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Central Automorphisms of Zappa–Szép Products

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Abstract

In this paper, the central automorphism group of the Zappa–Szép product of two groups is obtained.

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1 Introduction

Let G be a group and $\mathrm{Aut}(G)$ be the group of automorphisms of G. An automorphism $\theta \in \mathrm{Aut}(G)$ is called a central automorphism of G if it commutes with every inner automorphism of group G or equivalently $g^{-1}\theta(g) \in Z(G)$, for all $g \in G$, where $Z(G)$ denotes the center of the group G. Central automorphisms are important to investigate the whole automorphism group $\mathrm{Aut}(G)$. In fact, the set $\mathrm{Aut}_c(G)$ of all central automorphisms of G is a normal subgroup of $\mathrm{Aut}(G)$. It is obvious that if we have an abelian automorphism group, then it coincides with the central automorphism group. The automorphism group of a p-group (where p is a prime) coinciding with the central automorphism group are studied in [2], [5] and [12]. The study of central automorphisms of a group has been an interest to the algebraists (see [2], [3], [5], [7], [8], [9], [10] and [12]).

Let H and K be two subgroups of a group G. Then, G is called the *internal Zappa–Szép product* of H and K if $G = HK$ and $H \cap K = \{1\}$. The Zappa–Szép product is a natural generalization of the semidirect product of two groups. If G is the internal Zappa–Szép product of H and K, then K appears as a right transversal to H in G. Let $h \in H$ and $k \in K$. Then

$$kh = \sigma(k, h)\tau(k, h),$$

where $\sigma(k, h) \in H$ and $\tau(k, h) \in K$. This determines the maps

$$\sigma : K \times H \to H \qquad \text{and} \qquad \tau : K \times H \to K$$

defined by

$$\sigma(k, h) = \sigma_k(h) \qquad \text{and} \qquad \tau(k, h) = \tau_h(k),$$

for all $h \in H$ and $k \in K$, respectively. These maps are called the *matched pair* of groups and satisfy the following conditions:

(C1) $\sigma_1(h) = h$ and $\tau_1(k) = k$,

(C2) $\sigma_k(1) = 1 = \tau_h(1)$,

(C3) $\sigma_{kk'}(h) = \sigma_k(\sigma_{k'}(h))$,

(C4) $\tau_h(kk') = \tau_{\sigma_{k'}(h)}(k)\tau_h(k')$,

(C5) $\sigma_k(hh') = \sigma_k(h)\sigma_{\tau_h(k)}(h')$,

(C6) $\tau_{hh'}(k) = \tau_{h'}(\tau_h(k)$,

for all $h, h' \in H$ and $k, k' \in K$ (see [4]).

Now, let H and K be two groups, $\sigma : K \times H \to H$ and $\tau : K \times H \to K$ be two maps which satisfy the above conditions. Then, the set $H \times K$ with the binary operation defined by

$$(h, k)(h', k') = (h\sigma_k(h'), \tau_{h'}(k)k')$$

forms a group called the *external Zappa–Szép product* of H and K. Note that the internal Zappa–Szép product is isomorphic to the external Zappa–Szép product (see [4, Proposition 2.4]). We will identify the external Zappa–Szép product with the internal Zappa–Szép product.

G. Zappa [18] introduced the Zappa–Szép product of two groups which was also studied by J. Szép in a series of papers (few of them are [14], [15], [16], and [17]). Finite simple groups are studied as a Zappa–Szép product of two groups with coprime orders [1]. Also, it was observed that a finite group is solvable if and only if it is a Zappa–Szép product of a Sylow p-subgroup and a Sylow p-complement [6].

H. Mousavi and A. Shomali [13] have studied the central automorphisms of the semidirect product of groups. Extending this work, in this paper, we have found the structure of the central automorphism group of the Zappa–Szép product of two groups. As an application, we have computed the central automorphism group of a group of order p^5 which is the Zappa–Szép product of two abelian groups of orders p^2 and p^3, where p is an odd prime.

Throughout the paper, $\mathbb{Z}_n$ denotes the cyclic group of order n. Let U and V be groups. Then $\mathrm{Hom}(U,V)$ denotes the group of all group homomorphisms from U to V. The group of all the surjective group homomorphisms of a group G is denoted by $\mathrm{Epi}(G)$ and i_g defined by

$$x \mapsto gxg^{-1},$$

for all $g \in G$, denotes the inner automorphism of a group G. Moreover, $M_{r\times s}(\mathbb{Z}_n)$ denotes the set of all matrices of order $r \times s$ with entries in $\mathbb{Z}_n$.

2 Structure of the central automorphism group

Let G be the Zappa–Szép product of two group H and K. Let U, V and W be any groups. Then $\mathrm{Map}(U,V)$ denotes the set of all maps between the group U and V. If $\phi, \psi \in \mathrm{Map}(U,V)$ an $\eta \in \mathrm{Map}(V,W)$, then the maps $\phi+\psi, \sigma_\phi(\psi), \tau_\phi(\psi) \in \mathrm{Map}(U,V)$ are defined by

$$(\phi+\psi)(u) = \phi(u)\psi(u),$$

$$(\sigma_\phi(\psi))(u) = \sigma_{\phi(u)}(\psi(u)),$$

$$(\tau_\phi(\psi))(u) = \tau_{\phi(u)}(\psi(u)),$$

respectively, for all $u \in U$, and the map $\eta\phi \in \mathrm{Map}(U,W)$ is defined

by $\eta\phi(u) = \eta(\phi(u))$, for all $u \in U$. Let

$$\ker(\sigma) = \{k \in K \mid \sigma_k(h) = h, \text{ for all } h \in H\}$$

and

$$\mathrm{Fix}(\sigma) = \{h \in H \mid \sigma_k(h) = h, \text{ for all } k \in K\}.$$

Similarly, we define the sets $\ker(\tau)$ and $\mathrm{Fix}(\tau)$. Now, we find the center of the group G.

Theorem 1 *Let G be the Zappa–Szép product of two groups H and K. Then*

$$Z(G) = \big\{(x, y) \in H \times K \mid x \in \mathrm{Fix}(\sigma), y \in \mathrm{Fix}(\tau), \sigma_y = i_{x^{-1}}, \tau_x = i_y\big\}.$$

Proof — Let

$$\Gamma = \big\{(x, y) \in H \times K \mid x \in \mathrm{Fix}(\sigma), y \in \mathrm{Fix}(\tau), \sigma_y = i_{x^{-1}}, \tau_x = i_y\big\}.$$

Let $(x, y) \in Z(G)$. Then, for all $h \in H$, we have $(x, y)(h, 1) = (h, 1)(x, y)$. Therefore, $(x\sigma_y(h), \tau_h(y)) = (hx, y)$. Thus, we have

$$x\sigma_y(h) = hx \qquad \text{and} \qquad y \in \mathrm{Fix}(\tau). \tag{2.1}$$

Using a similar argument, for $k \in K$, we get

$$\tau_x(k)y = yk \qquad \text{and} \qquad x \in \mathrm{Fix}(\sigma). \tag{2.2}$$

Using the Equation (2.1), we get

$$\sigma_y(h) = x^{-1}hx = i_{x^{-1}}(h),$$

for all $h \in H$. Thus $\sigma_y = i_{x^{-1}}$. Using a similar argument and (2.2), we get $\tau_x = i_y$. Hence, $(x, y) \in \Gamma$.

Conversely, let $(x, y) \in \Gamma$ and $(h, k) \in G$. Then

$$\begin{aligned}(x, y)(h, k) &= (x\sigma_y(h), \tau_h(y)k) = (xi_{x^{-1}}(h), yk)\\ &= (hx, (yky^{-1})y) = (h\sigma_k(x), i_y(k)y)\\ &= (h\sigma_k(x), \tau_x(k)y) = (h, k)(x, y).\end{aligned}$$

Thus $(x, y) \in Z(G)$. Hence, $Z(G) = \Gamma$. □

Let us define the sets

$$H^* = Fix(\sigma) \cap \ker(\tau) \cap Z(H)$$

and

$$K^* = Fix(\tau) \cap \ker(\sigma) \cap Z(K).$$

Corollary 2 *Let* G *be the Zappa–Szép product of two abelian groups* H *and* K. *Then* $Z(G) = H^* \times K^*$.

Proof — Let $(x, y) \in Z(G)$. Then using the Theorem 1,

$$x \in Fix(\sigma), \quad y \in Fix(\tau), \quad \sigma_y = i_{x^{-1}}, \quad \text{and} \quad \tau_x = i_y,$$

for all $x \in H$ and $y \in K$. Since H and K are abelian groups, $i_x = I_H$ and $i_y = I_K$, for all $x \in H$ and $y \in K$. Therefore,

$$\sigma_y(h) = h \qquad \text{and} \qquad \tau_x(k) = k,$$

for all $h \in H$ and $k \in K$. Thus $y \in \ker(\sigma)$ and $x \in \ker(\tau)$. Hence,

$$x \in Fix(\sigma) \cap \ker(\tau) \qquad \text{and} \qquad y \in Fix(\tau) \cap \ker(\sigma)$$

and the result holds. □

Corollary 3 *Let* $G = H \rtimes_\phi K$ *be the semidirect product of groups* H *and* K *with the group homomorphism* $\phi : K \longrightarrow Aut(H)$. *Then*

$$Z(G) = \left\{(x, y) \in H \times K \mid x \in Fix(\phi), y \in Z(K), \phi_y = i_{x^{-1}}\right\}.$$

Proof — By taking $\sigma = \phi$ and τ the trivial action of H on K in the Theorem 1, the proof follows immediately. □

Let $\mathcal{A}_c$ be the set of all matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\alpha \in Map(H, H), \ \beta \in Hom(K, H), \ \gamma \in Hom(H, K), \ \delta \in Map(K, K)$$

satisfy the following conditions:

(A1) $\alpha(hh') = \alpha(h)\sigma_{\gamma(h)}(\alpha(h'))$,

(A2) $h^{-1}\alpha(h) \in \mathrm{Fix}(\sigma)$ and $\gamma(h) \in \mathrm{Fix}(\tau)$,

(A3) $\sigma_{\gamma(h)} = i_{(h^{-1}\alpha(h))^{-1}}$ and $\tau_{h^{-1}\alpha(h)} = i_{\gamma(h)}$,

(A4) $\beta(k) \in \mathrm{Fix}(\sigma)$ and $\tau_{\beta(k)}(k^{-1})\delta(k) \in \mathrm{Fix}(\tau)$,

(A5) $\sigma_{\tau_{\beta(k)}(k^{-1})\delta(k)} = i_{(\beta(k))^{-1}}$ and $\tau_{\beta(k)} = i_{\tau_{\beta(k)}(k^{-1})\delta(k)}$,

(A6) $\delta(kk') = \tau_{\beta(k')}(\delta(k))\delta(k')$,

(A7) $\beta(k)\sigma_{\delta(k)}(\alpha(h)) = \alpha(\sigma_k(h))\beta(\tau_h(k))$,

(A8) $\tau_{\alpha(h)}(\delta(k))\gamma(h) = \gamma(\sigma_k(h))\delta(\tau_h(k)$,

(A9) for any $h'k' \in G$, there exists a unique $h \in H$ and $k \in K$ such that $h' = \alpha(h)\beta(k)$ and $k' = \gamma(h)\delta(k)$.

for all $h, h' \in H$ and $k, k' \in K$. Then, the set $\mathcal{A}_c$ forms a group with the binary operation defined below,

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha'\alpha + \beta'\gamma & \alpha'\beta + \beta'\delta \\ \gamma'\alpha + \delta'\gamma & \gamma'\beta + \delta'\delta \end{pmatrix}.$$

Remark 4 Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}_c$ and $h \in \mathrm{Fix}(\sigma)$ and $k \in \mathrm{Fix}(\tau)$. Then using (C4), (C5) and (A2), (A4), we get

$$\alpha(h) \in \mathrm{Fix}(\sigma), \quad \delta(k) \in \mathrm{Fix}(\tau), \quad \alpha(h)\beta(k') = \beta(k')\alpha(h)$$

and

$$\gamma(h')\delta(k) = \delta(k)\gamma(h'),$$

for all $h' \in H$ and $k' \in K$.

Remark 5 Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathcal{A}_c$. Then using (A2) and (A4), we get

$$\gamma(h)\beta(k) = \beta(k)\gamma(h),$$

for all $h \in H$ and $k \in K$. Also, using (A2) and (A5), we get

$$h^{-1}\alpha(h)\beta(k) = \beta(k)h^{-1}\alpha(h)$$

and

$$\gamma(h)\tau_{\beta(k)}(k^{-1})\delta(k) = \tau_{\beta(k)}(k^{-1})\delta(k)\gamma(h).$$

Theorem 6 *Let* G *be the Zappa–Szép product of two groups* H *and* K *and let* $\mathcal{A}_c$ *be as above. Then there exists an isomorphism of groups between* $\mathrm{Aut}_c(G)$ *and* $\mathcal{A}_c$ *given by*

$$\theta \longleftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\theta(h) = \alpha(h)\gamma(h)$ *and* $\theta(k) = \beta(k)\delta(k)$, *for all* $h \in H$ *and* $k \in K$.

Proof — Let $\theta \in \mathrm{Aut}_c(G)$. Then define the maps α, β, γ and δ by means of

$$\theta(h) = \alpha(h)\gamma(h) \quad \text{and} \quad \theta(k) = \beta(k)\delta(k),$$

for all $h \in H$ and $k \in K$. Now, $h^{-1}\alpha(h)\gamma(h) = h^{-1}\theta(h) \in Z(G)$. Therefore, by Theorem 1, (A2) and (A3) holds. Similarly,

$$\sigma_{k^{-1}}(\beta(k))\tau_{\beta(k)}(k^{-1})\delta(k) = k^{-1}\beta(k)\delta(k) = k^{-1}\theta(k) \in Z(G).$$

Therefore, by Theorem 1,

$$\sigma_{k^{-1}}(\beta(k)) \in \mathrm{Fix}(\sigma) \quad \text{and} \quad \tau_{\beta(k)}(k^{-1})\delta(k) \in \mathrm{Fix}(\tau).$$

Now,

$$\begin{aligned} \sigma_{k^{-1}}(\beta(k)) &= \sigma_k(\sigma_{k^{-1}}(\beta(k))) = \sigma_{kk^{-1}}(\beta(k)) \\ &= \sigma_1(\beta(k)) = \beta(k). \end{aligned}$$

Thus (A4) and (A5) holds. Now, for all $h, h' \in H$, we have

$$\begin{aligned}\alpha(hh')\gamma(hh') = \theta(hh') = \theta(h)\theta(h') &= (\alpha(h)\gamma(h))(\alpha(h')\gamma(h'))\\ &= \alpha(h)\sigma_{\gamma(h)}(\alpha(h'))\tau_{\alpha(h')}(\gamma(h))\gamma(h')\\ &= \alpha(h)\sigma_{\gamma(h)}(\alpha(h'))\gamma(h)\gamma(h'),\end{aligned}$$

where in the last equality we use that $\gamma(h) \in \mathrm{Fix}(\tau)$. Using the uniqueness of representation, we get

$$\alpha(hh') = \alpha(h)\sigma_{\gamma(h)}(\alpha(h'))$$

and

$$\gamma(hh') = \gamma(h)\gamma(h').$$

Thus $\gamma \in \mathrm{Hom}(H, K)$ and $(A1)$ holds. Using a similar argument for $\theta(kk') = \theta(k)\theta(k')$, we get $\beta \in \mathrm{Hom}(K, H)$ and $(A6)$ holds. Now,

$$\begin{aligned}\beta(k)\sigma_{\delta(k)}(\alpha(h)\tau_{\alpha(h)}(\delta(k))\gamma(h) = \beta(k)\delta(k)\alpha(h)\gamma(h) &= \theta(k)\theta(h)\\ &= \theta(kh) = \theta(\sigma_k(h)\tau_h(k))\\ &= \theta(\sigma_k(h))\theta(\tau_h(k))\\ &= \alpha(\sigma_k(h))\gamma(\sigma_k(h))\beta(\tau_h(k))\delta(\tau_h(k))\\ &= \alpha(\sigma_k(h))\beta(\tau_h(k))\gamma(\sigma_k(h))\delta(\tau_h(k)).\end{aligned}$$

Using the uniqueness of representation, we get $(A7)$ and $(A8)$. Since θ is onto, $(A9)$ holds. Thus, to every $\theta \in \mathrm{Aut}_c(G)$, we can associate a unique matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{A}_c$. This defines a map $\eta : \mathrm{Aut}_c(G) \longrightarrow \mathcal{A}_c$ given by

$$\theta \mapsto \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

Now, let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{A}_c$ satisfies the conditions $(A1)$–$(A9)$. Then, define $\theta : G \longrightarrow G$ by

$$\theta(hk) = \alpha(h)\gamma(h)\beta(k)\delta(k),$$

for all $h \in H$ and $k \in K$. Using $(A1)$ and $(A6)$–$(A9)$, one can easily check that θ is an endomorphism of G. Also, by $(A9)$, θ is a bijection. Thus $\theta \in \mathrm{Aut}(G)$. Now, let $g = hk \in G$. Then

$$\begin{aligned}
g^{-1}\theta(g) &= k^{-1}h^{-1}\alpha(h)\beta(k)\gamma(h)\delta(k)\\
&= \sigma_{k^{-1}}(h^{-1}\alpha(h)\beta(k))\tau_{h^{-1}\alpha(h)\beta(k)}(k^{-1})\gamma(h)\delta(k)\\
&= \sigma_{k^{-1}}(h^{-1}\alpha(h))\sigma_{\tau_{h^{-1}\alpha(h)}(k^{-1})}(\beta(k))\\
&\quad \tau_{\beta(k)}(\tau_{h^{-1}\alpha(h)}(k^{-1}))\gamma(h)\delta(k)\\
&= h^{-1}\alpha(h)\beta(k)\tau_{\beta(k)}(i_{\gamma(h)}(k^{-1}))\gamma(h)\delta(k)\\
&= h^{-1}\alpha(h)\beta(k)\tau_{\beta(k)}(\gamma(h)k^{-1}\gamma(h)^{-1})\gamma(h)\delta(k)\\
&= h^{-1}\alpha(h)\beta(k)\tau_{\sigma_{\gamma(h)^{-1}}(\beta(k))}(\gamma(h)k^{-1})\\
&\quad \tau_{\beta(k)}(\gamma(h)^{-1})\gamma(h)\delta(k)\\
&= h^{-1}\alpha(h)\beta(k)\tau_{\beta(k)}(\gamma(h)k^{-1})\gamma(h)^{-1}\gamma(h)\delta(k)\\
&= h^{-1}\alpha(h)\beta(k)\tau_{\sigma_{k^{-1}}(\beta(k))}(\gamma(h))\tau_{\beta(k)}(k^{-1})\delta(k)\\
&= h^{-1}\alpha(h)\beta(k)\gamma(h)\tau_{\beta(k)}(k^{-1})\delta(k).
\end{aligned}$$

Using $(A2)$, $(A4)$, $(C4)$ and $(C5)$, we get

$$h^{-1}\alpha(h)\beta(k) \in \mathrm{Fix}(\sigma) \quad \text{and} \quad \gamma(h)\tau_{\beta(k)}(k^{-1})\delta(k) \in \mathrm{Fix}(\tau).$$

Also, using the Remark 5, we get

$$\sigma_{\gamma(h)\tau_{\beta(k)}(k^{-1})\delta(k)} = i_{(h^{-1}\alpha(h)\beta(k))^{-1}}$$

and

$$\tau_{h^{-1}\alpha(h)\beta(k)} = i_{\gamma(h)\tau_{\beta(k)}(k^{-1})\delta(k)}.$$

Therefore, by the Theorem 1, $g^{-1}\theta(g) \in Z(H)$. Hence, $\theta \in \mathrm{Aut}_c(G)$. Therefore, η is a bijection. Now, for all $g = hk \in G$, we have

$$\begin{aligned}
\theta'\theta(hk) &= \theta'(\alpha(h)\beta(k)\gamma(h)\delta(k)) \\
&= \alpha'(\alpha(h)\beta(k))\beta'(\gamma(h)\delta(k))\gamma'(\alpha(h)\beta(k))\delta'(\gamma(h)\delta(k)) \\
&= \alpha'(\alpha(h))\sigma_{\gamma'(\alpha(h))}(\alpha'(\beta(k)))\beta'(\gamma(h)\delta(k)) \\
&\quad \gamma'(\alpha(h)\beta(k))\tau_{\beta'(\delta(k))}(\delta'(\gamma(h)))\delta'(\delta(k)) \\
&= \alpha'(\alpha(h))\alpha'(\beta(k))\beta'(\gamma(h))\beta'(\delta(k)) \\
&\quad \gamma'(\alpha(h))\gamma'(\beta(k))\delta'(\gamma(h))\delta'(\delta(k)) \\
&= \alpha'\alpha(h)\beta'\gamma(h)\alpha'\beta(k)\beta'\delta(k)\gamma'\alpha(h)\delta'\gamma(h)\gamma'\beta(k)\delta'(\delta(k)) \\
&= (\alpha'\alpha+\beta'\gamma)(h)(\alpha'\beta+\beta'\delta)(k)(\gamma'\alpha+\delta'\gamma)(h)(\gamma'\beta+\delta'\delta)(k).
\end{aligned}$$

where in the third last equality we use $(A2)$, $(A4)$ and Remark 4 and in the second last equality we use Remark 4. Writing hk as $\begin{pmatrix} h \\ k \end{pmatrix}$ and the map θ as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we get

$$\theta(hk) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \alpha(h)\beta(k) \\ \gamma(h)\delta(k) \end{pmatrix}$$

and

$$\theta'\theta(hk) = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} \alpha(h)\beta(k) \\ \gamma(h)\delta(k) \end{pmatrix} = \begin{pmatrix} \alpha'\alpha+\beta'\gamma & \alpha'\beta+\beta'\delta \\ \gamma'\alpha+\delta'\gamma & \gamma'\beta+\delta'\delta \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}.$$

Therefore,

$$\eta(\theta'\theta) = \begin{pmatrix} \alpha'\alpha+\beta'\gamma & \alpha'\beta+\beta'\delta \\ \gamma'\alpha+\delta'\gamma & \gamma'\beta+\delta'\delta \end{pmatrix} = \eta(\theta')\eta(\theta).$$

Hence, η is an isomorphism of groups. □

We will identify the central automorphisms of G with the corresponding matrices in $\mathcal{A}_c$. Note that, if $h^{-1}\alpha(h) \in H^*$, then

$$\tau_{\alpha(h)}(k) = \tau_h(k),$$

for all $h \in H$ and $k \in K$. Also, if $k^{-1}\delta(k) \in K^*$, then

$$\sigma_{\delta(k)}(h) = \sigma_k(h),$$

for all $h \in H$ and $k \in K$. Let

$$\begin{aligned}
P = \big\{ & \alpha \in Aut_c(H) \mid \sigma_k(\alpha(h)) = \alpha(\sigma_k(h)), \\
& h^{-1}\alpha(h) \in H^* \ \ \forall\, h \in H, k \in K \big\},
\end{aligned}$$

$$Q = \big\{ \beta \in Hom(K, H^*) \mid \beta(k) = \beta(\tau_h(k)) \ \ \forall\, h \in H, k \in K \big\},$$

$$R = \big\{ \gamma \in Hom(H, K^*) \mid \gamma(\sigma_k(h)) = \gamma(h) \ \ \forall\, h \in H, k \in K \big\},$$

$$\begin{aligned}
S = \big\{ & \delta \in Aut_c(K) \mid \tau_h(\delta(k)) = \delta(\tau_h(k)), \\
& k^{-1}\delta(k) \in K^* \ \ \forall\, h \in H, k \in K \big\}.
\end{aligned}$$

Then one can easily check that P, Q, R and S are all subgroups of the group $Aut_c(G)$. Let

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; \alpha \in P \right\}, \qquad B = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \;\middle|\; \beta \in Q \right\},$$

$$C = \left\{ \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \;\middle|\; \gamma \in R \right\}, \qquad D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \;\middle|\; \delta \in S \right\}.$$

be the corresponding subsets of $\mathcal{A}_c$, where 0 is the trivial group homomorphism and 1 is the identity automorphism of groups. Then one can easily check that A, B, C and D are subgroups of $\mathcal{A}_c$. Note that A and D normalize B an C. By the similar argument as in the proof of Theorem 2.4 in [11], we have the following result.

Theorem 7 *Let* G *be the Zappa–Szép product of two group* H *and* K. *Let* A, B, C *and* D *be defined as above. Then, if* $1 - \beta\gamma \in P$, *for all maps* β *and* γ, *then* $ABCD = \mathcal{A}_c$.

Corollary 8 *Let* G *be the Zappa–Szép product of two abelian groups* H *and* K. *Let* A, B, C *and* D *be defined as above. Then, if* $1 - \beta\gamma \in P$, *for all maps* β *and* γ, *then* $ABCD = \mathcal{A}_c$, *where*

$$\mathcal{A}_c = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \begin{array}{l} \alpha \in \mathrm{Epi}(H),\ \beta \in \mathrm{Hom}(K, H^*), \\ \gamma \in \mathrm{Hom}(H, K^*),\ \delta \in \mathrm{Epi}(K) \end{array} \right\}$$

and the maps α, β, γ *and* δ *satisfies*

$$h^{-1}\alpha(h) \in H^*, \quad k^{-1}\delta(k) \in K^*$$

and (A8)–(A10).

Computation

Consider a group

$$G = \left\langle a, b, c, d, e \;\middle|\; \begin{array}{c} a^p = b^p = c^p = d^p = e^p = 1,\ ab = ba, \\ ac = ca,\ ad = da,\ ae = eac,\ bc = cb,\ bd = db, \\ be = ebd,\ cd = dc,\ ce = ec,\ de = ed \end{array} \right\rangle,$$

where p is an odd prime. Note that $|G| = p^5$. Let

$$H = \langle a, b, d \mid a^p = b^p = d^p = 1,\ ab = ba,\ ad = da,\ bd = db \rangle,$$

$$K = \langle c, e \mid c^p = e^p = 1,\ ec = ce \rangle.$$

The mutual actions of H and K are defined for the generators of H and K by

$$\sigma_j(i) = \begin{cases} bd^{-1}, & \text{if } i = b, j = e \\ i, & \text{otherwise} \end{cases} \tag{2.3}$$

and

$$\tau_i(j) = \begin{cases} ec^{-1}, & \text{if } i = a, j = e \\ j, & \text{otherwise.} \end{cases} \tag{2.4}$$

Then G is a Zappa–Szép product of groups H and K. Clearly,

$$\mathrm{Fix}(\sigma) = \langle a, d \rangle,\ \ker(\sigma) = \langle c \rangle,\ \mathrm{Fix}(\tau) = \langle c \rangle \quad \text{and} \quad \ker(\tau) = \langle b, d \rangle$$

Therefore, $H^* = \langle d \rangle$, $K^* = \langle c \rangle$ and so $Z(G) = \langle c, d \rangle$.

Now, let $\alpha \in P$. Then $h^{-1}\alpha(h) \in H^* = \langle d \rangle$, for all $h \in H$. Suppose that

$$\alpha(a) = ad^{\nu_1}, \quad \alpha(b) = bd^{\nu_2} \quad \text{and} \quad \alpha(d) = d^{\nu_3},$$

where

$$0 \leqslant \nu_1, \nu_2 \leqslant p-1 \qquad \text{and} \qquad 1 \leqslant \nu_3 \leqslant p-1.$$

Then the map α can be identified with an element of $GL(3,p)$. Also, using the relations in (2.3) and (2.4), we see that

$$\sigma_e(\alpha(a)) = \alpha(\sigma_e(a)),\ \ \sigma_e(\alpha(d)) = \alpha(\sigma_e(d)),\ \ \sigma_c(\alpha(h)) = \alpha(\sigma_c(h))$$

hold trivially, for all $h \in H$. Now, using $\sigma_e(\alpha(b)) = \alpha(\sigma_e(b))$, we get

$$\begin{aligned} bd^{\nu_2 - 1} &= \sigma_e(\alpha(b)) = \alpha(\sigma_e(b)) = \alpha(bd^{-1}) \\ &= bd^{\nu_2}d^{-\nu_3} = bd^{\nu_2 - \nu_3}. \end{aligned}$$

Thus $\nu_2 - 1 \equiv \nu_2 - \nu_3 \pmod{p}$ which implies that $\nu_3 \equiv 1 \pmod{p}$. Therefore, α can be identified with the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \nu_1 & \nu_2 & 1 \end{pmatrix},$$

where $0 \leqslant \nu_1, \nu_2 \leqslant p-1$. Let

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \nu_2 & 1 \end{pmatrix} \qquad \text{and} \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \nu_1 & 0 & 1 \end{pmatrix}.$$

Then $A \simeq \langle x, y \mid x^p = y^p = 1, xy = yx \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$.

Let $\beta \in Q$. Then $\mathrm{Im}(\beta) \leqslant H^* = \langle d \rangle$. Let β be defined as

$$\beta(c) = d^{\lambda_1} \qquad \text{and} \qquad \beta(e) = d^{\lambda_2},$$

where $0 \leqslant \lambda_1, \lambda_2 \leqslant p-1$. Since $K \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ and $H^* \simeq \mathbb{Z}_p$, the map β can be identified with an element $\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix}$ of $M_{1\times 2}(\mathbb{Z}_p)$.

Clearly, using the relations in (2.3) and (2.4),

$$\beta(\tau_a(c)) = \beta(c) \qquad \text{and} \qquad \beta(\tau_h(k)) = \beta(k)$$

hold trivially, for all $h \in \{b, d\}$ and $k \in K$. Now, using $\beta(\tau_a(e)) = \beta(e)$, we get

$$\beta(e) = \beta(\tau_a(e)) = \beta(ec^{-1}) = \beta(e)\beta(c^{-1}).$$

Thus, $\beta(c) = 1$ which implies that $\lambda_1 = 0$. Hence, the map β can be identified with the matrix $\begin{pmatrix} 0 & \lambda_2 \end{pmatrix} \in M_{1\times 2}(\mathbb{Z}_p)$. Let $u = \begin{pmatrix} 0 & \lambda_2 \end{pmatrix}$. Then,

$$B \simeq \langle u \mid u^p = 1 \rangle \simeq \mathbb{Z}_p.$$

Now, let $\gamma \in R$. Then $\mathrm{Im}(\gamma) \in K^* = \langle c \rangle$. So, we define

$$\gamma(a) = c^{\mu_1}, \quad \gamma(b) = c^{\mu_2} \quad \text{and} \quad \gamma(d) = c^{\mu_3},$$

where $0 \leqslant \mu_1, \mu_2, \mu_3 \leqslant p-1$. Since

$$H \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \quad \text{and} \quad K^* \simeq \mathbb{Z}_p,$$

then the map γ can be identified with an element $\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \end{pmatrix}$ of $M_{1\times 3}(\mathbb{Z}_p)$. Clearly, using the relations in (2.3) and (2.4),

$$\gamma(\sigma_e(a)) = \gamma(a), \quad \gamma(\sigma_e(d)) = \gamma(d), \quad \gamma(\sigma_c(h)) = \gamma(h)$$

hold trivially, for all $h \in H$. Now, using

$$\gamma(b) = \gamma(\sigma_e(b)) = \gamma(bd^{-1}) = \gamma(b)\gamma(d^{-1}),$$

we get $\gamma(d) = 1$. Therefore, $\mu_3 = 0$ and γ can be identified with the matrix $\begin{pmatrix} \mu_1 & \mu_2 & 0 \end{pmatrix} \in M_{1\times 3}(\mathbb{Z}_p)$. Let

$$z = \begin{pmatrix} \mu_1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad w = \begin{pmatrix} 0 & \mu_2 & 0 \end{pmatrix}.$$

Then,

$$R \simeq \langle z, w \mid z^p = w^p = 1, zw = wz \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p.$$

Let $\delta \in S$. Then $k^{-1}\delta(k) \in K^* = \langle c \rangle$, for all $k \in K$. Suppose that

$$\delta(c) = c^{\rho_1} \qquad \text{and} \qquad \delta(e) = ec^{\rho_2},$$

where $1 \leqslant \rho_1 \leqslant p-1$ and $0 \leqslant \rho_2 \leqslant p-1$. Then the map δ can be

identified with an element of $GL(2,p)$. Clearly, using the relations in (2.3) and (2.4),

$$\tau_b(\delta(e)) = \delta(\tau_b(e)),\ \tau_d(\delta(e)) = \delta(\tau_d(e))$$

and

$$\tau_h(\delta(c)) = \delta(\tau_h(c))$$

hold trivially, for all $h \in H$. Now, using $\tau_a(\delta(e)) = \delta(\tau_a(e))$, we get

$$ec^{-1+\rho_2} = \tau_a(\delta(e)) = \delta(\tau_a(e)) = \delta(ec^{-1}) = ec^{\rho_2-\rho_1}.$$

Thus $\rho_1 \equiv 1 \pmod p$. Therefore,

$$\delta(c) = c \qquad \text{and} \qquad \delta(e) = ec^{\rho_2},$$

where $0 \leqslant \rho_2 \leqslant p-1$. Hence, the map δ can be identified with the matrix

$$\begin{pmatrix} 1 & \rho_2 \\ 0 & 1 \end{pmatrix} \in GL(2,p).$$

Let

$$t = \begin{pmatrix} 1 & \rho_2 \\ 0 & 1 \end{pmatrix}.$$

Then $D \simeq \langle t \mid t^p = 1 \rangle \simeq \mathbb{Z}_p$.

Also, one can easily verify that

$$\alpha\beta = \beta = \beta\delta, \qquad \gamma\alpha = \gamma = \delta\gamma, \qquad \text{and} \qquad \beta\gamma = 0 = \gamma\beta.$$

Therefore, $Aut_c(G) \simeq A \times B \times C \times D$. Hence,

$$Aut_c(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p.$$

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