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An Overview of Some New Classes of Abelian p-Groups

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Abstract

This paper surveys results obtained in the last twelve years that gave origin to several new classes of abelian p-groups. These results have their source in the definition of the intrinsic algebraic entropy of the endomorphisms of p-groups, and are related to the notions of fully inert and uniformly fully inert subgroups, groups with minimal full inertia, countably totally projective groups, small pairs of classes of p-groups, thick-thin groups, fully-thick and fully-thin groups. All these notions are linked by a *fil rouge*, which connects results apparently far from each other. The solution to the uniformly fully inert subgroups problem and the connection between fully-thin groups and their completion in the $\oplus_c$-topology are also presented.

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1 Introduction

The book *Abelian Groups* [18] by László Fuchs was published in 2015. It was an up-to-date version of the two volumes *Infinite Abelian Groups* published in 1971 and 1973, respectively, providing a comprehensive account of the development of the theory since then. Two chapters

are devoted to p-groups, and one chapter to endomorphism rings. We refer to this book for the notation and every unexplained notion.

A nice metaphor of the theory of abelian p-groups is that of a wide forest including a large number of variety of different families of trees, each family representing a class of p-groups with specific properties, whose knowledge is far from being complete. So the discovery of new families of trees in this forest is always a pleasant event.

In the last twelve years, a dozen of papers devoted to p-groups and their endomorphism rings by Dikran Dikranjan, Brendan Goldsmith, Pat Keef, Paolo Zanardo and myself, developed topics and techniques containing new ideas, that cannot be found in Fuchs' book and are the object of this survey. The results obtained in those papers started from the notion of fully inert subgroups and gave answer to a chain of problems and questions linked by a *fil rouge* leading to small pairs of classes of p-groups, new noteworthy classes of p-groups, namely, countably totally projective and finitely fully transitive groups, thick-thin groups, fully-thick and fully-thin groups, and new results on the structure of endomorphism rings of p-groups.

The goal of this paper is to present the development of the ideas and the results contained in those papers. The solution to the uniformly fully inert subgroups problem given by Chekhlov–Danchev is also presented, as well as the connection between fully-thin groups and their completion in the $\oplus_c$-topology.

Recall that some topics on p-groups very close to those illustrated in this survey paper and not discussed here have been obtained in the last years by Breaz, Calugareanu, Chekhlov, Danchev, Goldsmith and Keef, alone or in different combinations as coauthors, focusing on the socles of the groups and often considering automorphisms instead than endomorphisms (even outside of the realm of p-groups). See [4] and the references there for these topics.

2 From the intrinsic algebraic entropy to fully inert subgroups and groups with minimal full inertia

Our story starts in 2011 when, looking at the following formula used in [15] for the computation of the algebraic entropy $ent(\phi)$ of an

endomorphism ϕ of an abelian group G:

$$\mathrm{ent}(\phi) = \sup_{F \in \mathcal{F}(G)} \lim_{n \to \infty} \log \frac{|T_{n+1}(\phi, F)|}{|T_n(\phi, F)|}$$

where $T_n(\phi, F) = F + \phi F + \phi^2 F + \ldots + \phi^{n-1} F$ is the n-th ϕ-trajectory of F and $\mathcal{F}(G)$ denotes the family of the finite subgroups of G, Dikranjan realized that, instead than using the finite subgroups, one can use the subgroups H such that $(H + \phi H)/H$ is finite; this is due to the following facts:

$$\frac{T_{n+1}(\phi, H)}{T_n(\phi, H)} \simeq \frac{\frac{T_{n+1}(\phi, H)}{H}}{\frac{T_n(\phi, H)}{H}} \quad ; \quad \frac{T_2(\phi, H)}{H} \text{ finite} \Rightarrow \frac{T_n(\phi, H)}{H} \text{ finite}$$

for all n.

In [14] we called these groups ϕ-inert. Using ϕ-inert subgroups in the formula above, one obtains a new kind of entropy, denoted by $\widetilde{\mathrm{ent}}(\phi)$ and called *intrinsic algebraic entropy*. This entropy, investigated in [14], has the advantage to coincide with the algebraic entropy when G is a torsion group, but to be non-trivial on torsion-free groups (while $\mathrm{ent}(\phi) = 0$ if G is torsion-free).

Later on, we noticed in [19] that, for certain classes of groups, including groups of finite torsion-free rank and groups which are finitely generated $\mathbb{Z}[X]$-modules (with the module structure induced by ϕ), for the computation of $\widetilde{\mathrm{ent}}(\phi)$ one can use subgroups not depending on ϕ, but only groups which are fully inert, that is, ψ-inert for all $\psi \in \mathrm{End}(G)$. Notice that fully inert subgroups are a slight generalization of the well known class of the fully invariant subgroups, that a celebrated theorem by Kaplansky characterized for the large class of fully transitive groups, which contains all separable groups and the totally projective groups (see [28]).

There is a simple way to find fully inert subgroups of a group G, using the notion of commensurability. Two subgroups H and K of a group G are *commensurable* if $H \cap K$ has finite index both in H and in K. It is easily seen that a subgroup commensurable with a fully inert subgroup is still such. Therefore, starting from a fully invariant subgroup H of G, we can obtain a fully inert subgroup just taking a finite extension of a finite subgroup of H.

These results gave the motivation for the investigation of fully in-

ert subgroups of particular classes of groups. Thus fully inert subgroups of divisible and free groups, and fully inert submodules of free and complete torsion-free J_p-modules, have been characterized in [13],[16] and [23], respectively. From these papers emerged the fact that for free groups (and similarly for free or complete torsion-free J_p-modules) every fully inert subgroup is commensurable with a fully invariant subgroup. We must notice that, while the fully invariant subgroups of a free group F are very simple, since they are of the form nF for some $n \in \mathbb{N}$ (and similarly for free J_p-modules), the fully invariant subgroups of p-groups are more complicated. Actually, the Kaplansky characterization for fully transitive groups G states that they are of the form $G(\underline{u})$, where $\underline{u}$ is a strictly increasing sequence of ordinals and symbols ∞ (for the basic notions of fully transitive groups and of subgroups of type $G(\underline{u})$ see [18]). Thus the following problem naturally arises.

Problem 2.1 *Characterize the reduced* p*-groups* G *such that every fully inert subgroup of* G *is commensurable with a fully invariant subgroup.*

It is useful at this point to introduce the following notation: given a reduced p-group G, we denote by $\mathcal{Inv}(G)$ the set of its fully invariant subgroups, by $\mathcal{Inv}\widetilde{\ }(G)$ the set of the subgroups commensurable with fully invariant subgroups, by $\mathcal{I}(G)$ the set of the fully inert subgroups of G. The set $\mathcal{I}(G)$ is called the *full inertia* of G.

Of course, the following inequalities hold:

$$\mathcal{Inv}(G) \subseteq \mathcal{Inv}\widetilde{\ }(G) \subseteq \mathcal{I}(G).$$

Thus Problem 2.1 asks for a characterization of groups G that satisfy $\mathcal{Inv}\widetilde{\ }(G) = \mathcal{I}(G)$. These groups are named *groups with minimal full inertia*, and the groups G such that $\mathcal{Inv}\widetilde{\ }(G) \subsetneq \mathcal{I}(G)$ are called *groups without minimal full inertia.*

A first step to give an answer to Problem 2.1 was made in 2014 with the paper [24], were it was shown that direct sums of cyclic p-groups have minimal full inertia. This result had a long and elaborate proof.

After eight years, a simplified version was provided in [21], using results by Chekhlov [3] and Chekhlov–Danchev–Goldsmith [6]. A further step was made when it was proved in [20], even with a long elaborate proof, that also torsion-complete p-groups have minimal full inertia.

An impressive improvement was made by Keef [30] when a third large class of p-groups of arbitrary length, strictly containing the

class of totally projective groups, and consequently the direct sums of cyclic groups, was proved to have also minimal full inertia . This new class consists of the reduced groups G, called *countably totally projective*, satisfying the property that, whenever C is a countable subgroup of G, then there is a totally projective group H and a homomorphism

$$\mu : G \to H$$

such that μ restricts to a valuated embedding on C. In [30] it is proved that the class of countably totally projective groups is closed under taking direct sums and isotype subgroups, it contains the totally projective groups, the $\aleph_1$-separable groups (we refer to [40] and [17] for these groups), and the p^{ω_1}-bounded C_{ω_1}-groups described by Hill [26] and Cutler [9]; furthermore, every countably totally projective group is fully transitive.

Thus the class of groups with minimal full inertia is quite large and we have the following result.

Theorem 2.2 (Goldsmith–Salce–Zanardo [24],[20], Keef [30]) *The class of groups with minimal full inertia contains the countably totally projective groups and the torsion-complete groups.*

Another type of groups with minimal full inertia is furnished by the following result.

Theorem 2.3 (Goldsmith–Salce [21]) *Let* $X = \bigoplus_{i\in I} G_i$, *where each* G_i *is isomorphic to a fixed unbounded fully transitive* p*-group* G *with minimal full inertia. Then* X *has minimal full inertia.*

Theorem 2.3 applies in particular when the group G is an unbounded torsion-complete group.

3 The uniformly fully inert subgroups problem and its solution

In this section we enlarge the frame of p-groups and consider arbitrary abelian groups.

In [11] the notion of fully inert subgroup was strengthened by defining H to be a *uniformly fully inert* subgroup of the abelian group G if

there exists a positive integer n such that $|(H + \phi H)/H| \leqslant n$ for every $\phi \in \mathrm{End}(G)$. It is easily seen that a subgroup commensurable with a fully invariant subgroup is uniformly fully inert. Thus we have the chain of inclusions:

$$\mathcal{I}nv(G) \subseteq \mathcal{I}nv\tilde{}(G) \subseteq \mathcal{I}_u(G) \subseteq \mathcal{I}(G)$$

where $\mathcal{I}_u(G)$ denotes the class of uniformly fully inert subgroups. The following natural question arises.

Question 3.1 *For which groups G is a uniformly fully inert subgroup necessarily commensurable with a fully invariant subgroup?*

In other terms, Question 3.1 asks whether the equality

$$\mathcal{I}nv\tilde{}(G) = \mathcal{I}_u(G)$$

holds for a group G. In [10, Conjecture 5.2] and [11, Conjecture 1.6] it was conjectured that the question has a positive answer for all groups. In [11] a positive answer was proved for all divisible groups D and for two more classes of groups: the groups compact in the natural topology, whose structure was determined by Orsatti in [36], and the groups with finite ranks.

These results are trivial in the case of reduced p-groups, since only finite p-groups belong to the two above classes. Therefore the only reduced p-groups providing a positive answer to the above question were, before 2023, those which have minimal full inertia.

In the attempt of giving an answer to Question 3.1 in the frame of p-groups, Keef introduces in [31] the class of *finitely fully transitive groups*, which are those groups G such that, given any finite subgroup F of G, any valuated homomorphism $F \to G$ extends to an endomorphism of G. If F is a singleton, we have the old notion of fully transitive groups. Keef proves that the class of finitely fully transitive groups contains all countably totally projective groups and the universally fully transitive groups investigated in [7] (hence all separable groups). In this remarkable paper Keef gives a positive answer to Question 3.1 for finitely fully transitive groups, giving support to the conjecture in [11].

The final step concerning Question 3.1 was given in summer 2023 by Chekhlov and Danchev in [5], where a complete positive solution to the conjecture for arbitrary abelian groups is provided. Their solution is based on the brilliant idea of shifting the problem from a

group G to the direct sum $G \oplus G$, where, by a result of Henriksen, characteristic subgroups are fully invariant, and then apply the solution of the analogous problem for uniformly characteristically inert subgroups, where automorphisms replace endomorphisms, already solved by Dardano–Dikranjan–Rinauro in [10, Corollary 1.9]. We enter now in more details in the proof of Chekhlov–Danchev.

The root of the solution relies in an old result by Bergman and Lenstra Jr. on non-commutative groups. Recall that, given a non-commutative group G, a subgroup H is *uniformly inert* in G if there exists a positive integer n such that $|H/(H^g \cap H)| \leqslant n$ for all $g \in G$, where H^g denotes the conjugate gHg^{-1} of H.

Theorem 3.2 (Bergman–Lenstra Jr [2]) *A subgroup K of the non-commutative group G is uniformly inert if and only if K is commensurable with a normal subgroup of G.*

Applying Theorem 3.2 to an abelian group G and using its holomorph, Dardano–Dikranjan–Rinauro obtained in [10, Corollary 1.9] the following consequence.

Corollary 3.3 (Dardano–Dikranjan–Rinauro [10]) *Every uniformly characteristically inert subgroup of an abelian group is commensurable with a characteristic subgroup.*

Recall that a characteristic subgroup of the abelian group G is sent into itself by any automorphism of G, and that a subgroup H is called uniformly characteristically inert if there exists a positive integer n such that $|H/(H \cap \xi(H))| \leqslant n$ for all automorphisms ξ of G.

The trick to pass from characteristic subgroups to fully invariant subgroups was furnished by the following matrix-theoretical result by Henriksen.

Theorem 3.4 (Henriksen [25]) *Let R be a (even non-commutative) ring with 1. Given an integer $n > 1$, every $n \times n$ matrix with coefficients in R is the sum of three invertible matrices.*

From this result it immediately follows that, if G is a non-zero abelian group, every endomorphism of $G \oplus G$ is the sum of three automorphisms (just apply Theorem 3.4 to $R = \mathrm{End}(G)$ and $n = 2$), and that a characteristic subgroup of $G \oplus G$ is fully invariant.

Remark 3.5 Theorem 3.4 is a consequence of the facts that every diagonal matrix of order $\geqslant 2$ over a ring with 1 is the sum of two

invertible matrices and every square matrix is the sum of a diagonal and an invertible matrix. This last result is credited to Kaplansky in [25].

From the above results it is not difficult to derive the announced solution.

Theorem 3.6 (Chekhlov–Danchev [5]) *A uniformly fully inert subgroup* H *of an abelian group* G *is commensurable with some fully invariant subgroup of* G.

After the Chekhlov–Danchev solution, Keef's paper [31] becomes useless for the uniformly fully inert subgroups problem. However the results obtained in [31] for the class of finitely fully transitive groups are of independent interest, providing an important substantial new contribution to the theory of p-groups.

4 From groups without minimal full inertia to thick-thin groups and small pairs

Before passing to groups without minimal full inertia, we must recall that Pierce proved in [37, Theorem 7.5] that, given two p-groups A and C, one has the group direct decomposition

$$\mathrm{Hom}(A,C) = \overline{F} \oplus \mathrm{Hom}_s(A,C),$$

where $\mathrm{Hom}_s(A,C)$ consists of the *small* homomorphisms, i.e., those morphisms $\phi : A \to C$ such that, for every $k \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $\phi(p^n A[p^k]) = 0$, and $\overline{F}$ is the completion in the p-adic topology of a free J_p-module F. If either A or C are bounded, then $\mathrm{Hom}(A,C) = \mathrm{Hom}_s(A,C)$, but this equality may hold even if both A and C are unbounded; the typical example is when A is a torsion-complete group and C is a direct sum of cyclic groups. Small homomorphisms and the related notion of large subgroups have been investigated by Megibben in [35].

In case $A = G = C$ is an unbounded group, F is always different from zero and contains $J_p \cdot 1_G$, which is the center of the ring $\mathrm{End}(G)$. Furthermore, $\mathrm{End}_s(G)$ is a two sided ideal, so $\overline{F}$ is a J_p-algebra isomorphic to the factor ring $\mathrm{End}(G)/\mathrm{End}_s(G)$. The decomposition

$$\mathrm{End}(G) = \overline{F} \oplus \mathrm{End}_s(G)$$

is called the *Pierce decomposition* of $\mathrm{End}(G)$. As we will see in a moment, a central role will be played by the size of F, more precisely, by its rank as J_p-module.

A further approach to the study of the endomorphism ring of a p-group different from the Pierce decomposition, originally presented in [37], was recently investigated in [22].

The first example of a separable p-group without minimal full inertia appeared in [24]; we used a group G constructed by Pierce (see [37, Theorem 15.4]), whose endomorphism ring, modulo the ideal of small endomorphisms, is "as small as possible", that is, equal to $J_p \cdot 1_G$.

Elaborating on the proof of that example, a general theorem ensuring that a group G does not have minimal full inertia was obtained in [21], provided its endomorphism ring $\mathrm{End}(G)$ modulo the ideal $\mathrm{End}_s(G)$ is "small". Here the term "small" means that the p-rank of the J_p-module $\overline{F}$ in the Pierce decomposition is strictly smaller than the continuum $2^{\aleph_0} = \mathfrak{c}$. Recall that the p-rank of $\overline{F}$ coincides with the rank of F as J_p-module. Actually, in the original proof in [21] this result had the stronger hypothesis that the p-rank of $\overline{F}$ was countable, but the weaker hypothesis of the continuum was proved to be enough in [22]. Here is the exact statement of that result, where the hypothesis of semi-standardness was needed; recall that a group is semi-standard if its Ulm–Kaplansky invariants of finite index are all finite.

Theorem 4.1 (Goldsmith-Salce [21] [22]) *Let G be an unbounded separable semi-standard p-group such that in the Pierce decomposition*

$$\mathrm{End}(G) = A \oplus \mathrm{End}_s(G),$$

A is the completion in the p-adic topology of a free J_p-module F of rank strictly smaller than the continuum. If H is a countably infinite subgroup of G, then the higher socles $H^F[p^k]$ $(k \geqslant 1)$ of the subgroup $H^F = \sum_{\alpha \in F} \alpha(H)$ are fully inert in G but not commensurable with any fully invariant subgroup.

The condition in Theorem 4.1 that the free J_p-module F has rank strictly smaller than the continuum, which is sufficient to ensure that the group G does not have minimal full inertia, is not necessary.

After Theorem 4.1 was proved, we realized that the Pierce decomposition of the endomorphism ring of a p-group G is a crucial tool in the investigation of our questions. We became more convinced in this

idea when, surprisingly enough, a similar hypothesis on the Pierce decomposition furnished in [32] the key tool to obtain groups which are at the same time thick and thin, consequently called *thick-thin* groups.

Recall that a p-group A is *thick* if $\mathrm{Hom}(A,C) = \mathrm{Hom}_s(A,C)$ for every direct sum of cyclic groups C, and a p-group C is *thin* if we have that $\mathrm{Hom}(A,C) = \mathrm{Hom}_s(A,C)$ for every torsion-complete group A. The easiest examples of thick groups are the torsion-complete groups, and the easiest examples of thin groups are the direct sums of cyclic p-groups. Keef noted in [30, p. 4] that every totally projective group is thin and proved in [29] that every subprojective group is such. For more information on thick and thin groups see [1],[18] and [32]. The easiest examples of thick-thin groups are the proper quasi-complete groups, as proved in [39, p. 258] and [35]. Quasi-complete groups are an old knowledge of p-groups scholars, investigated in the middle of the '60's; we refer to [18],[27] and [39] for the notions of (proper) quasi-complete groups. The application of the Pierce decomposition to thick-thin groups is given by the following result.

Theorem 4.2 (Keef–Salce [32]) *Suppose* G *is an unbounded separable group and* $\mathrm{End}(G) = A \oplus \mathrm{End}_s(G)$ *the Pierce decomposition of its endomorphism ring. If* A *is the completion in the* p*-adic topology of a free* J_p*-module* F *of rank strictly less than the continuum, then* G *is thick-thin.*

A particular case of the preceding theorems is when the J_p-module A has finite rank, so that $A = F$. If G is a thin group, this is ensured by the assumption that $\overline{G}/G$ ($\overline{G}$ denotes the torsion-completion of G) is isomorphic to a finite direct sum of λ_G copies of $\mathbb{Z}(p^\infty)$, as proved by the following

Theorem 4.3 (Goldsmith–Salce [22]) *Let* G *be a separable thin group such that* λ_G *is finite, and let* $\mathrm{End}(G) = \widehat{F} \oplus \mathrm{End}_s(G)$ *the Pierce decomposition of* $\mathrm{End}(G)$*. Then the* J_p*-rank of* F *is finite, consequently* G *is also thick.*

At the moment our knowledge of thick-thin groups does not allow to say that the condition in Theorem 4.2 that the free J_p-module F has rank strictly smaller than the continuum, which is sufficient to ensure that the group G is thick-thin, is not necessary.

Concrete examples of groups satisfying the common hypotheses of Theorems 4.1 and 4.2 are furnished by one of the celebrated realization theorems by Corner.

Theorem 4.4 (Corner [8]) *Let A be a ring whose additive group is the completion of a free p-adic module of rank $\leqslant \aleph_0$, and let μ be an infinite cardinal $\leqslant \mathfrak{c} = 2^{\aleph_0}$. Then there exists a separable p-group G with basic subgroups of final rank μ such that $\mathrm{End}(G) = A \oplus \mathrm{End}_s(G)$. Moreover, there is a family $\{G_i\}_{i<I}$ with $|I| = 2^{\mathfrak{c}}$ of such groups with the property that every homomorphism between two distinct members is small.*

In view of Theorems 4.1 and 4.2, the groups $\{G_i\}_{i<I}$ in Corner's theorem have no minimal full inertia and are thick-thin.

Note that, if $B = \oplus_{n\in\mathbb{N}}\mathbb{Z}(p^n)$ denotes the standard direct sum of cyclic groups, the group $G = B \oplus \overline{B}$ is an example of group without minimal full inertia (see [21, Example 3.6]) neither thick, nor thin. A more general result was furnished in the next

Proposition 4.5 (Goldsmith–Salce [21]) *Let $G = A \oplus C$ be a separable p-group which is the direct sum of two unbounded groups A and C, such that A is semi-standard and every homomorphism from A to C is small. Then G does not have minimal full inertia.*

By the way, a fully inert subgroup of the group G in Proposition 4.5 which fails to be commensurable with any fully invariant subgroup of G is $B[p]$, the socle of a basic subgroup B of A.

Examples of pairs (A, C) of p-groups satisfying the hypothesis of Proposition 4.5 and with minimal full inertia can be found easily, as noted above. Proposition 4.5 gave the motivation for introducing the new idea of small pairs of classes of p-groups, which are strictly related, as we will see in the next section, to thick groups and thin groups.

5 From small pairs to fully-thick and fully-thin groups

Having in mind ideas related to the generation of torsion theories and cotorsion pairs (see [38]), the definition of small pairs of classes of p-groups arose naturally and was developed in [32].

Definition 5.1 Let $\mathcal{A}$ and $\mathcal{C}$ be two families of p-groups. The ordered pair $(\mathcal{A}, \mathcal{C})$ is called a *small pair* if for every $A \in \mathcal{A}$ and every $C \in \mathcal{C}$ the equality $\mathrm{Hom}(A, C) = \mathrm{Hom}_s(A, C)$ holds and $\mathcal{A}$ and $\mathcal{C}$ are maximal with respect to this property.

The latter condition means that, if $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathrm{Hom}(A', C) = \mathrm{Hom}_s(A', C)$ holds for every $A' \in \mathcal{A}'$ and every $C \in \mathcal{C}$, then $\mathcal{A}' = \mathcal{A}$, and similarly for $\mathcal{C}$.

The couple $(\mathcal{A}, \mathcal{C})$ is a small pair if and only if the following two conditions are satisfied:

$$A \in \mathcal{A} \Leftrightarrow \mathrm{Hom}(A, C) = \mathrm{Hom}_s(A, C) \text{ for all } C \in \mathcal{C}$$

$$C \in \mathcal{C} \Leftrightarrow \mathrm{Hom}(A, C) = \mathrm{Hom}_s(A, C) \text{ for all } A \in \mathcal{A}.$$

There is a technique to get small pairs, similar to that used to obtain cotorsion pairs in [38]. Given a family of p-groups $\mathcal{A}$ and denoting by $\mathcal{G}$ the class of all p-groups, let

$$S(\mathcal{A}) = \{C \in \mathcal{G} \mid \mathrm{Hom}(A, C) = \mathrm{Hom}_s(A, C) \text{ for all } A \in \mathcal{A}\}.$$

Dually, given a family of groups $\mathcal{C}$, let

$$T(\mathcal{C}) = \{A \in \mathcal{G} \mid \mathrm{Hom}(A, C) = \mathrm{Hom}_s(A, C) \text{ for all } C \in \mathcal{C}\}.$$

If $\mathcal{A} = \{A\}$ we will simply write $S(A)$; similarly, if $\mathcal{C} = \{C\}$ we will simply write $T(C)$.

Remark 5.2 The two classes $S(\mathcal{A})$ and $T(\mathcal{C})$ have already been introduced and investigated by Keef with different notation in [29], where he also proved the next Proposition 5.4. So Keef had at disposal half of the ingredients to build up a small pair. In that paper Keef also defined the class of *cothin* groups, which are those groups G such that an homomorphism $\phi : G \to \oplus_{i \in I} A_i$ is small ($\{A_i\}_i$ any family of groups) if and only if $\pi_i \cdot \phi$ is small for all $i \in I$ (π_i is the canonical projection). In [29] it is proved that cothin groups are thick; it is still an open question whether the converse is true.

Proposition 5.3 (Keef–Salce [32]) (1) *Given an arbitrary family of groups $\mathcal{A}$, the ordered pair $(T(S(\mathcal{A})), S(\mathcal{A}))$ is a small pair, called the small pair* cogenerated *by $\mathcal{A}$.*

(2) *Given an arbitrary family of groups $\mathcal{C}$, the ordered pair $(T(\mathcal{C}), S(T(\mathcal{C})))$ is a small pair, called the small pair* generated *by $\mathcal{C}$.*

The proof of these facts is quite obvious and depends on the fact that $S(T(S(\mathcal{A}))) = S(\mathcal{A})$ and $T(S(T(\mathcal{C}))) = T(\mathcal{C})$. We can put an order on the small pairs, accordingly with the inclusion of the left classes,

i.e., $(\mathcal{A}, \mathcal{C}) < (\mathcal{A}', \mathcal{C}')$ whenever $\mathcal{A} \subsetneq \mathcal{A}'$ or, equivalently, $\mathcal{C}' \subsetneq \mathcal{C}$. There are three *trivial* small pairs:

$$(\mathcal{B}, \mathcal{G}) < (\mathcal{B} \oplus \mathcal{D}, \mathcal{R}) < (\mathcal{G}, \mathcal{B})$$

where $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{R}$ denote the classes of bounded p-groups, of divisible p-groups, and of reduced p-groups, respectively.

A key point to understand non-trivial small pairs is furnished by an old results proved by Keef. In order to state Keef's result, it is useful to introduce the following notation: the classes of thick groups and thin groups are denoted by $\mathcal{TK}$ and $\mathcal{TN}$, respectively.

Proposition 5.4 (Keef [29]) *Let* G *be a reduced unbounded* p*-group. Then* $S(G) \subseteq \mathcal{TN}$ *and* $T(G) \subseteq \mathcal{TK}$.

Thus, if $(\mathcal{A}, \mathcal{C})$ is a non-trivial small pair, then the class $\mathcal{A}$ consists of thick groups and the class $\mathcal{C}$ consists of thin groups. However, the pair $(\mathcal{TK}, \mathcal{TN})$ is not a small pair, since for a small pair $(\mathcal{A}, \mathcal{C})$ it is easily seen that $\mathcal{A} \cap \mathcal{C} = \mathcal{B}$, and, as we have seen below, there are unbounded p-groups which are thick-thin.

From Proposition 5.4 one immediately deduces that there are two small pairs which are maximal and minimal among the non-trivial small pairs. They are, respectively, the small pair $(\mathcal{TK}, S(\mathcal{TK}))$, called *thick small pair*, and $(T(\mathcal{TN}), \mathcal{TN})$ called *thin small pair*. Every non-trivial small pair lies between these two pairs, so we have the following

Question 5.5 *How many different non-trivial small pairs exist?*

An answer to the above question follows from Theorem 4.4. The groups $\{G_i\}_{i<I}$ in that theorem are thick-thin. Therefore, if G_i and G_j are two different groups in the above family, then $G_i \in S(G_j) \setminus S(G_i)$, and similarly, $G_i \in T(G_j) \setminus T(G_i)$. So there are 2^c different small pairs $(T(S(G_i)), S(G_i))$ and 2^c different small pairs $(T(G_i), S(T(G_i)))$. But we can say more: it is enough to replace the single groups G_i by arbitrary subsets of the family $\{G_i\}_{i<I}$.

Proposition 5.6 (Keef–Salce [32]) *The family* $\{G_i\}_{i<I}$ *of groups in Corner's theorem gives rise to* 2^{2^c} *different small pairs cogenerated by the subsets of* $\{G_i\}_{i<I}$*, and to* 2^{2^c} *different small pairs generated by the subsets of* $\{G_i\}_{i<I}$.

The p-groups of the right class $S(\mathcal{TK})$ of the thick small pair are called *fully-thin*, and the p-groups in the left class $T(\mathcal{TN})$ of the thin

small pair are called *fully-thick*. Thus a group G is fully-thin if

$$\mathrm{Hom}(A, G) = \mathrm{Hom}_s(A, G)$$

for every thick group A, and it is fully-thick if

$$\mathrm{Hom}(G, C) = \mathrm{Hom}_s(G, C)$$

for every thin group C. Of course, the following problem arises.

Problem 5.7 *Characterize fully-thick and fully-thin groups, or, at least, determine classes of groups which are fully-thick or fully-thin.*

Probably the answer to the first part of this problem is not easy; on the other side, the following two propositions exhibit classes of groups which are fully-thick or fully-thin.

Recall that if μ denotes the smallest measurable cardinal (if it exists), a $p^{<\mu}$-*injective group* is a group G that is p^α-injective for some $\alpha < \mu$ (in the category of p-groups), that is, $p^\alpha \mathrm{Ext}(X, G) = 0$ for all p-groups X. Recall that the p^ω-injective groups are exactly the torsion-complete groups. Then, from a result by Keef (see [29, Corollary 8]) one can deduce the following result.

Proposition 5.8 (Keef–Salce [32]) *Every pure quotient of a* $p^{<\mu}$*-injective group is fully-thick.*

We have a dual result, but requiring the separability hypothesis, where p^α-projective groups are considered, i.e., p-groups G such that $p^\alpha \mathrm{Ext}(G, X) = 0$ for every p-group X. Recall that the p^ω-projective groups are exactly the direct sums of cyclic groups.

Proposition 5.9 (Keef–Salce [32]) *Given an arbitrary ordinal* α*, every subgroup of a separable* p^α*-projective group is fully-thin.*

The next proposition, whose proof is quite direct and simple, provides examples of thick groups not fully thick, and thin groups not fully thin.

Proposition 5.10 (Keef–Salce [32]) *Suppose* T *is an unbounded reduced thick-thin group and* B *is an unbounded direct sum of cyclic groups. Then*

(1) *the thick group* $T \oplus \overline{B}$ *is neither fully-thick, nor thin;*

(2) *the thin group* $T \oplus B$ *is neither fully-thin, nor thick.*

6 More on thin and fully-thin groups

In this section we illustrate the main results obtained in [33] concerning thin and fully-thin groups, which make evident that these two classes of groups are surprisingly large.

The main question whether every totally projective group is fully-thin is still open, while it is easy to prove, by transfinite induction on the length of the group, that each totally projective group is thin, as noted by Keef in [30]. This fact is generalized in the following

Theorem 6.1 (Keef–Salce [33]) *The class of countably totally projective groups is properly contained in the class of thin groups.*

The proper inclusion depends on the fact that countably totally projective groups have minimal full inertia, by Theorem 2.2, while there exist thin groups (actually thick-thin, see Theorems 4.1 and 4.2) that fail to have minimal full inertia.

In a similar way as separability forces a p^σ-projective group to be fully-thin, we have the following improvement of Theorem 6.1.

Proposition 6.2 (Keef–Salce [33]) *A separable countably totally projective group is fully-thin.*

From this last result one can improve the result proved in [32] stating that $\aleph_1$-separable groups are thin, deriving that they are actually fully-thin.

With the aid of the Continuum Hypothesis the next theorem provides plenty of fully-thin groups. It is conceivable that it holds even without assuming CH.

Theorem 6.3 (Keef–Salce [33], CH) *If* M, N *are arbitrary separable groups, then* $\mathrm{Tor}(M, N)$ *is countably totally projective, therefore it is fully-thin and has minimal full-inertia.*

Another way to establish that certain separable groups are fully-thin is offered by topological groups. Recall that D'Este introduced in [12] the $\oplus_c$-topology for a p-group G. A basis of neighborhoods of 0 for this topology consists of the subgroups H such that G/H is a direct sum of cyclic groups. A group complete with respect to this topology is separable and is called $\oplus_c$-*complete*. D'Este proved in [12, Theorem 2.5] that separable p^σ-projective groups are $\oplus_c$-complete for every ordinal σ. Mader found in [34] an error in D'Este's paper, where she proves that the $\oplus_c$-completion of a group is $\oplus_c$-complete; this error invalidates several results in D'Este's paper, including Theorem 2.5.

Let $\check{G}$ denote the completion of G in the $\oplus_c$-topology. Mader constructed in [34] a separable $p^{\omega+1}$-projective group G such that $\check{G}/G$ is non-zero elementary, and so, reduced. This property of the factor group $\check{G}/G$ is crucial in determining when a thin group is fully-thin.

Theorem 6.4 (Keef–Salce [33]) *Suppose* G *is a separable thin group such that* $\check{G}/G$ *is reduced. Then* G *is fully-thin.*

Combining this theorem with the classical Megibben's characterization of separable thin groups (see [35]), we have the following result, which gives us a complete description of the $\oplus_c$-complete groups that are fully-thin.

Corollary 6.5 (Keef–Salce [33]) *Suppose* G *is separable and* $\check{G}/G$ *reduced. Then* G *is fully-thin if and only if it does not contain a subgroup that is an unbounded torsion-complete group.*

An application of Theorem 6.4 provides a large class of separable fully-thin groups.

Theorem 6.6 (Keef–Salce [33]) *Suppose* A *is any separable group and* M *is a reduced group such that* $M/p^{\omega}M$ *is a direct sum of cyclic groups. Then* $G = \mathrm{Tor}(A, M)$ *is fully-thin.*

The proof of Theorem 6.6 uses that the separable group G is thin and demonstrates that $\check{G}/G$ is reduced, so the conclusion follows from Theorem 6.4. More results on the completion $\check{G}$ in the $\oplus_c$-topology of a group G and its relationship with p^{σ}-projective groups can be found in [33].

Suppose now that $(\mathcal{A}, \mathcal{C})$ is a non-trivial small pair. By [29, Lemma 4], if C is a group and $p^{\omega+n}C = 0$, then $C \in \mathcal{C}$ if and only if $C/p^{\omega}C \in \mathcal{C}$. So we can deduce that totally projective groups of length strictly smaller than $\omega \cdot 2$ are fully-thin. On the other hand, we don't know whether fully-thin groups of length at least $\omega \cdot 2$ exist at all.

The following proposition states that several condition concerning fully-thin groups are equivalent, but it is not known if they actually hold true.

Proposition 6.7 (Keef–Salce [33]) *The following statements are equivalent:*

(1) *There is a fully-thin group of length* $\omega \cdot 2$.

(2) *The class of thick groups is closed under large subgroups.*

(3) *The class of fully-thin groups is closed under extensions.*

(4) *Totally projective groups of length* $\omega \cdot 2$ *are fully-thin.*

The next result lists four conditions, each one implying the next one and representing open questions. The first condition gives an answer to the question on cothin groups mentioned in Remark 5.2. The main question still open is in item (c).

Proposition 6.8 (Keef-Salce [33]) *Consider the following conditions:*

(a) *Every thick group is cothin.*

(b) *The class of fully-thin groups is closed under direct sums.*

(c) *Every totally projective group is fully-thin.*

(d) *Each one of the equivalent conditions in* Proposition 6.7*.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

7 Conclusion

Seven new classes of p-groups have been introduced and investigated in the last years, which are illustrated in the previous sections. In this section we give a short summary of these classes, recalling the first year of appearance and the main related facts.

- Groups G with minimal full inertia (2014), i.e., $\mathcal{Inv}^{\sim}(G) = \mathcal{I}(G)$. This class contains the basic classes of direct sums of cyclic groups and of torsion-complete groups, and the countably totally projective groups
- Groups G without minimal full inertia (2014), i.e., $\mathcal{Inv}^{\sim}(G) \subsetneq \mathcal{I}(G)$. This class contains the semi-standard groups with "small" endomorphism rings, that is, the p-rank of the J_p-summand A of $End(G) = A \oplus End_s(G)$ is strictly smaller than the continuum.
- Class of countably totally projective groups (2022). It is a wide extension of the class of totally projective groups and is contained in the class of groups with minimal full inertia, as well as in the class of thin groups.

- Class of finitely fully transitive groups (2023). It is a proper subclass of fully transitive groups, but large enough to contain all countably totally projective groups and the universally fully transitive groups, hence all separable groups.
- Class of thick-thin groups (2023). It contains the proper quasi-complete groups and the groups G such that the p-rank of the J_p-summand A of $End(G) = A \oplus End_s(G)$ is strictly smaller than the continuum
- Class of fully-thin groups (2023). It is a proper subclass of the thin groups, disjoint from the class of thick-thin groups. It contains the separable p^σ-projective groups and their subgroups, and certain $\oplus_c$-complete groups.
- Class of fully-thick groups (2023). It is a proper subclass of the thick groups, disjoint from the class of thick-thin groups. It contains the $p^{<\mu}$-injective groups and their pure quotients.

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