_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{\geqslant 0}}$ (resp. by $\mathfrak{h}$ and the commuting Drinfeld generators $\{t_{i,r}\}_{i\in\mathbf{I},r\in\mathbb{Z}_{>0}}$) equipped with the standard grading over the root lattice Q. The definition of $\mathcal{R}^-(s)$ follows from the existence of a unique formal series

$$\mathcal{R}^-(s) = \sum_{\beta\in Q_+} \mathcal{R}^-(s)_\beta \in (Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g}))[\![s^{-1}]\!]\,,$$

satisfying the following three properties:

(P1) *Normalization:* $\mathcal{R}^-(s)_0 = 1 \otimes 1$.

(P2) *Triangularity:* for any $\beta \in Q_+$,

$$\mathcal{R}^-(s)_\beta \in (Y^-_\hbar(\mathfrak{g})_{-\beta} \otimes Y^+_\hbar(\mathfrak{g})_\beta)[\![s^{-1}]\!]\,.$$

(P3) *Intertwining equation:* for any $x \in Y_\hbar(\mathfrak{g})$,

$$\mathcal{R}^-(s)\,\underset{KM,s}{\Delta}(x) = \underset{D,s}{\Delta}(x)\,\mathcal{R}^-(s)\,.$$

In finite-type, one notes that the conditions (P1), (P2) and (P3), for $x = t_{i,1}$, where i is an arbitrary node of the Dynkin diagram, already have a unique solution. By a reduction to rank one,

the condition (P3) for arbitrary $x \in Y_\hbar(\mathfrak{g})$ reduces to $x = x_0^\pm$ in $Y_\hbar(\mathfrak{sl}_2)$ (see [29, Section 4]). In affine-type, however, the problem is harder.

Roughly, the proof carried out in [29] relies on a linear map $T : \mathfrak{h}' \to Y_\hbar^0(\mathfrak{g})$ given by $T(d_i h_i) = t_{i,1}$, where $\mathfrak{h}' \subset \mathfrak{h}$ is the span of the coroots $\{h_i\}_{i\in\mathbf{I}}$, which in affine-type is a codimension one subspace. This is a natural choice, since the formulae of the standard coproduct are explicit and fairly manageable in this case. Whenever $h \in \mathfrak{h}'$ and $\beta \in Q_+$ are such that $\beta(h) \neq 0$, the intertwining equation (P3) for $T(h)$ produces an explicit expression of the block $\mathcal{R}^-(s)_\beta$ in terms of lower blocks $\mathcal{R}^-(s)_\gamma$, $\gamma < \beta$. Specifically, let $\rho^\vee \in \mathfrak{h}$ such that $\alpha_i(\rho^\vee) = 1$, for all $i \in \mathbf{I}$. Then, one obtains a recursive equation

$$\mathcal{R}^-(s)_\beta = \hbar \sum_{k\geqslant 0} \frac{\mathrm{ad}(\square\, T(\rho^\vee))^k}{(s\ \mathrm{ht}\,\beta)^{k+1}} \sum_{\alpha\in\Phi_+} \mathrm{ht}\,(\alpha)\mathcal{R}^-(s)_{\beta-\alpha}\Omega_\alpha\,, \quad (6.3)$$

where $\square(y) = y \otimes 1 + 1 \otimes y$, for all $y \in Y_\hbar(\mathfrak{g})$.

For Yangians of finite types, this is enough to determine $\mathcal{R}^-(s)$ entirely. In affine types, however, this fails completely since we have that $\rho^\vee \notin \mathfrak{h}'$, thus there is no $T(\rho^\vee)$. Moreover, since the imaginary root δ vanishes on $\mathfrak{h}'$, the resulting system, for any $h \in \mathfrak{h}'$, is unable to determine recursively the imaginary blocks $\mathcal{R}^-(s)_{n\delta}$, $n \in \mathbb{Z}_{>0}$, and, consequently, any block $\mathcal{R}^-(s)_\beta$ with $\beta > n\delta$, for some $n \in \mathbb{Z}_{>0}$.

This issue is solved by constructing a suitable extension of T to include the *full* Cartan, and not just $\mathfrak{h}' \subset \mathfrak{h}$. This yields a recursive formula in the spirit of (6.3), and, consequently, an intertwiner

$$\mathcal{R}^-_{V_1,V_2}(s) : V_1 \underset{\mathrm{KM},s}{\otimes} V_2 \to V_1 \underset{\mathrm{D},s}{\otimes} V_2$$

which is readily seen to depend rationally on s and to satisfy the cocycle equation (M2); interestingly, in affine types, the cocycle equation only holds as an identity of rational functions valued in $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$, and does not seem to possess a natural

lift to $Y_\hbar(\mathfrak{g})$.

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